4. Calculating $ds$ in a different coordinate system

Cylindrical polar coordinates are defined by

$$
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{align*}
$$

(a) Confirm that $dx = d\rho \cos \phi - \rho \sin \phi d\phi$.

(b) Calculate a similar expression for $dy$.

(c) Starting from $ds^2 = dx^2 + dy^2 + dz^2$ show that $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$.

(d) Having warmed up with that calculation, repeat with spherical polar coordinates which are defined by

$$
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
$$

and show that $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$.

*Hint:* The spherical result is easier to get starting from the cylindrical result and using $\rho = r \sin \theta$.

4. Solution: Calculating $ds$ in a different coordinate system

(a) This is a simple application of the product rule $dx = d\rho \cos \phi - \rho \sin \phi d\phi$.

(b) $dy = d\rho \sin \phi + \rho \cos \phi d\phi$.

(c) Now

$$
\begin{align*}
dx^2 + dy^2 &= (d\rho \cos \phi - \rho \sin \phi d\phi)^2 + (d\rho \sin \phi + \rho \cos \phi d\phi)^2
\end{align*}
$$

The cross terms cancel so

$$
\begin{align*}
dx^2 + dy^2 &= d\rho^2 (\cos^2 \phi + \sin^2 \phi) + \rho^2 d\phi^2 (\sin^2 \phi + \cos^2 \phi) = d\rho^2 + \rho^2 d\phi^2
\end{align*}
$$

Adding $dz^2$ gives the desired result.

(d) Start with $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$. Now $\rho^2 = r^2 \sin^2 \theta$ and the product rule applied to $d\rho$ gives

$$
\begin{align*}
d\rho &= dr \sin \theta + r \cos \theta d\theta
\end{align*}
$$

and also

$$
\begin{align*}
dz &= dr \cos \theta - r \sin \theta d\theta
\end{align*}
$$

Thus substituting into $ds^2$ from above gives

$$
\begin{align*}
ds^2 &= (dr \sin \theta + r \cos \theta d\theta)^2 + r^2 \sin^2 \theta d\phi^2 + (dr \cos \theta - r \sin \theta d\theta)^2
\end{align*}
$$

Once again the cross-terms cancel leaving

$$
\begin{align*}
ds^2 &= dr^2 (\sin^2 \theta + \cos^2 \theta) + r^2 (\cos^2 \theta + \sin^2 \theta) d\theta^2 + r^2 \sin^2 \theta d\phi^2
\end{align*}
$$

which simplifies to the desired result $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$. 
5. Geodesics on the Sphere

The equation of a sphere in spherical polar coordinates is particularly simple: it is \( r = a \), where \( a \) is a constant.

(a) Starting with \( ds \) in spherical polar coordinates, write down the simplified form of \( ds \) when \( r = a \) is a constant.

(b) Use this expression for \( ds \) to write down an integral that represents the distance between two points connected by a path that lies on the surface of a sphere. Write the integral in the form where \( \phi \) is a function of \( \theta \).

(c) Write down a first integral for this integrand.

(d) Show that

\[
\phi - \phi_0 = \sin^{-1}[\alpha \cot \theta]
\]

satisfies the first integral, where \( \phi_0 \) and \( \alpha \) are two independent constants.

(e) The equation of a plane through the origin is \( Ax + By + Cz = 0 \). Rewrite this equation in spherical polar coordinates. Rearrange the equation to make it look like the solution above and find \( \alpha \) and \( \phi_0 \) in terms of \( A, B \) and \( C \).

(f) Thus give a simple geometric description and method of finding geodesics on a sphere.

5. Solution: Geodesics on the Sphere

(a) If \( r = a \) is a constant then

\[
ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2.
\]

(b) The integral is

\[
I = \int ds = a \int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2 \theta \phi'^2} d\theta.
\]

Thus \( F(\theta, \phi, \phi') = \sqrt{1 + \sin^2 \theta \phi'^2} \).

(c) Since \( \theta F / \partial \phi = 0 \) a first integral is

\[
\frac{\sin^2 \theta \phi'}{\sqrt{1 + \sin^2 \theta \phi'^2}} = C
\]

or

\[
\phi' = \pm \left( \frac{C}{\sin \theta \sqrt{\sin^2 \theta - C^2}} \right).
\]

(d) Direct differentiation and some algebra, yields the result.

(e) In spherical coordinates this becomes

\[
A r \sin \theta \cos \phi + B r \sin \theta \sin \phi + C r \cos \theta = 0
\]

The \( r \) cancels, the \( \cos \theta \) can be moved to the other side and both sides divided by \( \sin \theta \) to give

\[
A \cos \phi + B \sin \phi = -C \cot \theta
\]

Trig identities can be used to rewrite the left hand side as

\[
\sqrt{A^2 + B^2} \sin(\phi - \phi_0) = -C \cot \theta
\]

where \( \phi_0 = -\tan^{-1}(A/B) \) and \( \alpha = -C / \sqrt{A^2 + B^2} \).

(f) In other words, the curve with the shortest distance lies simultaneously on the surface of a sphere AND on a plane through the origin. The intersection of such a plane and a sphere is called a great circle.