Math 401 HW#10, due Monday 11/25/19 **NAME:**

1. Read the question and hint for exercise §7 (11). You don't need to write anything down for this problem (everyone gets 10 points).

2. Exercise §7 (20)
   Consider the surface given by a right cylinder over a simple curve C. (See §3 (7) for the precise definition). Calculate all of its geodesics and show that the exponential map of this surface is defined on the whole of TS, provided that C is compact or has infinite length. Check, however, that there are curves C for which the resulting surface S is not closed. (This shows that the converse of Corollary 7.41 is not true.)

3. §8.9 Consider the surface of $\mathbb{R}^3$, given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = r^2\}$$

   where $0 < r < a$, that is, the torus of revolution of Example 2.17. Show that if K is its Gauss curvature and $(x, y, z) \in S$ then

$$\begin{aligned} K(x,y,z) > 0 &\quad \text{if and only if} \quad x^2 + y^2 > a^2, \\ K(x,y,z) < 0 &\quad \text{if and only if} \quad x^2 + y^2 < a^2, \end{aligned}$$

   and that the image of the set of points of S with vanishing Gauss curvature through a Gauss map N of S consists only of the north and south poles of the unit sphere. Show also that each point of the sphere, except these two poles, has exactly two preimages through N, one of them with positive Gauss curvature and the other one with negative Gauss curvature. From this, determine the function $\deg(N, -)$ defined on $S^2 \setminus \{\text{north pole, south pole}\}$ and compute $\deg(N)$.

4. Do Carmo §4.5 (6)
   Show that $(0, 0)$ is an isolated singular point and compute the index at $(0, 0)$ of the following vector fields in the plane:
   (a) $V = (x, y)$
   (b) $V = (-x, y)$
   (c) $V = (x, -y)$
   (d) $V = (x^2 - y^2, -2xy)$
   * extra credit (2 points): $V = (x^3 - 3xy^2, y^3 - 3x^2y)$

5. Do Carmo §4.5 (8)
   Prove that an orientable compact surface $S \subset \mathbb{R}^3$ has a differentiable vector field without singular points if and only if S is homeomorphic to a torus.

* Assignment Reflections
  How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience?