Math 444/539 HW#10, due Monday 11/22/19 NAME:

1. Let M be a smooth manifold with a Riemannian metric $g : TM \otimes TM \to \mathbb{R}$. If $f : M \to \mathbb{R}$ is a smooth function, the *gradient* of f with respect to g is the vector field ∇f defined by

$$df = g(\nabla f, \cdot).$$

   (a) In local coordinates $\{x^i\}$, if $g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij}$, explain how to compute ∇f in terms of g_{ij} and $\partial f/\partial x^i$. *Hint:* See HW#6.

   (b) Let $f : M \to \mathbb{R}$ and let $p \in M$. Show that if $V \in T_pM$ satisfies $df_p(V) > 0$, then there exists a Riemannian metric g on M with $\nabla f(p) = V$.

2. Lee 11.9 [SECOND].
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

and let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the following map (which is the inverse of stereographic projection):

$$F(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right)$$

Compute one of F^*df and $d(f \circ F)$, your choice. Verify with a classmate (who computed the other) that they are equal.

3. Lee 11-14 [SECOND].
Consider the following two covector fields on $\mathbb{R}^3$.

$$\omega = -\frac{4z}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2x}{x^2+1}dz$$

$$\eta = -\frac{4xz}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2}{x^2+1}dz$$

   (a) Set up an evaluate the line integral of each covector field along the straight line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

   (b) Determine whether either of these covector fields is exact. As in multivariable calculus, a covector field α is exact whenever there exists a potential function f such that $df = \alpha$.

   (c) For each one that is exact, find a potential function and use it to recompute the line integral.

Note: This is a multivariable calculus problem in our fancy language, which will help us make sense of Chapter 16 and 17.

math* Lee 11.15 [Line Integrals of Vector Fields].
Let X be a smooth vector field on an open subset $U \subset \mathbb{R}$. Given a piecewise smooth curve segment $\gamma : [a, b] \to U$, define the **line integral of** X **over** γ,

$$\int_\gamma X \cdot ds := \int_a^b X_{\gamma(t)} \cdot \gamma'(t)dt,$$

where the dot on the right-hand side denotes the Euclidean dot product between tangent vectors at $\gamma(t)$, identified with elements of $\mathbb{R}^n$. A **conservative vector field** is one whose line integral around every piecewise smooth closed curve is zero.

(a) Show that X is conservative if and only if there exists a smooth function $f \in C^\infty(U)$ such that $X = \operatorname{grad} f$. *Hint: consider the covector field* ω *defined by* $\omega_x(v) = X_x \cdot v$.

(b) Suppose $n = 3$. Show that if X is conservative, then curl $X = 0$, where

$$\operatorname{curl} X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$

(c) Show that if $U \subset \mathbb{R}^3$ is star-shaped, then X is conservative on U if and only if curl $X = 0$.

everyone: How difficult was this assignment? How many hours did you spend on it?