1. Lee 14.6 SECOND

Define a 2-form $\omega$ on $\mathbb{R}^3$ by

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$ 

(a) Compute $\omega$ in spherical coordinates $(\rho, \varphi, \theta)$ defined by

$$(x, y, z) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi).$$

(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

(c) Compute the pullback $\iota^* S^2 \omega$ to $S^2$, using coordinates $(\varphi, \theta)$ on the open subset where these coordinates are defined.

(d) Show that $\iota^* S^2 \omega$ is nowhere zero.

2. Lee 14.7 SECOND

Let $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$, $\omega = ydz \wedge dx$, and $F : M \to N$ is the smooth map defined by

$$F(\theta, \varphi) = ((\cos \varphi + 2) \cos \theta, (\cos \varphi + 2) \sin \theta, \sin \varphi).$$

Compute $d\omega$ and $F^* \omega$, and verify by direct computation that $F^*(d\omega) = d(F^* \omega)$.

3. Define a 1-form $\alpha$ on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by

$$\alpha = \left( -\frac{y}{x^2 + y^2} \right) dx + \left( \frac{x}{x^2 + y^2} \right) dy.$$ 

(a) Calculate $\int_C \alpha$ for any circle $C$ of radius $r$ around the origin.

(b) Prove that in the half plane $\{x > 0\}$, $\alpha$ is the differential of a function.

4. Lee 16.10 SECOND

Let $D$ denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(r - 2)^2 + z^2 = 1$ around the $z$-axis (example 5.17), with its induced Riemannian metric and with the orientation determined by the outward unit normal.

(a) Compute the surface area of $D$

(b) Compute the integral over $D$ of the 2-form $\omega = zdx \wedge dy$.

everyone: How difficult was this assignment? How many hours did you spend on it?