Math 401 HW#2, due Wednesday 9/11/24 NAME:

For full credit, pick one of 1 (theory) or 2 (computation) to upload to gradescope.
Do not turn in ⋆ in class problems. But please finish and understand them!
UPDATE 8/29: New torsion/curvature problem. Two exercises were moved to HW 3.

⋆ **Multi Warm Up:** Find an equation for the tangent plane to the surface at the point $(1, -2, 1)$:

$$S = \begin{cases} x &= e^{u-v} \\ y &= u - 3v \\ z &= \frac{1}{2}(u^2 + v^2) \end{cases}$$

⋆ **Multi Warm Up**
Find a parametrization for each of the following surfaces (perhaps involving an angular variable that is defined only up to multiples of 2π).

(a) The surface obtained by revolving the curve $z = f(x)$, $a < x < b$ in the xz-plane around the z-axis, where $a > 0$.

(b) The surface obtained by revolving the curve $z = f(x)$, $a < x < b$ in the xz-plane around the x-axis, where $f(x) > 0$.

(c) The lower sheet of the hyperboloid $z^2 - 2x^2 - y^2 = 1$.

(d) The cylinder $x^2 + z^2 = 9$.

1. **Exercise 1.9, Theory, in lieu of (2)**
Let $\alpha : I \to \mathbb{R}^3$ be a curve and $[a, b] \subset I$. Prove that

$$|\alpha(a) - \alpha(b)| \leq L_b^a(\alpha).$$

In other words, straight lines are the shortest curves joining two given points.

2. **Computation, in lieu of (1)**
For the following curve $\alpha : \mathbb{R} \to \mathbb{R}$, compute $\mathbf{T}, \mathbf{N}, \mathbf{B}$, and $k(t)$.

$$\alpha(t) = \langle e^t, e^{-t}, t\sqrt{2} \rangle.$$

3. **Exercise 1.37** Let $\alpha : I \to \mathbb{R}^3$ be a curve p.b.a.l. with curvature $k(s) > 0$ for every $s \in I$. Prove that α is an arc of a circular helix or an arc of a circle if and only if both the curvature and the torsion of α are constant. (See example 1.5 for their parametrizations.)

4. **Remembering Multi:** For each of the following maps $f : \mathbb{R}^2 \to \mathbb{R}^3$,

- Describe or sketch the (possibly singular) surface $S = f(\mathbb{R}^2)$
- Find a description of S as the locus of an equation $F(x, y, z) = 0$.
- Find the points where $\partial_u f$ and $\partial_v f$ are linearly dependent, and describe the singularities of S (if any) at these points.

(a) $f(u, v) = (2u + v, u - v, 3v)$
(b) $f(u, v) = (au \cos v, bu \sin v, u)$ with $a, b > 0$
(c) $f(u, v) = (u \cos v, u \sin v, u^2)$

* Assignment Reflections
How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience?