Math 444/539 HW#2, due Monday 9/9/19 NAME:

1. Show that a space X is simply connected if and only if there exists a unique homotopy class of paths connecting any two points in X.

2. Show that for a space X, the following three conditions are equivalent:

   (a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

   (b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

   (c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

   Deduce that a space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints.']

3. Let $\varphi : X \to Y$ be a continuous map and let γ be a class of paths in X from x_0 to x_1. Prove that the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\varphi_*} & \pi_1(Y, \varphi(x_0)) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \pi_1(X, x_1) & \xrightarrow{\varphi_*} & \pi_1(Y, \varphi(x_1)) \end{array}$$

The isomorphism u is defined by $u(\alpha) = \gamma^{-1}\alpha\gamma$ and v is defined similarly using $\varphi_*(\gamma)$ instead of γ. *Note: A important special case occurs if* $\varphi(x_0) = \varphi(x_1)$. *Then,* $\varphi_*(\gamma)$ *is an element of the group* $\pi_1(Y, \varphi(x_0))$.

4. Show that if G is a topological group (a topological space with a group structure such that inversion and multiplication are continuous), then $\pi_1(G, 1)$ is abelian.

5. Prove that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n \neq 2$. *Hint: Consider the complement of a point in* $\mathbb{R}^2$ *or* $\mathbb{R}^n$*. You do not need to prove* $\pi_1(S^n) = 0$ *for* $n \geq 2$*.*

*math: Let $\{U_i\}$ be an open covering of the space X having the following properties:

(a) There exists $x_0 \in X$ such that $x_0 \in U_i$ for all i.

(b) Each U_i is simply connected.

(c) If $i \neq j$, then $U_i \cap U_j$ is path connected.

Prove that X is simply connected.
Hint: To prove that any loop $f : I \to X$ based at x_0 is trivial, consider the open covering $\{f^{-1}(U_i)\}$ of the compact metric space I and use the Lebesgue number[1] *of this covering.*

Remark: The two most important cases of this exercise are (1) a covering by two open sets and (2) when the sets U_i are linearly ordered by inclusion.

[1]See page 48 of Massey for more on the Lebesgue number of coverings of intervals.

*math: Using the result of the previous exercise in the case that there is a covering consisting of two open sets, prove that the n-sphere, S^n, for $n \geq 2$ is simply connected.

everyone: How difficult was this assignment? How many hours did you spend on it?