UPDATED Math 401 HW#3, due Wednesday 9/18/19 NAME:

1. Multi Warm Up
   Let $(u, v) = f(x, y) = (x - 2y, 2x - y)$.
   (a) Compute the inverse transformation $(x, y) = f^{-1}(u, v)$.
   (b) Find the image (e.g. sketch it) in the uv-plane of the triangle bounded by the lines $Y = x, y = -x, y = 1 - 2x$.
   (c) Find the region in the xy-plane that is mapped to the triangle with vertices $(0, 0), (-1, 2)$, and $(2, 1)$ in the uv-plane.

2. Multi meets Math 401
   Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate map,

$$(x, y, z) = f(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

   (a) Without looking this up, describe the surfaces in xyz-space that are the images of the planes $\rho =$ positive constant, $\varphi =$ constant (check $(0, \frac{\pi}{2})$, $\frac{\pi}{2}$, $(\frac{\pi}{2}, \pi)$ separately), and $\theta =$ constant (in $[0, 2\pi)$).
   (b) Compute the derivative Df and show the Jacobian is $\det Df(\rho, \varphi, \theta) = \rho^2 \sin\varphi$.
   (c) What is the condition on the point $(\rho_0, \varphi_0, \theta_0)$ for f to be locally invertible about this point? What is the corresponding condition on $(x_0, y_0, z_0) = f(\rho_0, \varphi_0, \theta_0)$?

3. Exercise 2.43
   If $X : U \to S$ is a parametrization of a surface, show that $X : U \to X(U)$ is a diffeomorphism.

4. Exercise §2 (13)
   Let $f : S_1 \to S_2$ be a differentiable map between surfaces. If $p \in S_1$ and $\{e_1, e_2\}$ is an orthonormal basis of T_pS, we define the **absolute value of the Jacobian of f at p** by

$$|\text{Jac}f|(p) = |(df)_p(e_1) \wedge (df)_p(e_2)|.$$

   (a) Prove that this definition does not depend on the orthonormal basis.
   (b) Prove that $|\text{Jac}f|(p) \neq 0$ if and only if $(df)_p$ is an isomorphism.

5. Weird but true
   Given polar (r, θ) and rectangular $(x := r\cos\theta, y := r\sin\theta)$ coordinates on $\mathbb{R}^2$ we have that the coordinate vector fields transform by

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}, \end{aligned}$$

   thanks to my Math 444/539 students who proved this fun fact for arbitrary coordinate transformations in any finite dimension. Using this fact, consider $f(x, y) = x^2$ on $\mathbb{R}^2$ and let X be the vector field

$$X = \text{ grad } f = 2x\frac{\partial}{\partial x}$$

Compute the coordinate expression of X in polar coordinates (on some open subset on which they are defined) using the above proposition and show that it is *not* equal to

$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}.$$

* Assignment Reflections
  How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience?