Math 401 HW#3, due Wednesday 9/18/24 **NAME:**

Do not upload solutions to the **In Class Warm Up Problems** to gradescope.
UPDATE 9/10: The parametrization is a diffeo exercise was moved to HW 4.

⋆ **Multi Warm Up**
Let $(x, y) = F(u, v) = (u^2 - v^2, 2uv)$.

(a) Restricting to $u > 0$, sketch the image of the xy-plane of the coordinate grid.

(b) Draw the curves in the $u > 0$ half-plane that map to the coordinate grid in the xy-plane.

1. **Spherical coordinates**
   Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate map,

$$(x, y, z) = f(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

   (a) Describe and attempt to sketch by hand the surfaces in xyz-space that are the images of $\rho =$ positive constant, $\varphi =$ c (check $c \in (0, \frac{\pi}{2})$, $c = \frac{\pi}{2}$, $c \in (\frac{\pi}{2}, \pi)$ separately), and $\theta = k \in [0, 2\pi)$.

   (b) Compute the derivative Df and show the Jacobian is $\det Df(\rho, \varphi, \theta) = \rho^2 \sin\varphi$.

   (c) What is the condition on the point $(\rho_0, \varphi_0, \theta_0)$ for f to be locally invertible about this point? What is the corresponding condition on $(x_0, y_0, z_0) = f(\rho_0, \varphi_0, \theta_0)$?

2. **Folland §3.4**: Transformations and Coordinate Systems
   Let $(u, v) = F(x, y) = (x - y, xy)$.

   (a) Sketch some curves $x - y =$ constant and $xy =$ constant in the xy-plane. Which regions in the xy-plane map onto the rectangle in the uv-plane given by $0 \leq u \leq 1,\ 1 \leq v \leq 4$? There are two of them; draw a picture of them.

   (b) Compute the derivative DF and the Jacobian $J = \det DF$

   (c) When J vanishes at a point p, what can be said about the gradients $\nabla u(p)$ and $\nabla v(p)$?

   (d) Notice that $F(2, -3) = (5, -6)$. Compute explicitly the local inverse G of F such that $G(5, -6) = (2, -3)$ and compute its derivative DG.

   (e) Note that $DF(2, -3)$ and $DG(5, -6)$ are matrix inverses of each other.

3. **Exercise 2.24**
   Prove that one-sheeted cone is not a surface: $C = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2, z \geq 0\}$

4. **First part of Exercise 2 (7)**
   Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid e^{x^2} + e^{y^2} + e^{z^2} = a\}$, with $a > 3$. Prove that S is a surface.

* Assignment Reflections
How difficult was this assignment? How many hours did you spend on it? Which problems did you find helpful?