Math 401 HW#6, due Wednesday 10/23/24 NAME:

1. Exercise 3.22
   Consider as a surface S the hyperbolic paraboloid given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}.$$

   Show that the second fundamental form of S at the point $(0, 0, 0)$ is not a semi-definite bilinear form, i.e. the Gauss curvature of S is negative at this point.

2. Exercise 3.24
   Let S be a regular quadric - see Example 2.16 - given implicitly by the second degree equation

$$0 = (1, p^t)A \begin{pmatrix} 1 \\ p \end{pmatrix} = \langle Bp, p\rangle + 2\langle b, p\rangle + c, \quad p \in \mathbb{R}^3,$$

   where A is a symmetric matrix of order four, B is a non-null symmetric matrix of order three, $b \in \mathbb{R}^3$, and $c \in \mathbb{R}$. Show that

$$N(p) = \frac{Bp + b}{|Bp + b|}, \quad \text{for all } p \in S,$$

   is a Gauss map defined on S. As a consequence, check that the corresponding second fundamental form is given by

$$\sigma_p(v, v) = -\frac{1}{|Bp + b|}\langle Bv, v\rangle, \quad p \in S,\ v \in T_pS.$$

   Conclude that an ellipsoid has positive Gauss curvature at each of its points.

3. Exercise 3.25 (invariance under rigid motions) (10 points)
   Let S be an orientable surface and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the rigid motion given by $\phi(p) = Ap + b$ where $A \in O(3)$ and $b \in \mathbb{R}^3$. If N is a Gauss map for the surface S, prove that $N' = A \circ N \circ \phi^{-1}$ is a Gauss map for the image surface $S' = \phi(S)$. Conclude that

$$(dN')_{\phi(p)} = A \circ (dN)_p \circ A^{-1}$$

   and

$$\sigma'_{\phi(p)}((d\phi)_p(v), (d\phi)_p(w)) = \sigma'_{\phi(p)}(Av, Aw) = \sigma_p(v, w),$$

   for each $p \in S$, $v, w \in T_pS$, where σ and σ' stand, respectively, for the second fundamental forms of S and S'. Finally, find the relationship between the Gauss and mean curvatures of S and S'.

4. Exercise §3 (8)
   Use the methods of Section 3.6 to compute the Gauss and mean curvatures of the torus of revolution described in Example 2.17. Find its elliptic, hyperbolic, and parabolic points. (Recall your computations on HW 5 #1)

* Assignment Reflections
   How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience?