Math 444/539 HW#6, due Monday 10/21/19 NAME:

1. (Put a CW complex on $\mathbb{RP}^n$)
The n-dimensional real projective space, denoted by $\mathbb{RP}^n$, is the set of all 1-dimensional subspaces of $\mathbb{R}^{n+1}$. It may be topologized as a quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ or of the unit sphere S^n. Each 1-dimensional subspace of $\mathbb{R}^{n+1}$ intersects S^n in a pair of antipodal points. The inclusions
$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset ... \subset \mathbb{R}^{n+1}$$
give rise to corresponding inclusions of real projective spaces:
$$\mathbb{RP}^0 \subset \mathbb{RP}^1 \subset ... \subset \mathbb{RP}^n.$$
   (a) What common spaces are $\mathbb{RP}^0$ and $\mathbb{RP}^1$ respectively?
   (b) Show that $\mathbb{RP}^k$ is obtained from $\mathbb{RP}^{k-1}$ by attaching a single cell of dimension k.

   In (b), using homogenous coordinates, the map $f : B_1^k$ to $\mathbb{RP}^k$, defined by $f_k(x_1, ..., x_k) = (x_1, ..., x_k, \sqrt{1 - ||x||^2})$ where $x = (x_1, ..., x_k)$, may prove to be helpful.

2. Consider spherical coordinates on $\mathbb{R}^3$ (not including the line $x = y = 0$) ρ, ϕ, θ defined in terms of the Euclidean coordinates x, y, z by
$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$
   (a) Express $\partial/\partial\rho$, $\partial/\partial\phi$, and $\partial/\partial\theta$ as linear combinations of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$. (The coefficients in these linear combinations will be functions on $\mathbb{R}^3 \setminus (x = y = 0)$.)
   (b) Express $d\rho$, $d\phi$, and $d\theta$ as linear combinations of dx, dy, and dz.

3. Let V and W be finite dimensional vector spaces and let $A : V \to W$ be a linear map. Show that the dual map $A^* : W^* \to V^*$ is given in coordinates as follows. Let $\{e_i\}$ and $\{f_j\}$ be bases for V and W, and let $\{e^i\}$ and $\{f^j\}$ be the corresponding dual bases for V^* and W^*. If $Ae_i = A_i^j f_j$ then $A^* f^j = A_i^j e^i$.

4. Let V be a finite dimensional vector space and let $\langle\cdot,\cdot\rangle$ be an inner product on V. The inner product determines an isomorphism $\phi : V \to V^*$.
   (a) Show that the isomorphism ϕ is given in coordinates as follows. Let $\{e_i\}$ be a basis for V, let $\{e^i\}$ be the dual basis, and write $g_{ij} = \langle e_i, e_j\rangle$. Then $\phi(e_i) = g_{ij}e^j$.
   (b) The inner product, together with the isomorphism ϕ, define an inner product on V^*. Write this in coordinates as $g^{ij} = \langle e^i, e^j\rangle$. Show that the matrix (g^{ij}) is the inverse of the matrix (g_{ij}).

5. Lee 1-6 [First Edition] = Lee 1-8 [Second Edition]
By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

*math: Lee 1.7 [First Edition] = Lee 1.9 [Second Edition]

(Just do the case $n = 1$.) Also check that the projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ is smooth.

Complex projective n-space, denoted by $\mathbb{C}P^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$. Show that $\mathbb{C}P^n$ is a compact 2n-dimensional topological manifold, and show how to give it a smooth structure analogous to the one constructed for $\mathbb{R}\mathbb{P}^n$ (done in an earlier example in Chapter 1 in both versions). We use the correspondence

$$(x^i + iy^1, ..., x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, ...x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{n+2}$

everyone: How difficult was this assignment? How many hours did you spend on it?