Math 401 HW#7, due Wednesday 10/30/19 NAME:

1. Exercise §6 (1)
   Let S be an ovaloid in $\mathbb{R}^3$. Show that
   $$\int_S H^2 \geq 4\pi,$$
   and that equality occurs if and only if S is a sphere.

2. Exercise §6 (7)
   Show that the following integral formulas are always valid:
   $$\int_S \langle N, a\rangle H = 0 \quad \text{and} \quad \int_S \langle N, a\rangle K = 0$$
   where S is a compact surface, N its Gauss map, H and K its mean and Gauss curvatures, and $a \in \mathbb{R}^3$ is an arbitrary vector.

3. Exercise §6 (8)
   Let S be a compact surface contained in a closed ball of radius $r > 0$ and such that its mean curvature satisfies $|H| \leq 1/r$. Prove that S is a sphere of radius r.

4. Exercise §6 (16)
   Let S be a compact surface and $V : S \to \mathbb{R}^3$ a tangent vector field. Prove that
   $$\int_S (\mathrm{div} V)(p)dp = 0,$$
   $$\int_S \{k_1(p)\langle (dV)_p(e_1), e_1\rangle + k_2(p)\langle (dV)_p(e_2), e_2\rangle\}dp = 0.$$
   where $\{e_1, e_2\}$ is a basis of principal directions at T_pS for each $p \in S$.

5. Exercise §6 (17)
   Let $f : S \to \mathbb{R}$ be a differentiable function defined on a surface S. We use the term *gradient* of f for the vector field of tangent vectors denoted by $\nabla f : S \to \mathbb{R}^3$ and given by
   $$\begin{cases} \langle (\nabla f)(p), v\rangle = (df)_p(v) & \text{for all } v \in T_pS, \\ \langle (\nabla f)(p), N(p)\rangle = 0 \end{cases}$$
   where $N(p)$ is a unit normal to S at p. Prove that ∇f is a differentiable vector field and that, if it is identically zero, f is constant on each connected component of S. *This is HW 6 #4 rehashed (sorry), but now we have a definition of a gradient...*

* Assignment Reflections
How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience?