Math 444/539 UPDATED HW#8, due Tuesday 11/5/19   NAME:

1. Consider the function $f(x, y) = \sin(4\pi x)\cos(6\pi y)$ on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$.

   (a) Prove that f is a Morse function, e.g. that every critical point is nondegenerate. Calculate the number of minima, saddles, and maxima. You can appeal to the standard second derivative test from calculus.

   (b) Describe the evolution of the sublevel sets $f^{-1}((-\infty, c))$ as c varies from the lowest minimum value to the highest maximum value. You may use wolfram alpha or another computer aided means in your quest.

2. The *Hopf fibration* is the map $f : S^3 \to \mathbb{C}P^1$ sending $(z^1, z^2) \in \mathbb{C}^2$ with $|z^1|^2 + |z^2|^2 = 1$ to $[z^1 : z^2] \in \mathbb{C}P^1$. Show that the Hopf fibration is a submersion.

3. (Lee Second 4-6) Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

4. (Lee Second 5-1) Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$

   Show that $(0, 1)$ is a regular value of ϕ and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to S^2.

5. (Lee Second 5-10) For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \{(x, y) \mid y^2 = x(x - 1)(x - a)\}.$$

   For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

math* Let M be a smooth manifold and let $f : M \to M$ be a smooth map.

   (a) Define the *diagonal*

$$\Delta = \{(p, p) \mid p \in M\} \subset M \times M$$

   and the *graph*

$$\Gamma(f) = \{(p, f(p)) \mid p \in M\} \subset M \times M.$$

   Check that Δ and $\Gamma(f)$ are submanifolds of $M \times M$ which are canonically diffeomorphic to M.

   (b) A *fixed point* of f is a point $p \in M$ with $f(p) = p$. A fixed point p is *nondegenerate* if $1 - df_p : T_pM \to T_pM$ is invertible. Show that all fixed points of f are nondegenerate if and only if $\Gamma(f)$ is transverse to Δ.

   (c) The *Lefschetz sign* of a nondegenerate fixed point p, denoted by $\epsilon(p) \in \{\pm 1\}$, is the sign of the determinant of $1 - df_p$. If all fixed points are nondegenerate, and if there are only finitely many fixed points, define the signed count of fixed points by

$$\#\operatorname{Fix}(f) = \sum_{f(p)=p} \epsilon(p) \in \mathbb{Z}.$$

Let A be a 2×2 integer matrix. The map $A : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a map $f_A : T^2 \to T^2$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. If A does not have 1 as an eigenvalue, show that all fixed points of f_A are nondegenerate, and compute $\#\operatorname{Fix}(f)$ in terms of A.

Remark 1. If $\Gamma(f)$ is transverse to Δ, and if M is compact and oriented, then the intersection number[1] is given by

$$\Gamma(f) \cdot \Delta = \#\operatorname{Fix}(f).$$

For those of you who know what homology is, this can be used to prove the *Lefschetz fixed point theorem*

$$\#\operatorname{Fix}(f) = \sum_i (-1)^i \operatorname{Tr}(f_* : H_i(M; \mathbb{Q}) \to H_i(M; \mathbb{Q})).$$

everyone: How difficult was this assignment? How many hours did you spend on it?

[1] The intersection number is something we won't have a chance to discuss this semester.