Math 444/539 HW#9, due Friday 11/15/19 NAME:

1. (a) Suppose that $A : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and V is a vector subspace of $\mathbb{R}^n$. Check that $A \pitchfork V$ is equivalent to $A(\mathbb{R}^K) + V = \mathbb{R}^n$.
   (b) If V and W are linear subspaces of $\mathbb{R}^n$, check that $V \pitchfork W$ is equivalent to $V + W = \mathbb{R}^n$.

2. For which values of R does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = R$ transversely? What does the intersection look like for different values of R?

3. (Lee Second 6-9) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined by
$$F(x, y) = (e^y \cos x, e^y \sin x, e^{-y}).$$
(a) For which positive numbers r is F transverse to the 2-sphere of radius r, $S_r(0) \subset \mathbb{R}^3$?
(b) For which positive numbers r is $F^{-1}(S_r(0))$ an embedded submanifold of $\mathbb{R}^2$?

4. Lee 11.7 [SECOND].
In the following subproblems, M and N are smooth manifolds, $F : M \to N$ is a smooth map, and ω is a covector field on N. Compute $F^*\omega$ in each case.
   (a) $M = N = \mathbb{R}^2,\ F(s, t) = (st, e^t)$,
   $\omega = xdy - ydx$
   (b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3,\ F(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi)$,
   $\omega = z^2 dx$
   (c) $M = \{(s, t) \in \mathbb{R}^2 \mid s^2 + t^2 < 1\}$ and $N = \mathbb{R}^3 \setminus \{0\}$, $F(s, t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$,
   $\omega = (1 - x^2 - y^2)dz$

5. Lee Exercise 11.17 (page 280, SECOND)
Given polar (r, θ) and rectangular $(x := r\cos\theta, y := r\sin\theta)$ coordinates on $\mathbb{R}^2$. Using this fact, consider $f(x, y) = x^2$ on $\mathbb{R}^2$ and let X be the vector field
$$X = \text{ grad } f = 2x\frac{\partial}{\partial x}$$
Compute the coordinate expression of X in polar coordinates (on some open subset on which they are defined) using Equation (11.4) on page 275, and show that it is *not* equal to
$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}.$$
Takeaway: The partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field. However, they can be interpreted as the components of a covector field. This is the most important application of covector fields.

math* Lee SECOND 6-10: Suppose $F : N \to M$ is a smooth map that is transverse to an embedded submanifold $X \subset M$, and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_pW = (dF_p)^{-1}(T_{F(p)}X)$. Conclude that if two embedded submanifolds $X, X' \subset M$ intersect transversely, then $T_p(X \cap X') = T_pX \cap T_pX'$ for every $p \in X \cap X'$.

everyone: How difficult was this assignment? How many hours did you spend on it?