Math 402/500 HW#2, due Friday 2/12/21 NAME:

1. Lee 4-6 [SECOND]
   Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$. A submersion is a smooth map whose differential is surjective.

2. Lee 5-1 [SECOND]
   Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ defined by
   $$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$
   Show that $(0, 1)$ is a regular value of Φ. (Note: the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to S^2.)

3. Lee 8-11 [SECOND]
   For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.
   $$\text{(a)} \quad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$
   $$\text{(b)} \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

4. Lee Exercise 11.17 (page 280, SECOND)
   Given polar (r, θ) and rectangular $(x := r\cos\theta, y := r\sin\theta)$ coordinates on $\mathbb{R}^2$ we have that the coordinate vector fields transform, using Equation (11.4) on page 275, by
   $$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$
   $$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y},$$
   thanks to my Fall 2019 Math 444/539 students who proved this proposition for arbitrary coordinate transformations in any finite dimension. Using this fact, consider $f(x, y) = x^2$ on $\mathbb{R}^2$ and let X be the vector field
   $$X = \text{ grad } f = 2x\frac{\partial}{\partial x}$$
   Compute the coordinate expression of X in polar coordinates (on some open subset on which they are defined) using the above proposition and show that it is *not* equal to
   $$\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}.$$
   Takeaway: The partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field. However, they can be interpreted as the components of a covector field. This is the most important application of covector fields, aka differential 1-forms.

everyone: How difficult was this assignment? How many hours did you spend on it?