Math 402/500 HW #3, due Friday 2/26/21 NAME:

1. Let V and W be finite dimensional vector spaces and let $A : V \to W$ be a linear map. Show that the dual map $A^* : W^* \to V^*$ is given in coordinates as follows. Let $\{e_i\}$ and $\{f_j\}$ be bases for V and W, and let $\{e^i\}$ and $\{f^j\}$ be the corresponding dual bases for V^* and W^*. If $Ae_i = A_i^j f_j$ then $A^* f^j = A_i^j e^i$.

2. Let V be a finite dimensional vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on V. The inner product determines an isomorphism $\phi : V \to V^*$.

   (a) Show that the isomorphism ϕ is given in coordinates as follows. Let $\{e_i\}$ be a basis for V, let $\{e^i\}$ be the dual basis, and write $g_{ij} = \langle e_i, e_j \rangle$. Then $\phi(e_i) = g_{ij} e^j$.

   (b) The inner product, together with the isomorphism ϕ, define an inner product on V^*. Write this in coordinates as $g^{ij} = \langle e^i, e^j \rangle$. Show that the matrix (g^{ij}) is the inverse of the matrix (g_{ij}).

3. Math 401: Curves and Surfaces, HW problem
   We associate to each differentiable function $f : A \to \mathbb{R}$, defined on a subset A of Euclidean space, a vector field ∇f, also defined on A, by

   $$\langle (\nabla f)(p), v \rangle = (df)_p(v), \qquad \text{for all } v \in \mathbb{R}^3$$

   This vector field is called the *gradient* of f. Prove that the three components of ∇f are

   $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

   Show that, if the gradient of a differentiable function f vanishes everywhere, then f is constant on each connected component of A.

everyone: How difficult was this assignment? How many hours did you spend on it?