Math 402/500 HW #4, due Friday 3/5/21 NAME:

1. Consider the function $f(x, y) = \sin(4\pi x)\cos(6\pi y)$ on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$.
   (a) Prove that f is a Morse function, e.g. that every critical point is nondegenerate. Calculate the number of minima, saddles, and maxima. You can appeal to the standard second derivative test from calculus.
   (b) Describe the evolution of the sublevel sets $f^{-1}((-\infty, c))$ as c varies from the lowest minimum value to the highest maximum value. You may use wolfram alpha or another computer aided means in your quest.

2. Let M be a smooth manifold with a Riemannian metric $g : TM \otimes TM \to \mathbb{R}$. If $f : M \to \mathbb{R}$ is a smooth function, the *gradient* of f with respect to g is the vector field ∇f defined by
$$df = g(\nabla f, \cdot).$$
   (a) In local coordinates $\{x^i\}$, if $g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij}$, explain how to compute ∇f in terms of g_{ij} and $\partial f/\partial x^i$. *Hint:* Recall #1 and #2 on HW # 3.
   (b) Let $f : M \to \mathbb{R}$ and let $p \in M$. Show that if $V \in T_pM$ satisfies $df_p(V) > 0$, then there exists a Riemannian metric g on M with $\nabla f(p) = V$.

3. (a) Suppose that $A : \mathbb{R}^k \to \mathbb{R}^n$ is a linear map and V is a vector subspace of $\mathbb{R}^n$. Check that $A \pitchfork V$ is equivalent to $A(\mathbb{R}^k) + V = \mathbb{R}^n$.
   (b) If V and W are linear subspaces of $\mathbb{R}^n$, check that $V \pitchfork W$ is equivalent to $V + W = \mathbb{R}^n$.

4. For which values of R does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = R$ transversely? What does the intersection look like for different values of R?

everyone: How difficult was this assignment? How many hours did you spend on it?