Math 402/500 HW #6, due Monday 3/29/21 NAME:

1. Compute the Morse homology of $\mathbb{RP}^2$ using the Morse function in Exercise 6 of Audin-Damian. Try to work out the orientations (heuristic pictorial arguments suffice).

Exercise 6 (On the sphere and on the real projective plane). With essentially the same formula, let us now consider the function

$$f : S^2 \longrightarrow \mathbf{R}$$
$$(x, y, z) \longmapsto y^2 + 2z^2$$

and the resulting function $g : \mathbf{P}^2(\mathbf{R}) \to \mathbf{R}$ on the real projective plane that follows from it by passing to the quotient. Verify that f (and therefore g) is a Morse function and show that it has six critical points (and therefore that g has three):

- Two points of index 0, the points $(\pm 1, 0, 0)$, at level 0
- Two points of index 1, the points $(0, \pm 1, 0)$, at level 1
- Two points of index 2, the points $(0, 0, \pm 1)$, at level 2.

Figure 1.7 shows a few level sets of this function on the sphere. The critical level set containing the two points of index 1 consists of the two circles defined by intersecting the sphere with the planes $z = \pm x$.

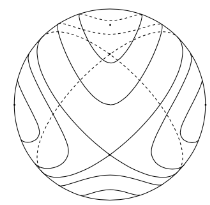

Fig. 1.7 A Morse function on the real projective plane

2. Use Morse theory to prove that if V is a compact connected manifold of dimension 1, then V is diffeomorphic to S^1 if $\partial V = \emptyset$ and diffeomorphic to $[0, 1]$ otherwise. See Audin-Damian 2.3.b if you get stuck. Please write the proof in your own words.

3. Define $f : [-1, 1]^n \to \mathbb{R}$ by

$$f(x_1, \ldots, x_n) = \frac{1}{4} \sum_{i=1}^{n} (x_i + 1)^2 (x_i - 1)^2$$

and let g be the Euclidean metric. Show that

$$-\nabla f = -\sum_{i=1}^{n} (x_i + 1) x_i (x_i - 1).$$

Deduce from this that f has a critical point of index k at the center of each k-face of the cube, and no other critical points. Then deduce that the descending manifold of a critical point is the interior of the corresponding face.

The compactified descending manifold of a critical point is diffeomorphic to a "fully truncated k-cube". Explain as much of this as you have interest in.

If $k = 2$, its boundary is an octagon. If $k = 3$, its boundary is a polyhedron whose faces consist of 6 octagons, 12 quadrilaterals, and 8 hexagons. What happens when $k = 4$? (You can google search this.)

optional Prove the Morse inequalities from Hutchings Theorem 3.1.

optional Use Hutchings Theorem 3.1 to prove the Poincaré-Hopf index theorem: If X is a closed oriented smooth manifold, then $\int_X e(TX) = \chi(X)$, where the left hand side describes the signed number of zeroes of a generic vector field on X. (See Guillemin-Pollack for a bit more about the VF perspective).

optional Use Hutchings Theorem 3.1 to prove Poincaré duality for closed oriented manifolds.

optional Show that the diagram in Hutchings commutes. (exercise 3, page 23)

everyone: How difficult was this assignment? How many hours did you spend on it?