A SYMPLECTIC PERSPECTIVE OF THE SIMPLE SINGULARITIES

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Abstract. We recall the notion of a simple singularity and derive the equations for its associated hypersurface $f_\Gamma^{-1}(0)$. We then define resolutions and link of a simple singularity. The contact structures for the link of a simple singularity and the Brieskorn manifolds are derived following a review of pseudoconvexity. In particular we demonstrate that the image of S^3/Γ in $f_\Gamma^{-1}(0)$ is contactomorphic to the link of its singularity, $S^5 \cap f_\Gamma^{-1}(0)$. The proof however relies on basic several complex variables.

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1. Introduction

The notion of a simple singularitiy (alternatively called a DuVal or Klein singularity) has existed since antiquity and there are a number of ways in which they can be characterized (see Reid [18]) . For the purpose of this note, we will primarily consider them as the absolutely isolated double point quotient singularity of $\mathbb{C}^2/\Gamma$, where $\Gamma \subset SU_2(\mathbb{C})$ is a finite subgroup. Klein proved the following theorem regarding the classification of finite subgroups of $SU_2(\mathbb{C})$

Theorem 1.1 (Klein). *Γ is classified by the Dynkin diagrams of type $(A_n)_{n\geq 1}$, $(D_n)_{n\geq 4}$, E_6, E_7, E_8. These correspond to the cyclic group $\mathbb{Z}_{n+1}$, the binary dihedral group $\mathbb{D}^*_{2(n-2)}$, the binary tetrahedral group $\mathbb{T}^*$, the binary octahedral group $\mathbb{O}^*$ and the binary icosahedral group $\mathbb{I}^*$ respectively.*

The Dynkin diagrams of the above types are diagrams with three branches of length p, q, and r. The rank of the Cartan matrix corresponding to Γ is given by $p+q+r-2$. More regarding this and the relationship to the minimal resolution of the singularity will follow. See also Table 1. Note that Γ acts on $\mathbb{C}^2$ holomorphically and the only fixed point of this group action is the origin $(0,0)$. Thus $\mathbb{C}^2/\Gamma$ is an algebraic variety with an isolated quotient singularity at the origin. A singularity obtained in this manner is considered to be a **simple singularity**.

Alternatively we can realize the variety $\mathbb{C}^2/\Gamma$ as a hypersurface $f_\Gamma^{-1}(0)$ embedded in $\mathbb{C}^3$. This is done by regarding the action of Γ on the generators u,v of $\mathbb{C}[u,v]$ and observing which monomials are invariant under the action of Γ. In this manner we obtain a basis consisting of 3 monomials, which generate the whole ring of Γ-invariant polynomials in u,v. Using these monomials one is then able to write the polynomial map

$$\varphi : \mathbb{C}^2 \to \mathbb{C}^3. \tag{1.1}$$

This descends to a map $\hat{\varphi} : \mathbb{C}^2/\Gamma \to \mathbb{C}^3$, which maps the orbit space $\mathbb{C}^2/\Gamma$ homeomorphically onto the hypersurface given by a single polynomial equation $\{f_\Gamma(z_0,z_1,z_2)=0\}$. Thus $\mathbb{C}^2/\Gamma$ is isomorphic to $\mathbb{C}[u,v]^\Gamma$, which equivalent to $\mathbb{C}[z_0,z_1,z_2]/(f_\Gamma=0)$. This manipulation allows us to realize $\mathbb{C}^2/\Gamma$ as the complex hypersurface cut out by $f_\Gamma^{-1}(0)$. The quotient singularity at the origin of $\mathbb{C}^2/\Gamma$ coresponds to the isolated singularity at the origin of $f_\Gamma^{-1}(0)$ which is precisely the simple singularity of type Γ. Note that away from the origin, the map φ restricts to a diffeomorphism between the orbit space $\mathbb{C}^2/\Gamma$ and the hypersurface $f_\Gamma^{-1}(0)$. The explicit equations are given below, following Burban [2] and Reid [19].

Example 1.2 (A_n)**.** Consider the cyclic subgroup $\mathbb{Z}_{n+1}$, which acts on $\mathbb{C}^2$ by $u \mapsto \varepsilon u,\ v \mapsto \varepsilon^{-1} v$, where $\varepsilon = e^{2\pi i/(n+1)}$, a primitive $(n+1)$-th order root of unity. Then the following monomials will generate the whole ring of invariants:

$$z_0 = uv, \quad z_1 = u^{n+1}, \quad z_2 = v^{n+1}$$

They satisfy the relation:

$$z_0^{n+1} - z_1 z_2 = 0$$

After a suitable change of coordinates in $\mathbb{C}^3$ we obtain:

$$f_\Gamma(z_0,z_1,z_2) = z_0^{n+1} + z_1 z_2 = 0$$

or equivalently,

$$f_\Gamma(z_0,z_1,z_2) = z_0^{n+1} + z_1^2 + z_2^2 = 0$$

Example 1.3 (D_n)**.** Consider the binary dihedral group $\mathbb{D}^*_{4(n-2)}$. We have $|\mathbb{D}^*_{4(n-2)}| = 8(n-2)$ and $\mathbb{D}^*_{4(n-2)} = \langle a,b \rangle$ with relations

$$\begin{cases} a^{(n-2)} &= b^2 \\ b^4 &= 1 \\ bab^{-1} &= a^{-1} \end{cases}$$

where

$$a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \ \varepsilon = e^{\pi i/(2(n-2))}, \ b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This acts on $\mathbb{C}^2$ as follows

$$\begin{cases} u & \mapsto & \varepsilon u \\ v & \mapsto & \varepsilon^{-1} v \end{cases} \qquad \begin{cases} u & \mapsto & -v \\ v & \mapsto & u \end{cases}$$

A bit of introspection yields the following invariant monomials

$$z_0 = u^{2(n-2)} + v^{2(n-2)}, \ z_1 = u^2v^2, \ z_2 = uv(u^{2(n-2)} - v^{2(n-2)})$$

They satisfy the relation:

$$z_2^2 = u^2v^2(u^{4(n-2)} + v^{4(n-2)} - 2u^{2(n-2)}v^{2(n-2)}) = z_0^2 z_1 - 4z_1^{(n-1)}$$

And a change of coordinates in $\mathbb{C}^3$ with a bit more algebraic manipulation yields:

$$f_\Gamma = z_0^2 z_1 + z_1^{n-1} + z_2^2 = 0$$

Example 1.4 (E_6, E_7, E_8)**.** For the exceptional groups E_6, E_7, E_8 we consider the action of the binary tetrahedral group $\mathbb{T}^*$, the binary octahedral group $\mathbb{O}^*$ and the binary icosahedral group $\mathbb{I}^*$ respectively. They are defined as follows:

$\mathbb{T}^*$: $|\mathbb{T}^*| = 24$ and $\mathbb{T}^* = \langle \sigma, \tau, \mu \rangle$, where

$$\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon^7 & \varepsilon^7 \\ \varepsilon^5 & \varepsilon \end{pmatrix}, \quad \varepsilon = e^{2\pi i/8}.$$

$\mathbb{O}^*$: $|\mathbb{O}^*| = 48$ and $\mathbb{O}^* = \langle \sigma, \tau, \mu, \kappa \rangle$, where σ, τ, μ are as in the case of $\mathbb{T}^*$, and

$$\kappa = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\mathbb{I}^*$: $|\mathbb{I}^*| = 120$ and $\mathbb{I}^* = \langle \sigma, \tau \rangle$, where

$$\sigma = -\begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \quad \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}, \quad \varepsilon = e^{2\pi i/5}.$$

It can be checked that we obtain the following equations for the simple singularities of execptional type:

$$\begin{aligned} E_6: \quad & z_0^4 + z_1^3 + z_2^2 &= 0 \\ E_7: \quad & z_0^3 y + z_1^3 + z_2^2 &= 0 \\ E_8: \quad & z_0^5 + z_1^3 + z_2^2 &= 0 \end{aligned}$$

In this manner we obtain the following table summarizing the relationship of Γ to f_Γ. The integer triple (p, q, r) correspond to the lengths of the 3 branches in the Dynkin diagrams, and the rank of the Cartan matrix corresponding to Γ is given by $p + q + r - 2$. Note also that the f_Γ are weighted homogeneous polynomials. The weights (a, b, c) wil play an important role in the compactification of the resolution of the singularity later.

Group Γ	Polynomial $f_\Gamma(x, y, z)$	branches (p, q, r)	weights (a, b, c)	rank Γ
A_n	$z_0^{n+1} + z_1^2 + z_2^2$	$(1, k, l)$	$(1, k, l)$	n
D_n	$z_0^2 z_1 + z_1^{n-1} + z_2^2$	$(2, 2, n-2)$	$(n-2, 2, n-1)$	n
E_6	$z_0^4 + z_1^3 + z_2^2$	$(2, 3, 3)$	$(3, 4, 6)$	6
E_7	$z_0^3 z_1 + z_1^3 + z_2^2$	$(2, 3, 4)$	$(4, 6, 9)$	7
E_8	$z_0^5 + z_1^3 + z_2^2$	$(2, 3, 5)$	$(6, 10, 15)$	8

Table 1.

In the A_n case, (k, l) is an arbitrary pair of positive integers satisfying $k + l = n + 1$.

2. Associated Topology and Geometry

Important objects considered in the study of hypersurface singularity theory include the minimal and Milnor resolutions and the link of a singularity. We begin with a discussion of the resolution of a singularity and prepare a few definitions. In the following sections we discuss the associated contact structure to the link of a singularity and how one can use the resolutions to create symplectic fillings of the link.

Let V^n be an analytic space with an isolated singularity at $\mathbf{0}$. The **resolution of the singularity** is proper map from a smooth analytic manifold $\widetilde{V} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ to V^n which is an isomorphism on $V^n \backslash \{\mathbf{0}\}$. A **minimal resolution** is a resolution such that there are no embedded exceptional divisors in $\tilde{V}$, i.e. embedded copies of $\mathbb{C}P^{n-1}$ with self intersection -1.

Hironaka proved that every analytic space admits a resolution of singularities. A natural question to ask is whether or not there exists a **unique minimal resolution** of a singular variety; that is if every other resolution can be mapped through it. This is a hard question to answer in general, and is not true in all dimensions. Castelnuovo and Enriques proved however, that every singular complex curve and every singular complex surface admit a unique minimal resolution.

A remarkable feature of simple singularities is that their minimal resolutions correspond precisely to their associated Dynkin diagrams. Each dot in the Dynkin diagram represents a curve (a $\mathbb{C}P^1$, which is technically a sphere) in the resolution that is contracted. Each line represents where the copies of $\mathbb{C}P^1$ intersect, and each has self intersection -2. This is why simple singularities are equivalently characterized as an absolutely isolated double point singularity. The computations for the resolutions of A_1 and E_6 can be found in Burban [2] and for D_4 in Reid [18].

For the special case of hypersurface singularities one can obtain another resolution using a family of deformations known as the **Milnor fibration**. The Milnor fibration is obtained by considering pieces of the nearby levels of the hypersurface $f_\Gamma^{-1}(0)$, namely $f^{-1}(\lambda)$, for $\lambda \neq 0$, contained in a ball of fixed radius

$$B_\epsilon^6 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 | \ |z_0|^2 + |z_1|^2 + |z_2|^2 \leq \epsilon^2\}.$$

This family $(F_{r,\lambda})_{|\lambda|=const}$ is called the Milnor fibration associated to the critical point, and is a locally trivial fibration over the circle. Its fibers are called **Milnor fibers**. An associated object of interest is the **link of a singularity**, which is the boundary of the Milnor fibers:

$$L = f_\Gamma^{-1}(0) \cap S_\epsilon^5 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 | \ f_\Gamma(z_0, z_1, z_2) = 0 \ \text{ and } \ z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 = 1\} \tag{2.1}$$

Thus the topology of the variety $f_\Gamma^{-1}(0)$ within the disk bounded by S_ϵ^5 will be closely related to the topology of the link.

Remark 2.1. Note that the link of a singularity is typically only well-defined in discussions of conical type singularities; however simple singularities are such a class of conical type singularities.

In fact for any algebraic set V, defined to be the locus of common zeros of some collection of polynomial functions of $\mathbb{C}^N$, the following conic structure theorem holds. Here we take D_ϵ to be the closed disk given by the set $\{\mathbf{z} \in \mathbb{C}^N \mid ||\mathbf{z} - \mathbf{z}^*|| \leq \epsilon\}$ where $\mathbf{z}^*$ is an isolated singular point of V.

Theorem 2.2. *For $\epsilon > 0$ the intersection of V with D_ϵ is homeomorphic to the cone over the link $L = V \cap S_\epsilon$. In fact the pair $(D_\epsilon, V \cap D_\epsilon)$ is homeomorphic to the pair consisting of the cone over S_ϵ and the cone over L.*

Remark 2.3. Brieskorn proved in the case of simple singularities, that amazingly the minimal resolution and the Milnor fiber are diffeomorphic. This is a consequence of the existence of a simultaneous resolution. Ohta and Ono proved in [17] a stronger theorem, by considering the Milnor fiber and minimal resolution as symplectic fillings of the link and then showing that the deformation type of any minimal syplectic filling is unique.

In the case of $A_n,\ E_6,$ and E_8 singularities, the link is an example of Brieskorn manifold; however the analysis of interest to us carries through to the D_n and E_7 cases. The construction of Brieskorn manifolds and results pertaining to their topological and contact structure are rather useful, so we shall digress slightly to discuss them.

Definition 2.4. Let $\mathbf{a} = (a_0, ..., a_n)$ be an $(n+1)$-tupel of integers $a_j > 1$, and set

$$V(\mathbf{a}) := \{\mathbf{z} := (z_0, ...z_n) \in \mathbb{C}^{n+1} |\ f(\mathbf{z}) := z_0^{a_0} + ... + z_n^{a_n} = 0\}.$$

Further, with S^{2n+1} denoting the unit sphere in $\mathbb{C}^{n+1}$, we define the **Brieskorn manifold**

$$\Sigma(\mathbf{a}) := V(\mathbf{a}) \cap S^{2n+1} \tag{2.2}$$

Proposition 2.5. $\Sigma(\mathbf{a})$ *is a smooth manifold of dimension* $2n-1$*. The link* L *of a simple singularity is smooth manifold of dimension 3.*

Proof. This follows from the fact that the origin is the only critical point of f and f_Γ, respectively. □

In Brieskorn's seminal paper [1], necessary and sufficent conditions for these manifolds to be topological spheres are given. The Brieskorn manifolds were first constructed to obtain exotic differentiable structures on the topological $(2n-1)$-sphere, and the underlying algebraic varieties $V(\mathbf{a})$ were used as the prototypical example of hypersurfaces containing an isolated singularity. The Brieskorn manifolds were also some of the less boring examples of contact manifolds first known and studied. In the following sections we will see that they also admit a contact form, which induces a Stein fillable contact contact structure on them.

Of interest to us will be smoothings $V_t(\mathbf{a})$, of the varieties $V(\mathbf{a})$ obtained via the aforementioned Milnor fibration. We define them, along with their Brieskornesque counterparts, as one might expect, to be:

$$V_t(\mathbf{a}) = \{\mathbf{z} \in \mathbb{C}^{n+1} |\ f(\mathbf{z}) = t\}$$

$$\Sigma_t(\mathbf{a}) = V_t(\mathbf{a}) \cap S^{2n+1}$$

with $t \in \mathbb{R}^+$. In the case of the D_n and E_7 singularities, the smoothing we consider will be $f_\Gamma^{-1}(t)$.

Next we recall a few important results from the theory of singular points of complex hypersurfaces, which extend to the Brieskorn varieties and the hypersurfaces corresponding to the simple singularities. These are originally due to Milnor, Brieskorn, Hirzebruch and Ehresmann. We refer the reader to Milnor's book on this subject [15] for further details. For the initial study of complex hypersurfaces one wants to introduce a fibration (as previously mentioned) which will be useful in describing the topology of the link. This so-called fibration theorem relies on an earlier result of Ehresmann, as well as some complex geometry .

Theorem 2.6 (Ehresmann)**.** *Let* E *and* B *be differentiable manifolds,* B *connected and* $\pi : E \to B$ *a differentiable surjective map, such that for all* $x \in B$ *the rank of the* *derivative $d\pi_x$ is equal to the dimension of B, and $\pi^{-1}(x)$ is compact and connected. Then $\pi : E \to B$ is a differentiable fiber bundle, comprised of diffeomorphic fibers $\pi^{-1}(x)$.*

Theorem 2.7 (Milnor's Fibration Theorem)**.** *If $\mathbf{z}^*$ is any point of the complex hypersurface $V = f^{-1}(0)$ and if S_ϵ is a sufficiently small sphere centered at $\mathbf{z}^*$, then the mapping*

$$\phi(\mathbf{z}) = \frac{f(\mathbf{z})}{|f(\mathbf{z})|}$$

from $S_\epsilon \setminus L$ to the unit circle is the projection map of a smooth fiber bundle. Each fiber

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon \setminus L$$

is a smooth parallelizable 2n-dimensional manifold.

An important observation in the cases of the simple and brieskorn singularities is that it is not necessary to choose the ϵ used in defining the link of a singularity to be sufficently small. This follows from the ability to "rescale" homogeneous and weighted homogeneous polynomials.

Proposition 2.8. *$V_1(\mathbf{a})$ is diffeomorphic to $V_t(\mathbf{a})$ for $t \neq 0$. Likewise, $f_\Gamma^{-1}(1)$ is diffeomorphic to $f_\Gamma^{-1}(t)$ for $t \neq 0$.*

Proof. Define the following bijective map of $\mathbb{C}^{n+1}$

$$(z_0, ...z_n) \mapsto (\sqrt[a_0]{t}\ z_0, ..., \sqrt[a_n]{t}\ z_n)$$

where $\sqrt[a_j]{t}$ is the a_j-th root of t. For the Milnor fibration corresponding to D_n and E_7 we take $(a_0, a_1, a_2) = (2\frac{(n-1)}{(n-2)}, (n-1), 2)$ and $(\frac{9}{2}, 3, 2)$, respectively. The result follows. ☐

Using Morse theory one is able to further study the locally trivial fibration

$$\phi : S_\epsilon \setminus L \to S^1$$

associated to a complex polynomial f which vanishes at the origin. This allows us to conclude that each fiber F_θ is diffeomorphic to an open subset of a non-singular complex hypersurface, consisting of all $\mathbf{z}$ with $||\mathbf{z}|| < \epsilon$ and $f(\mathbf{z}) =$ constant.

Theorem 2.9. *For sufficently small $t \neq 0$, $V_t(\mathbf{a})$ is diffeomorphic to $V_t(\mathbf{a}) \cap \mathrm{Int}(B_1^{2n+2})$*

Theorem 2.10. *For sufficently small t, $\Sigma_t(\mathbf{a}) \neq \emptyset$ and $\Sigma_t(\mathbf{a})$ is diffeomorphic to $\Sigma(\mathbf{a})$.*

Theorem 2.11. *$V_t(\mathbf{a}) \cap B_1^{2n+2}$ is parallelizable manifold.*

Proposition 2.12. *L is diffeomorphic to the rational homology sphere S^3/Γ.*

Proof. This can be proven directly by standard computations. If we go back to the map φ from (1.1), we see that φ descends to a map, $\hat{\varphi} : \mathbb{C}^2/\Gamma \to \mathbb{C}^3$, which maps the orbit space $\mathbb{C}^2/\Gamma$ homeomorphically onto the hypersurface $f_\Gamma^{-1}(0)$. Thus $\hat{\varphi}(S^3/\Gamma)$ is mapped homeomorphically onto a submanifold of $f_\Gamma^{-1}(0)$, which is diffeomorphic to the intersection $L = f_\Gamma^{-1}(0) \cap S_\epsilon^5$. ☐

To see that $\hat{\varphi}(S^3/\Gamma)$ is homeomorphic to $f_\Gamma^{-1}(0) \cap S_\epsilon^5$ one must go through a bit of algebraic manipulation and rescaling. We do this in the case of A_n directly; the computations for the cases with different Γ's follow in a similar fashion.

Example 2.13. Recall from Example 1.2 that the following monomials generate the whole ring of invariants for the A_n case:

$$z_0 = uv, \quad z_1 = u^{n+1}, \quad z_2 = v^{n+1}$$

They satisfy the relation:

$$f_{A_n}(z_0, z_1, z_2) = z_0^{n+1} - z_1 z_2 = 0$$

Find the image of S^3/A_n under $\hat{\varphi} : \mathbb{C}^2/A_n \to \mathbb{C}^3$.

$$\begin{aligned}
\hat{\varphi}(S^3/A_n) &= \varphi(\{|u|^2 + |v|^2 = 1\}/\mathbb{Z}_{n+1}) \\
&= \{(|z_1|^{2/(n+1)} + |z_2|^{2/(n+1)} = 1\} \cap f_{A_n}^{-1}(0) \\
&= \{(|z_1|^{4/(n+1)} + 2|z_1|^{2/(n+1)}|z_2|^{2/(n+1)} + |z_2|^{4/(n+1)} = 1\} \cap f_{A_n}^{-1}(0) \\
&= \{(|z_1|^{4/(n+1)} + 2|z_0|^2 + |z_2|^{4/(n+1)} = 1\} \cap f_{A_n}^{-1}(0)
\end{aligned}$$

By rescaling as follows:

$$\begin{aligned}
z_0 &\mapsto z_0/\sqrt{2} \\
z_1 &\mapsto |z_1|^{(n-1)/2} z_1 \\
z_2 &\mapsto |z_2|^{(n-1)/2} z_2,
\end{aligned} \tag{2.3}$$

we see that $\hat{\varphi}(S^3/A_n)$ will be homeomorphic to $L = f_{A_n}^{-1}(0) \cap S_\epsilon^5$. Since any two homeomorphic 3-manifolds are diffeomorphic, we have the desired result.

Remark. Finding an explicit diffeomorphism between $\hat{\varphi}(S^3/\Gamma)$ and L is possible by reparametrizing the flow of a suitable vectorfield on $f_\Gamma^{-1}(0)$. The existence and construction of this "mysterious" vector field, whose flow generates a contact structure, is the subject of the subsequent discussion in this paper.

3. Contact Considerations

This section begins with an overview of contact geometry and provide a few basic examples of contact manifolds. We wish to consider the link of a simple singularity as a contact manifold. The link is a real hypersurface sitting inside complex space, which is diffeomorphic to S^3/Γ. One is able to describe a contact structure as the set of complex tangencies to a real hypersurface in a complex manifold. We will recall the definition and useful properties of Liouville vector fields associated to hypersurfaces of contact type. The following section contains the construction of contact structure arising in such a manner, as well as a natural Liouville vector field that we will ultimately use to prove that $\varphi(S^3/\Gamma)$ and L are contactomorphic.

Definition 3.1. Let V be a manifold of dimension $2n+1$. A **contact structure** is a maximally non-integrable hyperplane field $\xi = \ker\,\alpha \subset TV$, that is, the defining differential 1-form α is required to satisfy

$$\alpha \wedge (d\alpha)^n \neq 0 \tag{3.1}$$

Note that the contact structure is unaffected when we multiply the contact form α by any postive or negative function on V. We say that two contact structures $\xi_0 = \ker\,\alpha_0$ and $\xi_1 = \ker\,\alpha_1$ on a manifold V are **contactomorphic** whenever there is a diffeomorphism $\psi : V \to V$ such that ψ sends ξ_0 to ξ_1:

$$\psi_*(\xi_0) = \xi_1$$

Note that the diffeomorphism $\psi : V \to V$ being a contactomorphism is equivalent to the existence of a non-zero function $g : V \to \mathbb{R}$ such that $\psi^*\alpha_1 = g\alpha_0$. Finding an explicit contactomorphism often proves to be a rather difficult and messy task, but an application of Moser's argument yields Gray's stability theorem, which essentially states that there are no non-trivial deformations of contact structures on a fixed closed manifold.

Theorem 3.2 (Gray's stability theorem)**.** *Let $\xi_t,\ t \in [0,1]$, be a smooth family of contact structures on a closed manifold V. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of V such that*

$$\psi_{t*}(\xi_0) = \xi_t \quad \textit{for each } t \in [0,1]$$

A proof of Gray's stability theorem is provided in the appendix to this paper, and can also be found in [8].

Here is another important concept of contact geometry.

Definition 3.3. For any contact manifold $(M, \xi = \ker\,\alpha)$ the **Reeb vector field** is defined as the unique vector field determined by α such that

$$\iota(R_\alpha)d\alpha = 0, \quad \alpha(R_\alpha) = 1.$$

The first condition says that R_α points along the unique null direction of the form $d\alpha$ and the second condition normalizes R_α. Because

$$\mathcal{L}_{R_\alpha}\alpha = d\iota_{R_\alpha}\alpha + \iota_{R_\alpha}d\alpha$$

the flow of R_α preserves the form α and hence the contact strucuture ξ. Note that if one chooses a different contact form $f\alpha$, the corresponding vector field $R_{f\alpha}$ is very different from R_α, and its flow may have quite different properties. If however, α and $f\alpha$ differ by a contactomorphism then so do the Reeb vector fields R_α and $R_{f\alpha}$

Example 3.4. Consider $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, ..., x_n, y_n, z)$ and the 1-form

$$\alpha = dz + \sum_{j=1}^{n} x_j dy_j.$$

Then α is a contact form for $\mathbb{R}^{2n+1}$. The contact structure $\xi = \ker\,\alpha$ is called the standard contact strucutre on $\mathbb{R}^{2n+1}$

As in symplectic geometry a variant of Darboux's theorem holds. This states that locally all contact structures are diffeomorphic to the standard contact structure on $\mathbb{R}^{2n+1}$. Next we discuss the contact structure equipped to everyone's favorite closed manifold, the sphere.

Example 3.5. Consider the unit (2n+1)-sphere, S^{2n+1}, in $\mathbb{R}^{2n+2}$. Let

$$\alpha = \sum_{j=1}^{n+1} x_j\ dy_j - y_j\ dx_j|_{S^{2n+1}} \tag{3.2}$$

where $(x_1, y_1, ..., x_{n+1}, y_{n+2})$ are the standard Cartesian coordinates on $\mathbb{R}^{2n+2}$ with $\xi = \ker\,\alpha$ the standard contact strucure on S^{2n+1}. To see that α satisfies the contact condition (3.1) we carry out the following computation

$$\begin{aligned} \alpha \wedge (d\alpha)^n &= \left(\sum_{j=1}^{n+1} x_j\ dy_j - y_j\ dx_j \right) \wedge \left(2 \sum_{j=1}^{2n} dx_j \wedge dy_j \right)^n \\ &= \sum_{j=1}^{2n+2} (-1)^{j-1} x_j\ dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_{2n+2} \\ &= *dr \end{aligned}$$

where $*$ is the Hodge star operator. Note that this is the volume form for S^{2n+1}. For a detailed derivation of the volume form for S^n see section 6.1 of [7].

Since contact geometry is an odd-dimensional sibling of symplectic geometry, one expects some natural setting where we might observe an interdependence between the two. The most useful constructions relating the two arise when we consider hypsersurfaces in symplectic manifolds. As it turns out, a hypersurface of a symplectic manifolds will admit a contact form whenever we have a Liouville vector field defined in a neighborhood that is transverse to the hypersurface. The quintessential example of this is the **symplectization** of a contact manifold $(V, \alpha = \ \ker\,\xi)$, which allows us to embed Q as a hypersurface in an exact symplectic manifold. We take $M = Q \times \mathbb{R}$ with symplectic form

$$\omega = e^t(d\alpha - \alpha \wedge dt) = d\lambda, \qquad \lambda = e^t\alpha.$$

Here t is the coordinate on $\mathbb{R}$, and the symplectization of Q is $(Q \times \mathbb{R}, d(e^t\alpha))$. At the end of this section we will show that all hypersurfaces of contact type in (M, ω) look locally in M like a contact manifold sitting inside its symplectization.

Definition 3.6. A **Liouville vector field** Y on a symplectic manifold (M, ω) is a vector field satisfying

$$\mathcal{L}_Y \omega = \omega$$

The flow ψ_t of such a vector field is conformal symplectic, i.e. $\psi_t^*(\omega) = e^t \omega$. Note that the flow of these fields are volume expanding, so such fields may only exist locally on compact manifolds.

Proposition 3.7. *When (M, ω) admits a Liouville vector field Y, one can define a 1-form $\alpha := \iota_Y \omega$, which will be a contact form on any hypersurface (a codimension 1 submanifold of M) transverse to Y (that is, with Y nowhere tangent to M). Such hypersurfaces are said to be of* ***contact type****.*

Proof. The Cartan formula

$$\mathcal{L}_Y = d \circ \iota_Y + \iota_Y \circ d$$

combined with the fact that ω is closed allows us to write the Liouville condition on Y as $d(\iota_Y \omega) = \omega$. Assuming M to be of dimension 2n, we compute:

$$\begin{aligned} \alpha \wedge (d\alpha)^{n-1} &= \iota_Y \omega \wedge (d(\iota_Y \omega))^{n-1} \\ &= \iota_Y \omega \wedge \omega^{n-1} \\ &= \tfrac{1}{n} \iota_Y (\omega^n) \end{aligned}$$

Since ω^n is a volume form on M, it follows that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form when restricted to the tangent bundle of any hypersurface transverse to Y in M. □

The following is a useful result regarding the existence of Liouville vector fields and the associated contact form.

Proposition 3.8. *Let (M, ω) be a symplectic manifold and $Q \subset M$ a compact hypersurface. Then the following are equivalent*
(i) There exists a contact form α on Q such that $d\alpha = \omega|_Q$.
(ii) There exists a Liouville vector field $Y : U \to TM$ defined in a neighborhood U of Q, which is transverse to Q.

Proof. First assume that (ii) is satisfied and define $\alpha = \iota_Y \omega$. Then

$$d\alpha = d(\iota_Y \omega) = \omega$$

Since T_qQ is odd dimensional, there exists a nonzero $\tilde{v} \in T_qQ$ such that $\omega_q(\tilde{v}, v) = 0$ for all $v \in T_qQ$. Since ω is nondegerate we have $\alpha_q(\tilde{v}) = \omega_q(Y(q), \tilde{v}) \neq 0$. Hence

$$\xi_q = \{v \in T_qQ \mid \omega_q(Y(q), v) = 0\}$$

is a hyperplane field on Q and $\tilde{v}$ is transversal to ξ_q. In fact, ξ_q is the symplectic complement of span$\{Y(q), \tilde{v}\}$. This implies $\omega = d\alpha$ is nondegenerate on ξ_q and hence α restricts to a contact form on Q.

Conversely suppose that $\alpha \in \Omega^1(Q)$ is a contact form such that $d\alpha = \omega|_Q$. Let $R_\alpha \in \chi(Q)$ be the Reeb vector field of α:

$$\iota_{R_\alpha} d\alpha = 0, \qquad \iota_{R_\alpha}\alpha = 1$$

Choose a vector field $Y \in \chi(M)$ such that

$$\omega(Y, R_\alpha) = 1, \qquad \omega(Y, \xi) = 0$$

on Q. This can be done by picking any vector field Y_0 such that $\omega(Y_0, R_\alpha) = 1$ on Q. Then for every $q \in Q$ there exists a unique vector $Y_1(q) \in \xi_q$ such that $\omega(Y_0 + Y_1, v) = 0$ for all $v \in \xi_q$. Define $Y = Y_0 + Y_1$ on Q and extend to a vector field on M. Next we define $\phi : Q \times \mathbb{R} \to M$ by

$$\phi(q,t) = exp_q(tY(q))$$

Then

$$\begin{aligned} \phi^*\omega|_{Q\times\{0\}} &= \phi^* d\alpha|_{Q\times\{0\}} \\ &= d(\phi^*\alpha)|_{Q\times\{0\}} \\ &= d(e^t\alpha)|_{Q\times\{0\}} \\ &= d\alpha - \alpha \wedge dt \end{aligned}$$

Now by Moser's argument there exists a local diffeomorphism $\psi : Q \times (-\epsilon, \epsilon) \to M$ such that

$$\psi(q,0) = q, \qquad \psi^*\omega = e^t(d\alpha - \alpha \wedge dt).$$

So the required Liouville vectorfield is $\psi_*(\frac{\partial}{\partial t})$.

□

Example 3.9. The radial vector field

$$X_0 = \frac{1}{2}\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$$

is a Liouville vector field on $\mathbb{R}^{2n}$. It is transverse along the unit sphere S^{2n-1}. The corresponding 1-form

$$\lambda_0 = \iota_{X_0}\omega_0 = \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j) \tag{3.3}$$

is the canonical contact form for S^{2n-1}.

Proposition 3.10. Considering S^3/Γ as a hypersurface in $\mathbb{C}^2/\Gamma$, the vector field

$$Y_0 = \frac{1}{2}\sum_{j=2}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}|_{\mathbb{C}^2/\Gamma}$$

is transverse along S^3/Γ and is a Liouville vector on $\mathbb{C}^2/\Gamma$ away from the origin.

Proof. Note that away from the origin $\mathbb{C}^2/\Gamma$ admits the structure of a smooth manifold. Recall that we may write

$$S^3/\Gamma = \{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2/\Gamma \mid x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\}.$$

Then the defining equation $h(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$ is a smooth function from $\mathbb{C}^2/\Gamma$ to $\mathbb{R}$, and we observe that S^3/Γ is a regular level set of h. Then after choosing a Riemannian metric on $\mathbb{C}^2/\Gamma$ away from the origin we have

$$Y_0 = \frac{1}{4}\text{grad } h.$$

So by this construction Y_0 is a transverse vector field along S^3/Γ.

Next we must prove that Y_0 is a Liouville vector field on $\mathbb{C}^2/\Gamma$ away from the origin. Away from the origin $\mathbb{C}^2/\Gamma$ inherits the flat standard Kähler form on $\mathbb{C}^2$, written as $d\lambda$ with

$$\lambda = \frac{i}{2}(ud\bar{u} - \bar{u}du + vd\bar{v} - \bar{v}dv).$$

After taking the canonical identification of $\mathbb{C}^2$ with $\mathbb{R}^4$ we see that $\lambda = \lambda_0$ as in (3.3). Then a simple computation reveals that the Liouville condition

$$\mathcal{L}_{Y_0} d\lambda = d(\iota_{Y_0} d\lambda) = d\lambda$$

is satisfied as desired. □

There is an equivalent definition of contact type due to Weinstein, which uses the characteristic line field of a compact hypersurface S in a symplectic manifold (M, ω). The **characteristic line field** L is the real line bundle over S given by the symplectic complement of TS in TM:

$$L_q := (T_qS)^\omega = \{v \in T_qM \mid \omega(v, w) = 0 \text{ for } w \in T_qS\}.$$

Recall that every hypersurface of a symplectic manifold is a coisotropic manifold, thus L_q is a one dimensional subspace of T_qS for all $q \in S$. L integrates to give a 1-dimensional foliation of S called the characteristic foliation. The leaves of this foliation are the integral curves of any Hamiltonian vector field X_H for which S is a regular level surface of H (or a component of such a surface). Thus we see that we can alternatively define L to be the span of the symplectic gradient X_H.

Proposition 3.11. *Let Q be a compact hypersurface in a symplectic manifold (M, ω) and denote the inclusion map $i : Q \to M$. Then Q has contact type if and only if there exists a 1-form α on Q such that*

(i) $d\alpha = i^*\omega$

(ii) *the form α is never zero on the characteristic line field.*

Proof. Exercise: Prove this.

□

4. Notions of Convexity in Complex and Symplectic Geometry

Convexity is an important notion in both analysis and geometry, which has great implications on symplectic and contact geometry. Strictly Levi pseudoconvex hypersurfaces carry a natural contact structure arising from the set of complex tangencies to their boundary. We explain this derivation and go through the calculations in obtaining this contact structure for Brieskorn manifolds and the link of a simple singularity. These derivations are a part of a deeper story involving the relationships between pseudoconvexity, Stein manifolds, contact type hypersurfaces, and fillings, which are elaborated in this section. Many of the symplectic and contact applications of pseudoconvexity were first noted and written about by Eliashberg, Gromov, and McDuff. Before delving into the details of these matters, we motivate our discussion with an alternate derivation of the contact structure on S^3.

Example 4.1. Let $f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$. Then $S^3 = f^{-1}(1)$. Moreover at a point (x_1, y_1, x_2, y_2) in S^3 the tangent space is given by

$$T_{(x_1,y_1,x_2,y_2)}S^3 = \ker\, df_{(x_1,y_1,x_2,y_2)} = \ker\, (2x_1dx_1 + 2y_1dy_1 + 2x_2dx_2 + 2y_2dy_2).$$

Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ we have the standard complex structure J. We have $Jx_i = y_i,\ \ Jy_i = -x_i$ for $i = 1, 2$. The complex structure J induces a complex structure on each tangent space: $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$ for $i = 1, 2$.

We now claim that $\xi = \ker\alpha$, where α is from (3.3) is equivalent to the set of complex tangencies. By this we mean

$$\xi = TS^3 \cap J(TS^3).$$

Since

$$J(T_{(x_1,y_1,x_2,y_2)}S^3) = \ker\, (df_{(x_1,y_1,x_2,y_2)} \circ J)$$

and

$$df_{(x_1,y_1,x_2,y_2)} \circ J = -2x_1dy_1 + 2y_1dx_1 - 2x_2dy_2 + 2y_2dx_2,$$

we have that $\alpha = -(df \circ J)|_{S^3}$ from (3.3) as claimed.

Complex structures are invariant under $SU_2(\mathbb{C})$, thus we can use the standard contact structure on S^3 to induce a contact structure on S^3/Γ. Thus the proposition follows from this example.

Proposition 4.2. *Regarding the link of a simple singularity L as*

$$S^3/\Gamma = \{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2/\Gamma |\ x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\},$$

and taking the standard complex structure $J_{\mathbb{C}^2}$, we obtain the hyperplane field of maximal complex tangencies:

$$\xi_{S^3/\Gamma} = TS^3/\Gamma \cap J_{\mathbb{C}^2}(TS^3/\Gamma)$$

Then $(S^3/g, \xi_{S^3/\Gamma})$ is a contact manifold with the canonical contact form α, given by:

$$\alpha = (x_1\ dy_1 - y_1\ dx_1 + x_2\ dy_2 - y_2\ dx_2)|_{S^3/\Gamma} \tag{4.1}$$

such that $\xi_{S^3/\Gamma} = ker\ \alpha$.

In general the set-up is as follows. Let X be a complex manifold with boundary and J the induced complex structure on TX. Supposing we can find a function ϕ defined in a neighborhood of the boundary such that $\phi^{-1}(0) = \partial X$, we have that the complex tangencies to $V = \partial X$ are given by ker $(d\phi \circ J)$. Thus the complex tangencies ξ to V form a contact structure whenever $-d(d\phi \circ J)$ is a non-degenerate 2-form on ξ. Of course, we are interested in more than just some abstract construction. One desires to explicitly figure out which manifolds admit these special functions, and how one might procure them. This is where the notions of pseudoconvexity and Stein manifolds come into play, so we shall take a few moments to acquaint ourselves with these objects, beginning with a few prelimary definitions.

An open connected subset $\Omega \subset \mathbb{R}^m$ has **differentiable boundary** $\partial\Omega$ if there exists a smooth function $\rho : \mathbb{R}^m \to \mathbb{R}$ such that

$$\Omega = \{x \in \mathbb{R}^m : \rho(x) < 0\}$$

and $d\rho_p \neq 0$ in a neighborhood of $\partial\Omega$. Then $\partial\Omega$ is a codimension 1 submanifold of $\mathbb{R}^m$, given by the zero set $\{\rho = 0\}$, with 0 a regular value of ρ.

The **tangent space** $T_p(\partial\Omega)$ to $\partial\Omega$ at a point $p \in \partial\Omega$ can be described as

$$T_p(\partial\Omega) = \{x \in \mathbb{R}^m : d\rho_p(x) = 0\}$$

Next we wish to investigate the relative uniqueness of defining functions for $\partial\Omega$. Given two definining functions $\rho_1,\ \rho_2$ for Ω and a point $p \in \partial\Omega$, choose local coordinates $(x_1, ..., x_m)$ near p such that $p = \mathbf{0}$ and

$$\rho_2(x_1, ..., x_m) = x_m.$$

Denote $x' := (x_1, ..., x_{m-1})$. Then we have $\rho_1(x', 0) = 0$ for x' near $\mathbf{0} \in \mathbb{R}^{m-1}$. The fundamental theorem of calculus applied to $t \mapsto \rho_1(x', 0)$ yields

$$\rho_1(x', tx_m) = \rho(x'_1, x_m) - \rho_1(x', 0) = x_m \int_0^1 \frac{\partial \rho_1}{\partial x_m}(x', tx_m)dt.$$

In other words, we have found a smooth function h such that $\rho_1 = h\rho_2$ on $\mathbb{R}^m$. Since $d\rho_1 = hd\rho_2$ on $\partial\Omega$, h is strictly positive. With this in mind, we can describe the (geometric) convexity of Ω in terms of the function ρ.

Lemma 4.3. *If Ω is convex near $p \in \partial\Omega$, then*

$$\sum_{j,k=1}^{m} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(p) X_j X_k \geq 0 \text{ for all } X = (X_1, ...X_m) \in T_p(\partial\Omega) \tag{4.2}$$

*Conversely if 4.2 holds with strict inequality for all $\mathbf{0} \neq X \in T_p(\partial\Omega)$ we say Ω is **strictly convex** at p. In this case Ω is convex in a neighborhood of p in $\mathbb{R}^m$*

Proof. Note that the statements in this lemma are independent of the choice for the defining function ρ and invariant under *linear* coordinate changes in $\mathbb{R}^m$. This can be concluded by means of computing derivatives, which we leave as an exercise. These considerations then allow one to assume without loss of generality that $p = \mathbf{0}$ and $T_p(\partial\Omega)$ is the linear subspace of $\mathbb{R}^m$ corresponding to the first $m-1$ coordinates. We may then take the function ρ near $p = \mathbf{0}$ to be of the form

$$\rho(x_1, ...x_m) = f(x_1, ..., x_{m-1}) - x_m$$

with $df_{\mathbf{0}} = 0$. Convexity of ρ at p thus translates into saying that the function $t \mapsto f(tX_1, .., tX_{m-1})$ is convex at $t = 0$ for any non-zero vector $(X_1, ...X_{m-1})$. ☐

For the remainder of the discussion, we wish to work in $\mathbb{C}^n$. We will take the canonical identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and J the complex bundle structure on the tangent bundle $T\mathbb{R}^{2n}$ induced by multiplication by i. Recall that this defines an isomorphism of the real tangent space at a point to the holomorphic tangent space at a point:

$$\begin{array}{lcl} T_p\mathbb{R}^{2n} & \to & T_p^{(1,0)}\mathbb{C}^n \\ X & \mapsto & Z = \frac{1}{2}(X - iJX) \end{array}$$

This brings us to the following definition:

Definition 4.4. Given Ω as before, consider the **complex tangent space** to $\partial\Omega$ at a point $p \in \partial\Omega$ as:

$$T_p^{(1,0)}(\partial\Omega) := T_p(\partial\Omega) \cap J(T_p(\partial\Omega))$$

This is the largest J-invariant subspace of $T_p(\partial\Omega) \subset T_p\mathbb{R}^{2n}$, and has real dimension $2n-2$.

In terms of the standard cartesian coordiantes $(z_1, ..., z_n)$ on $\mathbb{C}^n$ we define complex valued 1-forms on $\mathbb{C}^n$, which are the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear sections respectively of the bundle $T^*\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ by:

$$\begin{array}{lcl} \partial\rho & = & \sum_{j=1}^{n} \partial_{z_j}\rho \; dz_j \\ \bar{\partial}\rho & = & \sum_{j=1}^{n} \partial_{\bar{z}_j}\rho \; d\bar{z}_j \end{array}$$

so that

$$d\rho = \partial\rho + \bar{\partial}\rho$$

We are then able to state the following lemma.

Lemma 4.5. *The complex tangent space $T_p^{(1,0)}(\partial\Omega)$ may be described as*

$$\begin{array}{lcl} T_p^{(1,0)}(\partial\Omega) & = & \{X \in T_p\mathbb{R}^{2n} \mid \partial\rho_p(X) = 0\} \\ & = & \{Z \in T_p^{(1,0)}(\partial\Omega) \mid \partial\rho_p(Z) = 0\} \end{array}$$

Proof. Since ρ is real valued we have $\bar{\partial}\rho = \overline{\partial\rho}$. This implies that $d\rho(X) = 2\mathrm{Re}(\partial\rho(X))$. Moreover, the $\mathbb{C}$-linearity of the 1-form $\bar{\partial}\rho$ gives

$$\mathrm{Re}(\partial\rho(JX)) = \mathrm{Re}(i\partial\rho(X)) = -\mathrm{Im}(\partial\rho(X))$$

Since $T_p^{(1,0)}(\partial\Omega)$ can be characterized as the vector space of the $X \in \mathbb{R}^{2n}$ with

$$d\rho_p(X) = d\rho_p(JX) = 0$$

the first description of $T_p^{(1,0)}(\partial\Omega)$ follows. The second description is obtained by writing X in the form

$$X = \sum_{j=1}^{n} Z_j \partial_{z_j} + \bar{Z}_j \partial_{\bar{z}_j}.$$

□

We want to find a complex analogue of the convexity condition (4.2). However, in order for it to bear relation to other notions of convexity in analysis, which are typically defined in terms of the existence of certain holomorphic functions, it should be a condition that it be invariant under biholomorphisms. Unfortunately, our previous convexity condition fails this, but the following calculation gives us a portion of the quadratic form characterising convexity that will be preserved under biholomorphic mappings. This form is called the Levi form, and the collection of Levi pseudoconvex domains is, in a local sense, the smallest class of domains that contains the convex domains and is closed under increasing union and biholomorphic mappings.

Remark 4.6. Let $\Omega \subset\subset \mathbb{C}^n$ be a convex subset with C^2 boundary, and $\rho \in C^2$ the defining function as before. Let $\hat{\Omega}$ be a neighborhood of $\bar{\Omega}$ and let $\Phi : \hat{\Omega} \to \mathbb{C}^n$ be biholomorphic. Let $\Omega = \Phi(\Omega')$ and $\rho = \rho' \circ \Phi$. Let $p' = \Phi(p) \in \partial\Omega'$ and denote

$$Z' = \left(\sum_{j=1}^{n} \frac{\partial \Phi_1(p)}{\partial z_j} Z_j, ..., \sum_{j=1}^{n} \frac{\partial \Phi_n(p)}{\partial z_j} Z_j \right) \in T_p(\partial\Omega')$$

We first begin by rewriting the convexity condition (4.2) as:

$$2 \text{ Re } \left\{ \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) Z_j Z_k \right\} + 2 \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) Z_j \bar{Z}_k \tag{4.3}$$

However

$$\frac{\partial^2 \rho}{\partial z_j \partial z_k} = \frac{\partial}{\partial z_j} \sum_{l=1}^{n} \frac{\partial \rho'}{\partial z'_l} \frac{\partial \Phi_l}{\partial z_k} = \sum_{l,m=1}^{n} \frac{\partial^2 \rho'}{\partial z'_l \partial z'_m} \frac{\partial \Phi_l}{\partial z_k} \frac{\partial \Phi_m}{\partial z_j} + \sum_{l=1}^{n} \frac{\partial \rho'}{\partial z'_l} \frac{\partial^2 \Phi_l}{\partial z_j \partial z_k}$$

and

$$\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} = \frac{\partial}{\partial z_j} \sum_{l=1}^{n} \frac{\partial \rho'}{\partial \bar{z}'_l} \frac{\partial \bar{\Phi}_l}{\partial \bar{z}_k} = \sum_{l,m=1}^{n} \frac{\partial^2 \rho'}{\partial z'_m \partial \bar{z}'_l} \frac{\partial \Phi_m}{\partial z_j} \frac{\partial \bar{\Phi}_l}{\partial \bar{z}_k}$$

Therefore

$$\text{equation (4.3)} = 2\ \mathrm{Re}\left\{\sum_{l,m=1}^{n}\frac{\partial^2\rho'}{\partial z'_l\partial z'_m}Z'_lZ'_k+\sum_{j,k=1}^{n}\sum_{l=1}^{n}\frac{\partial\rho'}{\partial z'_l}\frac{\partial^2\Phi_l}{\partial z_j\partial z_k}Z_jZ_m\right\}+2\sum_{l,m=1}^{n}\frac{\partial^2\rho'}{\partial z'_m\partial\bar{z}'_l}Z'_m\bar{Z}'_l.$$

So we see that the "portion" of the quadratic form characterizing convexity that is preserved under biholomorphic mappings is the second half,

$$\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(p)Z_j\bar{Z}_k.$$

In this manner we obtain the following definition for pseudoconvexity.

Definition 4.7. The Hermitian form

$$L_p\rho(Z)=\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(p)Z_j\bar{Z}_k$$

is called the **Levi form** (or complex Hessian) of ρ at $p\in\partial\Omega$. One says that Ω is **(strictly) Levi pseudoconvex** if its boundary $\partial\Omega$ is smooth and $L_p\rho(Z)\geq 0$ for all $p\in\partial\Omega$ (with strict inequality for $z\neq 0$).

With this in mind, we are now able to sensibly speak of Levi pseudoconvex hypersurfaces in arbitrary complex manifolds, or pseudoconvex boundaries of complex manifolds. The reader is likely to surmise at this point that these pseudoconvex (smooth) boundaries are in fact contact manifolds. Accordingly, we would like to take a short interlude to discuss various forms of symplectic convexity, which we will then relate to pseudoconvexity. This is a story that is in and of itself rather complicated and not completely understood, but we explore the situation anyways. Interestingly, symplectic convexity has strong implications on the nature of contact structures in dimension 3, yet this does not completely carry through to higher dimensions. For a more detailed perspective regarding these matters see also Etnyre's expository article [6].

Yet another useful notion of convexity, is that of ω-**convexity**. This is particularly crucial in cut and paste constructions of symplectic manifolds. As seen in the below definition, ω-convexity is essentially a way to "bound" a contact manifold by a symplectic manifold of one higher dimension.

Definition 4.8. We say that a compact symplectic manifold (M^{2n},ω) is a **strong symplectic filling** of the contact manifold $(V^{2n-1},\xi=\ker\alpha)$ whenever

(1) $\partial M=V$ as oriented manifolds.

(2) There exists an extension $\tilde{\alpha}$ on a collar neighborhood of V such that $\omega=d\tilde{\alpha}$.

(V,ξ) is said to be the ω-**convex** boundary of (M,ω). With the appropriate changes of sign in the above conditions one obtains a **strong concave filling** or ω-**concave** boundary. One may then glue ω-concave to ω-convex manifolds along V to obtain closed symplectic manifolds.

Remark 4.9. In first condition the orientation is induced by ω^n on M and $\alpha \wedge (d\alpha)^{n-1}$ on V. The second condition can be equivalently formulated in terms of the existence of a Liouville vector field Y defined in a neighborhood of ∂M, pointing outwards along ∂M, and satisfying $\xi = \ker\,(\iota_Y \omega|_{TV})$.

There also exists weaker notions of ω-convexity, namely domination and weak symplectic filling. We say that ω **dominates** ξ when $\omega|_\xi$ is in the canonical conformal class of symplectic forms on ξ. This simply means that there exists a positive function f on V such that $\omega = f d\alpha$ when restricted to ξ. Recall that any contact form α multiplied by a positive function yields the same contact structure, thus $d\alpha$ defines a unique conformal class of symplectic forms on ξ. The definition of **weakly symplectically fillable** (sometimes referred to as just symplectically fillable) is obtained by weakening condition (2) of the above definition (4.8) by replacing it with the condition that $\omega|_\xi$ be nondegenerate.

Notice that if (M, ω) that dominates (V, ξ), then (M, ω) is also a weak symplectic filling of (V, ξ). In dimension three, it is fairly easy to check that these two notions are the same. When the dimension is greater than four, these two notions will not be the same. If (V, ξ) is a strong symplectic filling of (M, ω) then ξ will be dominated by ω. What is surprising is that in dimension greater than four, we have that domination and strong symplectic fillability are equivalent concepts, as proved by McDuff [13]. In dimension three though, strong symplectic fillability is stronger than domination, since strong symplectic fillability implies that the symplectic form is exact, but there exist dominating symplectic forms which are not exact.

Now we'd like to relate ω-convexity to strict pseudoconvexity, which addresses the existence of strong symplectic fillings. In fact, we can show what pseudoconvexity implies weak symplectic fillability and that domination implies pseudoconvexity in all dimensions. These three notions are equivalent concepts in dimension three. However, before we get into those details we show that strictly pseudoconvex hypersurfaces carry a natural contact structure.

Given a region $\Omega \subset \mathbb{R}^{2n} \equiv \mathbb{C}^n$ as before, define a real 1-form α on $T(\partial\Omega)$ by

$$\alpha := -d\rho \circ J|_{T(\partial\Omega)}$$

Since $d\bar{z}_j \circ J = i\ dz_j$ and $d\bar{z}_j \circ J = -i\ d\bar{z}_j$, this can be written as

$$\alpha = i \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_j - \frac{\partial \rho}{\partial z_j} dz_j$$

whose restriction to $T(\partial\Omega)$ is from now on understood. We then compute:

$$d\alpha = 2i \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_j$$

The purely holomorphic and antiholomorphic terms disappear since the second derivatives of ρ commute and the antisymmetry of the wedge product. Observe that

$$\alpha = i(\bar{\partial}\rho - \partial\rho) = 2\ \mathrm{Im}(\partial\rho).$$

Thus ker $\alpha_p = T_p^{(1,0)}(\partial\Omega)$. In this manner we obtain the following lemma.

Lemma 4.10. *For $X \in T_p^{(1,0)}(\partial\Omega)$ and with $Z = \frac{1}{2}(X - iJX)$ as before we have*

$$d\alpha_p(X, JX) = 4L_p\rho(Z)$$

Proof. Write $X = Z + \bar{Z}$ with $Z = \sum_j Z_j \partial_{z_j}$ and $\bar{Z} = \sum_j \bar{Z}_j \partial_{\bar{z}_j}$. Then $JX = iZ - i\bar{Z}$. Hence

$$\begin{aligned} d\alpha_p(X, JX) &= 2i \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_j (Z + \bar{Z}, iZ - i\bar{Z}) \\ &= 4i \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} Z_j \bar{Z}_k \\ &= 4L_p\rho(Z) \end{aligned}$$

□

These calculations allow us to conclude that $d\alpha$ will be symplectic on $T_p^{(1,0)}(\partial\Omega)$

Proposition 4.11. *Let Ω be a strictly pseudoconvex region. Then the hyperplane distribution ξ on $\partial\Omega$ defined by $T_p^{(1,0)}(\partial\Omega),\ p \in \partial\Omega$ is a contact structure.*

Proof. For all $v, w \in T\partial\Omega \cap J(\partial\Omega) = \ker\ (d\rho \circ J)$,

$$(d(d\rho \circ J))(v, Jw) > 0.$$

Thus $d(d\rho \circ J))|_\xi$ is symplectic, so ξ is a contact structure.

□

Earlier we demonstrated that all odd dimensional spheres admit a contact structure. This is possible since the map $\rho : \mathbb{C}^n \to \mathbb{R}$, given by

$$\rho(z) = \frac{||z||^2 - 1}{4},$$

allows us to write the open unit ball as $\Omega = \{\rho(z) > 0\}$. Then

$$d\alpha = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$$

is the standard symplectic form on $\mathbb{C}^n$. So we see that the unit ball has a strictly pseduconvex boundary (i.e. unit sphere) and thus the set of complex tangiences to this boundary define a contact structure.

A smooth function $\rho : X \to \mathbb{R}$ on a complex manifold X is called **strictly plurisubharmonic** if the Levi form $L\rho(v, w) = (d(d\rho \circ J))(v, Jw)$ is positive definite. Furthermore,

ρ is called an **exhausting function** if it is proper and bounded below. A related concept is that of a **Stein manifold**, which when regarded with the least number of bells and whistles, is a complex manifold that admits a proper holomorphic embedding into $\mathbb{C}^N$ for some large integer N. A more sophisticated way of defining a Stein manifold is done via holomorphic convexity or holomorphic separability, but for our purposes we may avoid such a discussion. Grauert [9] showed that the Stein condition is equivalent to the existence of strictly plurisubharmonic functions.

Theorem 4.12. *A complex manifold is Stein if and only if it admits an exhausting plurisubharmonic function.*

One direction of this theorem follows from the fact that the function $\rho = \sum_{j=1}^{N} |z_j|^2$ on $\mathbb{C}^N$ restricts to an exhausting plurisubharmonic function on any Stein manifold. In light of remark 4.6 we see that the pseudoconvex condition is a local condition that is preserved under biholomorphic mappings. The requirement that a manifold be Stein, means that there is a holomorphic embedding of X in $\mathbb{C}^N$. Thus we see that $\rho = \sum_{j=1}^{N} |z_j|^2$ restricts to an exhausting plurisubharmonic function on X.

Thus we see that any Stein manifold admits a symplectic structure coming from the exhausting plurisubharmonic function, $\omega_\rho = -d(d\rho \circ J)$. In fact, this symplectic structure is essentially unique, as Eliashberg and Gromov [5] proved that given 2 plurisubharmonic functions ϕ and ψ on a Stein manifold X, (X, ω_ϕ) and (X, ω_ψ) will be symplectomorphic. Remembering our earlier interest in Liouville vector fields and hypersurfaces of contact type, we now demonstrate that these arise naturally in Stein manifolds. We also see that given a strictly plurisubharmonic function $\rho : X \to \mathbb{R}$ on a complex manifold (X, J) and a regular value c of ρ, the complex tangencies define a contact structure on the level set $Q_c = \rho^{-1}(c)$. Such hypersurfaces also admit strong symplectic fillings.

Lemma 4.13. *The gradient vector field $\nabla\rho$ of a plurisubharmonic function ρ on a Stein manifold X is a Liouville vector field for ω_ρ. The gradient vector field $\nabla\rho$ is transverse to $Q_c = \rho^{-1}(c)$, when c is a regular value. Furthermore, for any regular value c of ρ, the manifold $X_c := \{p \in X |\ \rho(p) \leq c\}$ is in a natural way a strong symplectic filling of the contact manifold Q_c.*

Proof. Exercise: Prove this.

□

Remark. X_c is frequently called a **Stein filling** of Q_c

We are now almost able to see why the set of complex tangencies to the link gives us a contact structure on the link. However, one issue that remains to be "smoothed" over is that our ambient space $f_\Gamma^{-1}(0)$ is singular at the origin and not a Stein manifold. In Section 2 we saw that we can perturb this variety to a smooth Stein manifold embedded into $\mathbb{C}^3$ by taking the Milnor fiber given by $f_\Gamma^{-1}(t)$. We then proved that L is diffeomorphic to $f_\Gamma^{-1}(t) \cap S^5$ for sufficently small $t > 0$, which makes the aforementioned contact structure well-defined. This special situation, in fact tells us a little bit more, namely that we obtain a Stein fillable contact structure on the link, which we elaborate on. A remarkable theorem of Eliashberg and Gromov [5] is that any any weakly fillable contact structure on a 3-manifold is tight.

Theorem 4.14. *The real 1-form*

$$\alpha := \frac{i}{4}\sum_{j=0}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j)$$

induces a Stein fillable contact structure on any Brieskorn manifold $\Sigma(\mathbf{a})$ *(see 2.4 for the defining equation) or link* L *of a simple singularity. The contact structure for each of these manifolds, given by* $\xi = ker\ \alpha$ *is* $\xi_{\Sigma(\mathbf{a})} = T\Sigma(\mathbf{a}) \cap J_{\mathbb{C}^{n+1}}(T\Sigma(\mathbf{a}))$ *and* $\xi_L = TL \cap J_{\mathbb{C}^3}TL$ *respectively.*

Proof. The function

$$\rho(\mathbf{z}) = \frac{||\mathbf{z}||^2 - 1}{4}$$

is strictly plurisubharmonic on $\mathbb{C}^{n+1}$ as well as on its restriction to the complex submanifold $V(\mathbf{a}) \setminus \{\mathbf{0}\}$ or $f_\Gamma^{-1}(0) \setminus \{\mathbf{0}\}$. Notice that $\alpha = -d\rho \circ J$. Then from the previous calculations we see that ker α defines the complex tangencies of the strictly pseudoconvex boundary of $V(\mathbf{a}) \cap B^{2n+2}$ or $f_\Gamma^{-1}(0) \cap B^6$ and is thus a contact structure.

Unfortunately, as previously mentioned, we have that $V(\mathbf{a})$ and $f_\Gamma^{-1}(0)$ are not a complex manifolds since they each have a singular point at $\mathbf{0}$. Thus in order to obtain a Stein filling, we replace each of $V(\mathbf{a})$ and $f_\Gamma^{-1}(0)$ by $V_t(\mathbf{a})$ and $f_\Gamma^{-1}(t)$ respectively. Then $\Sigma(\mathbf{a})$ is replaced by $\Sigma_t(\mathbf{a})$, likewise we replace $L = f_\Gamma^{-1}(0) \cap S^5$ by $L_t = f_\Gamma^{-1}(t) \cap S^5$. Now each of $V_t(\mathbf{a})$ and $f_\Gamma^{-1}(t)$ are non-singular Stein manifolds. As shown in Section 2, there exists an $\epsilon > 0$ such that $\Sigma_t(\mathbf{a})$ is diffeomorphic to $\Sigma(\mathbf{a}) = \Sigma_0(\mathbf{a})$ for all $t \in [0, \epsilon]$; likewise we have that L_t is diffeomorphic to $L = L_0$ for all $t \in [0, \epsilon]$. In fact, the proof showed that each of the $\Sigma_t(\mathbf{a})$ and L_t formed the fibers of a differentiable fiber bundle.

Our previous argument shows that the 1-form α induces a contact structure on each $\Sigma_t(\mathbf{a})$ and L_t, with a Stein filling given by $V_t(\mathbf{a}) \cap B^{2n+2}$ or $f_\Gamma^{-1}(t) \cap B^6$ respectively, for $t > 0$. With the help of a connection on the fiber bundle formed by the $\Sigma_t(\mathbf{a})$ and L_t, we may regard these contact structures as a smooth family ξ_t, $t \in [0, \epsilon]$ of contact structures on $\Sigma(\mathbf{a})$ and L. By Gray's stability theorem, the ξ_t are isotopic and since ξ_t is Stein fillable for $t > 0$, so is ξ_0.

□

5. Considerations of the Contactomorphic

We have already seen that S^3/Γ and L are diffeomorphic, and in the previous sections we discussed the natural contact structures they admit. In this section we will prove that are contactomorphic. However, finding an explicit contactomorphism proves to be a rather difficult and messy task, so we will appeal to Gray's stability theorem, an application of Moser's argument. We obtain our 1-parameter family of diffeomorphic contact manifolds by using the flow of a Liouville vector field defined in a neighborhood of $\varphi(S^3/\Gamma)$. Before proceeding we recall Gray's stability theorem and prove a corollary.

Theorem 5.1 (Gray's stability theorem)**.** *Let $\xi_t,\ t \in [0,1]$, be a smooth family of contact structures on a closed manifold V. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of V such that*

$$\psi_{t*}(\xi_0) = \xi_t \quad \textit{for each } t \in [0,1]$$

Corollary 5.2. *Let $(V_t,\ \xi_t),\ t \in [0,1]$, be a smooth 1-parameter family of diffeomorphic contact manifolds. Then there exists a diffeomorphism $\hat{\psi} : V_0 \mapsto V_1$ such that*

$$\hat{\psi}_* \xi_0 = \xi_1$$

Proof. Let $\phi_t : V_0 \mapsto V_t$ be the 1-parameter family of diffeomorphisms, with $\phi_0 = \mathrm{id}_{V_0}$. Then $\phi_t^* \xi_t$is a 1-parameter family of contact structures on V_0. By Gray's stability theorem, there exists $\psi_t : V_0 \to V_0$ such that

$$\psi_{t*} \xi_0 = \phi_t^* \xi_t$$

We have the following sequence of maps

$$(V_0, \xi_0) \xrightarrow{\psi_1} (V_0, \phi_1^* \xi) \xrightarrow{\phi_1} (V_1, \xi_1),$$

which yields

$$(\phi_1 \circ \psi_1)_* \xi_0 = \phi_{1*} \phi_1^* \xi_1 = \xi_1.$$

Thus we obtain $\hat{\psi} = \phi_1 \circ \psi_1$.

□

Theorem 5.3. *$(S^3/\Gamma, \xi_{S^3/\Gamma})$ is contactomorphic to (L, ξ_L)*

Proof. As shown in lemma 4.13 we have that the gradient vector field $\nabla \rho$ of a plurisubharmonic function ρ on a Stein manifold X, is a Liouville vector field for ω_ρ that is transverse along $\rho^{-1}(0) = L$. Here we take our plurisubharmonic function to be

$$\rho(\mathbf{z}) = \frac{||\mathbf{z}||^2 - 1}{4} = \frac{z_0 \bar{z}_0 + z_1 \bar{z}_1 + z_2 \bar{z}_2 - 1}{4}.$$

Then

$$\omega_\rho = \frac{i}{2} \left(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \right),$$

and

$$Y := \nabla \rho = \frac{1}{2} \left(z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right).$$

Now we consider the flow ψ_t of this Liouville vector field, starting at $\hat{\varphi}(S^3/\Gamma)$ and reparametrize such that the time-one map $\psi_{\sigma(1)}(\hat{\varphi}(S^3/\Gamma)) = L$. In this manner we obtain a 1-parameter family of diffeomorphic manifolds, with a natural associated contact structure, arising from the set of complex tangencies to a real hypersurface in a complex manifold, which is given by $T\psi_{\sigma(t)}(\hat{\varphi}(S^3/\Gamma)) \cap J_{\mathbb{C}^3} T\psi_{\sigma(t)}(\hat{\varphi}(S^3/\Gamma))$.

First we need $\varphi(S^3/\Gamma)$ and L to be disjoint submanifolds in $f_\Gamma^{-1}(0)$. This is accomplished by choosing a suitable radius R for the S^5 used in the defining equation of the link L. Note also that our choice of $Y = \nabla\rho$ means that the Liouville vector field is still defined on the portion of $f_\Gamma^{-1}(0)$ between $\varphi(S^3/\Gamma)$ and L_R and still transversal to L_R. For any integral curve γ of Y we consider the following initial value problem:

$$\begin{aligned} \gamma'(t) &= Y(\gamma(t)) \\ \gamma(0) &= z \in \hat{\varphi}(S^3/\Gamma) \end{aligned} \tag{5.1}$$

Denote the flow of this vector field $\psi_t(z) = \gamma(t)$. This exists and is smooth by the fundamental theorem on flows. Write $\psi_t(z) = \gamma_z(t)$ as the unique integral curve of Y passing through z. By means of the implicit function theorem and the properties of the Liouville vector field Y we can prove the following claim.

Claim 5.4. For every γ_z there exists some $\tau(z) \in \mathbb{R}^+$ such that $\psi_{\tau(z)}(z) \in L$. Furthermore, the choice of $\tau(z)$ varies smoothly for each $z \in \hat{\varphi}(S^3/\Gamma)$.

Proof. This follows from the fact that the Liouville vector field Y has no singular points and any integral curve γ of Y must eventually pass through L_R exactly once before exiting the region of $f_\Gamma^{-1}(0)$ between $\hat{\varphi}(S^3/\Gamma)$ and L_R. This is due to the the conformal symplectic and transversal nature of the Liouville vector field Y, which is used in the application of the implict function theorem. Before proceeding with the full details, we recall the implicit function theorem.

Theorem 5.5. *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function with coordinates on $\mathbb{R}^{n+m}$ given by $(x, y) = (x_1, ...x_n, y_1, ..., y_m)$. Fix a point (a, b) with $f(a, b) = c$, where $c \in \mathbb{R}^m$. If $[(\frac{\partial f_j}{\partial y_j})(a, b)]$ is invertible, then there exists an open set U^n containing a, an open set V^m containing b, and a uniquely continuously differentiable function $g : U^n \to V^m$ such that*

$$\{(x, g(x)) \mid x \in U\} = \{(x, y) \in U \times V \mid f(x, y) = c\}$$

Furthermore, regarding the regularity of g, we have that whenever the additional hypothesis that f is continuously differentiable up to k times inside $U \times V$ holds, then the same is true for the explicit function g inside U.

In our situation we have:

$$\begin{aligned} \gamma : &\quad \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^6 \\ \rho|_{f_\Gamma^{-1}(0)} : &\quad \mathbb{R}^6 \to \mathbb{R} \\ \rho \circ \gamma : &\quad \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R} \end{aligned}$$

and are solving $\rho(\gamma(t, z)) = 0$ to find t for a given z (i.e., we want to find the map $\tau : \mathbb{R}^6 \to \mathbb{R}$). Thus to apply to implicit function theorem we must show that for all (t, z)

with $\rho \circ \gamma = 0$

$$\frac{\partial(\rho \circ \gamma)}{\partial t} \neq 0.$$

Note that $\rho \circ \gamma$ is smooth. By the chain rule,

$$\left.\frac{\partial(\rho \circ \gamma)}{\partial t}\right|_{(s,p)} = \text{grad } \rho|_{\gamma(s,p)} \cdot \dot{\gamma}|_{(s,p)},$$

where $\dot{\gamma}|_{(s,p)} = \frac{\partial \gamma}{\partial t}|_{(s,p)}$. But if grad $\rho|_{\gamma(s,p)} \cdot \dot{\gamma}|_{(s,p)} = 0$, then either grad ρ is not transverse along $\{(\rho \circ \gamma)\ (s,p) = 0\}$ or $\dot{\gamma}|_{(s,p)} = \mathbf{0}$, since grad $\rho \neq 0$. However by construction grad $\rho = \nabla \rho$ is a Liouville vector field transverse to L and also L_R. Furthermore the conformal symplectic nature of a Liouville vector field implies that for any integral curve γ satisfying the initial value problem given by equation (5.1), $\dot{\gamma}|_{(s,p)} \neq \mathbf{0}$. Thus we see that the conditions for the implicit function theorem are satisfied and our claim is proven. □

Now we normalize the time it takes the flow $\psi_t(z)$ of Y to reach L_R. This is done by constructing a reparametrization

$$\sigma_z : [0,1] \to [0, \tau(z)]$$

of the time it takes to travel along the integral curve. Note that

$$\frac{z}{\tau(z)} : [0, \tau(z)] \to [0,1],$$

so

$$\sigma(z,s) = \tau(z) \cdot s$$

is our desired map.

We are thus able to define the following 1-parameter family of diffeomorphic contact manifolds $(M_s, \xi_s)_{s \in [0,1]}$, where

$$M_s := \psi_{\tau(z)\cdot s}(z)(\hat{\varphi}(S^3/\Gamma)), \quad \xi_s := TM_s \cap J_{\mathbb{C}^3}(TM_s)$$

and

$$M_0 = \psi_0(z)(\hat{\varphi}(S^3/\Gamma)) = \hat{\varphi}(S^3/\Gamma), \quad M_1 = \psi_{\tau(z)\cdot 1}(\hat{\varphi}(S^3/\Gamma)) = L_R.$$

After applying Gray's stability theorem, it remains to check that $(S^3/\Gamma, \xi_{S^3/\Gamma})$ is contactomorphic to $(\hat{\varphi}(S^3/\Gamma), TM_0 \cap J_{\mathbb{C}^3}(TM_0))$. Since $\varphi : \mathbb{C}^2 \to \mathbb{C}^3$ is a holomorphism and

$$\begin{aligned} \varphi(S^3) &= \hat{\varphi}(S^3/\Gamma) \\ \varphi_*(TS^3) &= \hat{\varphi}_*(TS^3/\Gamma), \end{aligned}$$

we have

$$\begin{aligned}
\hat{\varphi}_*(TS^3/\Gamma \cap J_{\mathbb{C}^2}(TS^3/\Gamma)) &= \varphi_*(TS^3 \cap J_{\mathbb{C}^2}(TS^3)) \\
&= \varphi_*(TS^3) \cap \varphi_*(J_{\mathbb{C}^2}(TS^3)) \\
&= \varphi_*(TS^3) \cap J_{\mathbb{C}^3}\varphi_*(TS^3)) \\
&= T\varphi(S^3) \cap J_{\mathbb{C}^3}(T\varphi(S^3)) \\
&= TM_0 \cap J_{\mathbb{C}^3}(TM_0)
\end{aligned}$$

as desired. □

The previous argument gave us a one-parameter family of diffeomorphic contact manifolds from $(\hat{\varphi}(S^3/\Gamma), \hat{\varphi}_*\xi_{S^3/\Gamma})$ to (L, ξ_L) by means of reparametrizing the flow of $\nabla\rho$. Recall that we defined $\xi_{S^3/\Gamma} = TS^3/\Gamma \cap J_{\mathbb{C}^2}(TS^3/\Gamma) =$ ker $(\iota_{Y_0}\omega_{\mathbb{C}^2})$, where Y_0 is the Liouville vector field as given in proposition 3.10 and $\omega_{\mathbb{C}^2}$ is the standard symplectic form on $\mathbb{C}^2$. However there are two natural symplectic structures we may consider on $\mathbb{C}^2$, namely

$$\omega_{\mathbb{C}^2} := \frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v}),$$

and

$$\varphi^*\omega_{\mathbb{C}^3} := \frac{i}{2}\varphi^*\left(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\right).$$

We see that these are equivalent, using Moser's trick as follows.

Proposition 5.6. *$(\mathbb{C}^2, \omega_{\mathbb{C}^2})$ and $(\mathbb{C}^2, \varphi^*\omega_{\mathbb{C}^3})$ are symplectomorphic.*

Proof. We want to show that

$$\omega_t = (1-t)\omega_{\mathbb{C}^2} + t\varphi^*\omega_{\mathbb{C}^3}$$

gives us a family of symplectic forms with exact time derivative. The exactness of $\frac{d}{dt}\omega_t$ follows from the fact that

$$\omega_t = d\lambda_t = (1-t)d\lambda_{\mathbb{C}^2} + t\varphi^* d\lambda_{\mathbb{C}^3}.$$

To see that ω_t is in fact a symplectic form follows from direct computations - we check this in the case of A_n, where $\varphi(u,v) = (uv, u^{n+1}, v^{n+1})$. We have that

$$\tfrac{2}{i}\varphi^*\omega_{\mathbb{C}^3} = d(uv) \wedge d(\overline{uv}) + d(u^{n+1}) \wedge d(\bar{u}^{n+1}) + d(v^{n+1}) \wedge d(\bar{v}^{n+1})$$

$$= ((n+1)^2|u|^{2n} + |v|^2)du \wedge d\bar{u} + u\bar{v}\ dv \wedge d\bar{u} + v\bar{u}\ du \wedge d\bar{v}\ + ((n+1)^2|v|^{2n} + |u|^2)dv \wedge d\bar{v}$$

$$= ((n+1)^2|u|^{2n} + |v|^2)du \wedge d\bar{u} + 2\mathrm{Re}(u\bar{v}\ dv \wedge d\bar{u}) + ((n+1)^2|v|^{2n} + |u|^2)dv \wedge d\bar{v}$$

is clearly symplectic since $|2\mathrm{Re}(u\bar{v}\ dv \wedge d\bar{u})| \leq 2|u||v||dv \wedge d\bar{u}|$. □

Furthermore, since $\Gamma \subset SU_2$ and from the above computation we see that both of $\omega_{\mathbb{C}^2}$ and $\varphi^*\omega_{\mathbb{C}^3}$ are preserved by the action of Γ. We say that an action of a Lie group G **preserves the symplectic form** on a symplectic manifold (M, ω) whenever

$$\omega(Dl_g(x)(X), Dl_g(x)(Y) = \omega(X, Y) \text{ for } X, Y \in T_xM,\ x \in M,\ g \in G,$$

where l_g denotes the map $x \mapsto g \cdot x$. Note that in the our setting the group Γ is a compact 0-dimensional Lie group. So a question that remains is if the symplectic structure on $\mathbb{C}^2/\Gamma \setminus \{\mathbf{0}\}$, inherited from $\mathbb{C}^2$ is the same for each of these. $(\mathbb{C}^2/\Gamma \setminus \{\mathbf{0}\}, \omega_{\mathbb{C}^2})$ and $(\mathbb{C}^2/\Gamma \setminus \{\mathbf{0}\}, \varphi^*\omega_{\mathbb{C}^3})$ are symplectomorphic. It follows that $(S^3/\Gamma, \iota_{Y_0}\omega_{\mathbb{C}^2})$ and $(S^3/\Gamma, \hat{\varphi}^*(\iota_{\nabla\rho}\omega_{\mathbb{C}^3}))$ are contactomorphic.

Remark 5.7.

In fact, Caubel and Popescu-Pampu [3] prove a stronger statement regarding the uniqueness of contact structures associated to rational homology spheres which admit a Milnor filling. Namely they prove the following:

Theorem 5.8. *If the 3-manifold M is a rational homology sphere, then there is at most one isomorphism class of contact structures on it which admits a Milnor filling.*

6. Appendix

Gray's stability theorem states that there are no non-trivial deformations of contact strucutres on closed manifolds. Before proceeding we prove a prepatory lemma regarding derivatives of time dependent k-forms and recall the definition of the Reeb vector field.

Lemma 6.1. *Let $\omega_t,\ t \in [0, 1]$, be a smooth family of differential k-forms on a manifold M and $(\psi_t)_{t\in[0,1]}$ an isotopy of M. Define a time-dependent vector field X_t on M by $X_t \circ \psi_t = \dot{\psi}_t$, where the dot denotes derivative with respect to t (so that ψ_t is the flow of X). Then*

$$\frac{d}{dt}(\psi_t^*\omega_t)|_{t=t_0} = \psi_{t_0}^*(\dot{\omega}_t|_{t=t_0} + \mathcal{L}_{X_{t_0}}\omega_{t_0})$$

Proof. Recall that for a time-*indepdendent* k-form ω we have

$$\frac{d}{dt}(\psi_t^*\omega)|_{t=t_0} = \psi_{t_0}^*(\mathcal{L}_{X_{t_0}}\omega)$$

We then merrily compute

$$\begin{aligned}
\tfrac{d}{dt}(\psi_t^*\omega_t) &= \lim_{h\to 0} \frac{\psi_{t+h}^*\omega_{t+h} - \psi_t^*\omega_t}{h} \\
&= \lim_{h\to 0} \frac{\psi_{t+h}^*\omega_{t+h} + \psi_{t+h}^*\omega_t - \psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h} \\
&= \lim_{h\to 0} \psi_{t+h}^*\left(\frac{\omega_{t+h} - \omega_t}{h}\right) + \lim_{h\to 0} \frac{\psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h} \\
&= \psi_t^*(\dot{\omega}_t + \mathcal{L}_{X_t}\omega_t).
\end{aligned}$$

Thereby obtaining the desired result. □

Theorem 6.2 (Gray's stability theorem)**.** *Let $\xi_t,\ t \in [0,1]$, be a smooth family of contact structures on a closed manifold V. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of V such that*

$$\psi_{t*}(\xi_0) = \xi_t \quad \textit{for each } t \in [0,1]$$

Proof. The simplest proof of this result relies on Moser's trick regarding stability results for (equicohomologous) volume and symplectic forms. The idea behind Moser's trick is to assume that ψ_t is the flow of a time-dependent vector field X_t. The desired equation for ψ_t translates to an equation for X_t. If that equation can be solved, then the isotopy ψ_t is found by integrating X_t. Recall that on a closed manifold the flow of X_t will be globally defined.

Let α_t be a smooth family of 1-forms with $ker\alpha_t = \xi_t$. The equation we desire our isotopy of M to satisfy becomes:

$$\psi_t^*\alpha_t = \lambda_t\alpha_0,$$

where $\lambda_t : M \to \mathbb{R}^+$ is a suitable smooth family of smooth functions. The proceeding lemma together with differentiation of this equation with respect to t yields,

$$\psi_t^*(\dot{\alpha}_t + \mathcal{L}_{X_t}\alpha_t) = \dot{\lambda}_t\alpha_0 = \frac{\dot{\lambda}_t}{\lambda_t}\psi_t^*\alpha_t.$$

With the aid of Cartan's formula and writing $\mu_t = \frac{d}{dt}(log\lambda_t) \circ \psi_t^{-1}$, this becomes

$$\psi_t^*(\dot{\alpha}_t + d(\alpha_t(X_t)) + \iota_{X_t}d\alpha_t) = \psi_t^*(\mu_t\alpha_t).$$

Choosing $X_t \in \xi_t$ this is equation will be satisfied provided

$$\dot{\alpha}_t + \iota_{X_t}d\alpha_t = \mu_t\alpha_t. \tag{6.1}$$

Plugging in the Reeb vector field R_t of α_t gives

$$\dot{\alpha}_t(R_t) = \mu_t. \tag{6.2}$$

So we can use (6.2) to define μ_t, the non-degeneracy of $d\alpha_t|_{\xi_t}$, and the fact that $R_t \in ker(\mu_t|A_t - \dot{\alpha}_t)$ which allows us to find the unique solution $X_t \in \xi_t$ of (6.1).

□

References

[1] Brieskorn, E. *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1-14.

[2] Burban, I.; *Du Val Singularities*, `http://www.math.uni-bonn.de/people/burban/singul.pdf`

[3] Caubel, C., Popescu-Pampu, P.; *On the contact boundaries of normal surface singularities*, C.R. Acad. Sci. Paris, Ser. I **339** (2004) 43-48

[4] Eliashberg, Y.; *Symplectic geometry of plurisubharmonic functions*,

[5] Eliashberg, Y. and Gromov, M. *Convex symplectic manifolds*, Proc. of Symposia in Pure Math., **52** (1991), part 2, 135-162

[6] Etnyre, J; *Symplectic convexity in low dimensional topology*

[7] Flanders, Harley; *Differential forms with applications to the physical sciences*, Dover Publications, 1989.

[8] Geiges, H.; *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge University Press, 2008.

[9] Grauert, H. *On Levi's problem*, Ann. of Math. **68** (1958), 460-472.
[10] Gromov, M. *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307-347.
[11] Hirzebruch, F. and Mayer, K. H. *O(n)-Mannigfaltigkeiten, exotische Sphären und Singularitäten*, Lecture Notes in Mathematics 57, Springer-Verlag, 1968.
[12] McDuff, D.; *The structure of rational and ruled symplectic 4-manifolds*, Journal of the Amer. Math. Soc. **3** (1990)679-712; erratum, **5** (1992), 987-8
[13] McDuff, D.; *Symplectic manifolds with contact type boundary*, Invent. Math. **103** (1991), 651-671.
[14] McDuff, D. and Salamon, D.; *Introduction to Symplectic Topology*, Oxford University Press, 1995.
[15] Milnor, J. *Singular Points of Complex Hypersurfaces*, Annals of Mathematical Studies 61, Princeton University Press, Princeton 1968.
[16] Ohta, H. and Ono, K.; *Simple singularities and topology of symplectically filling 4-manifolds*, Comment. Math. Helv. **74** (1999), 575-590
[17] Ohta, H. and Ono, K.; *Simple singularities and symplectic fillings*, J. Differential Geom. **69** (2005), no 1., 1-42
[18] Reid, M.; *The Du Val singularities* A_n, D_n, E_6, E_7, E_8, Incomplete chapters on algebraic surfaces, `http://www.warwick.ac.uk/~masda/surf/more/DuVal.pdf`
[19] Reid, M.; *Surface cyclic quotient singularities and Hirzebruch-Jung resolutions*, Incomplete chapters on algebraic surfaces, `http://www.warwick.ac.uk/~masda/surf/more/cyclic.pdf`
[20] Saito, K.; *A new relation among cartan matrix and Coxeter matrix*, J. Algebra **105** (1987) 149-158, MR 0871750 (88c:14003), Zbl 0613.14003.

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