CAAP Intro to Proof Based Math Lectures

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1 Differential Equations in Physics

1. Start with basics
$$F = ma \to F = m\frac{d^2x}{dt^2} = m\ddot{x}$$
2. If $F = K$ then
$$x = \frac{K}{2m}t^2 + bt + c$$
b and c are called the initial conditions.
3. **Proposition:** The number of free variables in a differential equation is the maximum number of derivatives in the equation (most of the time)
4. Consider a drag equation
$$F = -mg - m\alpha v$$
5. how do we solve this equation?
$$\int \frac{dv}{g + \alpha v} = -\int dt \implies v = \frac{-g}{\alpha}(1 - e^{-\alpha t})$$
and so
$$y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}(1 - e^{-\alpha t})\right)$$
approximating
$$e^{-x} \cong 1 - x + \frac{x^2}{2}$$
we get that when $\alpha t \ll 1$, that $y(t) \cong h - gt^2/2$, which is the same result when the initial velocity is 0 and there is no drag.
6. Physicists like to do these types of approximations a lot. And they're good sanity checks
7. See here for a demo

   `https://phet.colorado.edu/sims/html/projectile-motion/latest/projectile-motion_en.html`
8. Now let's move on to oscillations. We have the picture of a spring system and the following differential equation for $k > 0$:
$$F = m\ddot{x} = -kx \implies x(t) = A\sin(\sqrt{k}t) + B\cos(\sqrt{k}t)$$
9. Solution is periodic
10. If we assumed $k > 0$, then our solution would be
$$x(t) = Ae^{\sqrt{k}t} + Be^{-\sqrt{k}t}$$
11. Really, the second equation handles both cases when $k > 0$ and $k < 0$ (why is $k = 0$ not included?)

12. Recall Euler's identity

$$e^{ix} = \cos x + i \sin x \implies \cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

If $k < 0$ then $\sqrt{k} = i\sqrt{|k|} = i\omega$ and

$x(t) = Ae^{i\omega t} + Be^{-i\omega t} = A(\cos(\omega t) + i\sin(\omega t)) + B(\cos(\omega t) - i\sin(\omega t)) = (A+B)\cos(\omega t) + i(A-B)\sin(\omega t) = A^* \cos(\omega t) + B^*$ sin

13. Our constants now are complex numbers, of the form $a + ib$, but this handles the most general case

14. What if we combine oscillation and drag?

$$F = ma \implies m\ddot{x} = -kx - mbv = -kx - mb\dot{x}$$

A lot of times in math, we figure out tricks and then see how far they go. This is one of those times. Let's try $x(t) = e^{\alpha t}$

15. Doing the work, we'll get

$$\alpha^2 + b\alpha + \frac{k}{m} = 0$$

let's use the quadratic equation for this!

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4k/m}}{2}$$

Because we're mature mathematicians, we can handle complex numbers, so the $\pm$ sign gives us two different values, α_1 and α_2 (write these out) and hence we have

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

right? Wrong!

16. What if $\alpha_1 = \alpha_2$. What conditions would lead us to this

17. Really, we have three cases

$$b^2 > 4k/m \to \text{overdamped}, \quad b^2 = 4k/m \to \text{critically damped}, \quad b^2 < 4k/m \to \text{underdamped}$$

18. Question: Amongst the three scenarios, which one goes to 0 the fastest? We rewrite the equation:

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$

where $\omega^2 = k/m$ and $2\gamma = b/m$.

19. To determine the fastest convergence to 0, we solve each case and take ratios of limits

20. Let $\Omega^2 = \gamma^2 - \omega^2$, then the general solution is given by

$$\alpha_\pm = -\gamma \pm \Omega = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4\omega^2}}{2}$$

$$x(t) = e^{-\gamma t}(Ae^{\Omega t} + Be^{-\Omega t})$$

21. Overdamped: our condition of $b^2 > 4k/m$ becomes exactly $\Omega^2 > 0$, and so Ω is the positive square root

22. Note that in the overdamped case, we get two, real exponential solutions.

23. Critically damped: Corresponds to $\Omega^2 = 0$ and so

$$x(t) = e^{-\gamma t}(Ae^0 + Be^0) = (A+B)e^{-\gamma t}$$

Here, we've violated our rule of "number of derivatives equals number of free variables." So I pull out of my ass the general solution

$$y(t) = e^{-\gamma t}(At + B)$$

why is this different? Because $Ate^{\alpha t}$ is not "similar" enough to $Be^{\alpha t}$, in the same way that $Ae^{\alpha t}$ and $Be^{\alpha t}$ are similar

24. Underdamped: We have $\Omega^2 < 0$ and so we let $\Omega = iw$ for $w > 0$. We get an exponential times a periodic function

$$x(t) = e^{-\gamma t}(Ae^{iwt} + Be^{-iwt}) = e^{-\gamma t} C \cos(wt + \phi)$$

This alternate form comes from cosine of a sum and $A + B = C\cos\phi$ and $A - B = iC\sin\phi$ is the equivalent transformation of constants

25. Note that our two free variables changed from A and B to C and ϕ.

26. C can be understood as the amplitude of the harmonic motion

27. ϕ is the "phase" which tells us that the motion doesn't start at the origin, but at some other point along the curve

28. See picture

29. Good videos of under damped, overdamped, and critically damped:

`https://www.youtube.com/watch?v=99ZE2RGwqSM`

30. We usually go for critically damped bc as the video shows, it's the fastest!

2 Secondary Application: Population Model

1. Another common application of differentiation is differential equations in population modeling

2. (Bunny model): The population grows proportionally to the population:

$$\frac{dp}{dt} = \dot{p} = ap \implies \frac{\dot{p}}{p} = a \implies p(t) = Ce^{at}$$

3. A more nuanced model has the following assumptions

   (a) If the population is small, the growth rate remains directly proportional to the size of population. Think of this as an excess of resources, and the only limiting factor is the number of participants which can reproduce

   (b) If the population grows too large, the growth rate becomes negative. I.e. the population is limited by resources

4. In this case, we have a logistic population growth model

$$\dot{x} = ax(1 - x/N) \implies \frac{\dot{x}}{ax(1 - x/N)} = \frac{N}{a}\frac{\dot{x}}{x(N - x)} = \frac{\dot{x}}{a}\left[\frac{1}{x} - \frac{1}{x - N}\right]$$

$$\frac{1}{a}\left[\ln(x) - \ln(x - N)\right] = t + C \implies \ln\left(\frac{x}{x - N}\right) = at + C \implies \frac{x}{x - N} = 1 + \frac{N}{x - N} = Ke^{at}$$

$$N = 1 \implies x(t) = \frac{Ke^{at}}{1 + Ke^{at}} = 1 - \frac{1}{1 + Ke^{at}}$$

5. a and N are positive parameters: a is the population growth rate, and N represents the max capacity.

6. Here's the slope field and a plot of the solution for various values of a

7. Note that when $K = x(0) > 0$ the solutions tend to 1 (or N equivalently). When $x(0) < 0$, the solution diverge off to $-\infty$, but these aren't physically relevant

8. A modified system is

$$\dot{x} = x(1 - x) - h$$

Using a trig sub we have

$$x(t) = \frac{1}{2}\left[\sqrt{4h - 1}\tan\left(\frac{1}{2}\left(c_1\sqrt{4h - 1} - \sqrt{4h - 1}t\right)\right) + 1\right]$$

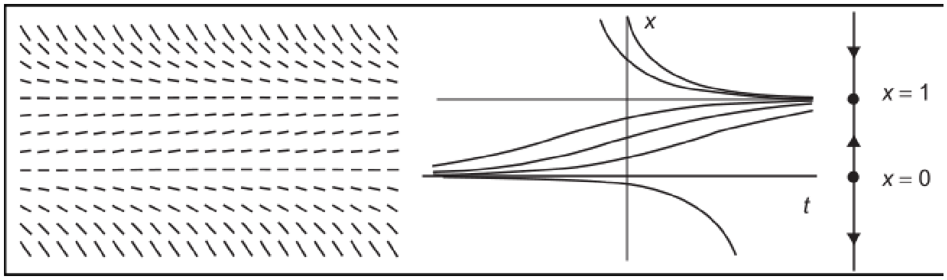

Figure 1.3 Slope field, solution graphs, and phase line for $x' = ax(1-x)$.

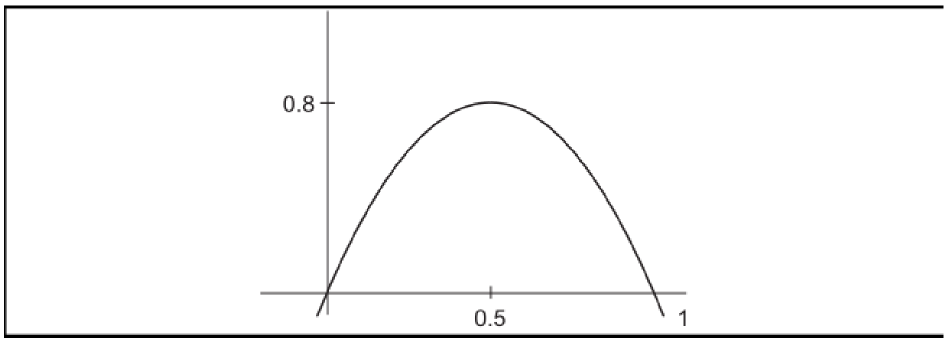

Figure 1.4 The graph of the function $f(x) = ax(1-x)$ with $a = 3.2$.

Figure 1

9. This is nice and general, but it doesn't tell us much about what happens, so let's plot the graph of the derivative

$$\dot{x} = x(1-x) - h = -x^2 + x - h = -(x - 1/2)^2 + (1/4 - h) = f_h(x)$$

   for $h > 1/4$, $h < 1/4$, and $h = 1/4$

10. $f_h(x)$ has two roots when $0 \leq h \leq 1/4$, one root when $h = 1/4$ and no roots when $h > 1/4$

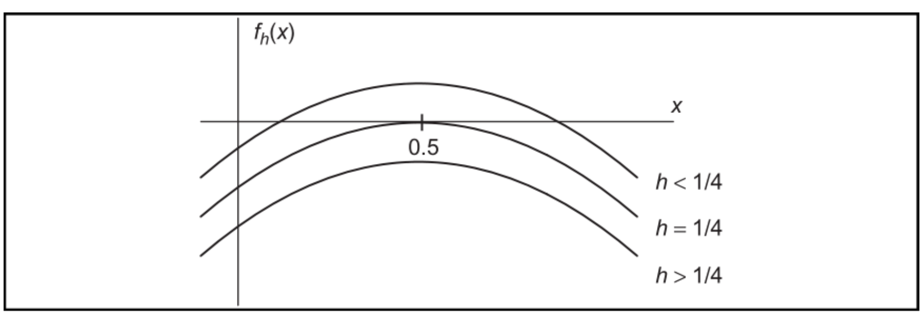

Figure 1.6 The graphs of the function $f_{h(x)} = x(1-x) - h$.

Figure 2

11. Looking at the graph, when $f_h(x) > 0$ our change in x is positive, so we move to the right on the graph and vice versa

12. Via the graph, we have that $h < 1/4$ has a source and a sink, meaning that if we make small perturbations away from that value x_l, then the population will not return for a while/in a short period of time

3 Random Cool Things

1. Proving that the harmonic series diverges
2. Proving the p-series test, generalizes the above example
3. **Theorem:** Consider the sequence
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
Then this series converges iff $p > 1$ and diverges else.
4. Proving the limit comparison test
5. **Theorem:** Consider $\sum_n a_n$ and $\sum_n b_n$, and define
$$c = \lim_{n \to \infty} \frac{a_n}{b_n}$$
If $0 < c < \infty$, then either both series converge or diverge.
6. Proof idea is that there exists A, B with $0 < A < c < B < \infty$, so that for N sufficiently large, we have
$$n \geq N \implies A < \frac{a_n}{b_n} < B \implies a_n < b_n B, \qquad Ab_n < a_n$$
and so taking the sum from N to ∞ tells us that both converge via this bound and the fact that $c > 0$ (i.e. same sign in the long run)