Reading with Otis Chodosh on the Allen Cahn Equation

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1 Introduction to Allen-Cahn Equation

1.1 Questions

1. Why isn't a point a perimeter minimizer in definition 3.9?
2. Don't understand "$2D^2 u_\epsilon(\nabla u_\epsilon, \cdot)$ on p. 13 - what implicit pairing is happening?
3. Not sure how achieving Rayleight quotient is ewquivalent to solving that PDE p. 17
4. P. 18, why does $\lim_{\tau\to\infty} u_\tau = 1$ uniformly? Resolved
5. Why is $|\nabla u| \leq C$ for a u a monotone solution to Allen-Cahn on $\mathbb{R}^3$? See middle of p. 22
6. Some sus moves on p.24: how does one go from inequality 2 to 3 in the first chain?

1.2 Miscellaneous References

Otis told us to check out these links

1. Poincare Bendixson theorem
2. Coarea formula
3. Good reference on other aspects of Allen Cahn
4. Other reference
5. Neves article - something about the Gromov-Guth families connecting to Otis' work on trying to generalize the Weyl law but for minimal surfaces. Very min-max oriented.
6. Gaspar and Guaraco article - same thing as above, relates to Weyl law and min-max theory

More notes to look into

1. Guaraco's notes on Allen Cahn and construction via min-max
2. A comparison between Allen-Cahn and Almgren-Pitts Min-max theory
3. here A catalonian professor with good work on Allen-Cahn
4. here One of the papers from the previous source
5. here Another one of the papers from Cabre

Another list of notes

1. Overview of Allen Cahn from Frank Pacard - pretty good! Paper
2. Informal write up of Min-Max theory Yangyang Li
3. The best write up of Min-Max Colding and De Lellis
4. Guaraco's notes, might have linked these before Guaraco
5. Time dependent allen-cahn as an imitation of mean-curvature flow this paper

1.3 Exercise 2.1

(a) This should be an elliptic bootstrapping argument. I found the analogous problem in Evans PDE (second edition). Mention of the non-linear poisson equation on p.457

Actually let's just do this: let $x \in M$. We have that $f(x) = \frac{1}{\epsilon}u(x)(u(x)^2 - 1) \in L^2(N_x)$ for some neighborhood $N_x \ni x$ because $u \in H^1(\Omega) \cap L^\infty(\Omega)$ for all Ω compact. Pushing this into real coordinate charts, we can apply

Lemma 1.1 (Higher Interior Elliptic Regularity). *Assume*

$$Lu = -(a^{ij}(x)u_{x_i})_{x_j} + b^k u_{x_k} + c(x)u$$

and $a^{ij}, b^i, c \in C^{m+1}(U)$ and $f \in H^m(U)$ for U some open subset of $\mathbb{R}^n$. Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \qquad \forall x \in U$$

Then $u \in H^{m+2}_{loc}(U)$

If we apply this to the image of N_x (under the image of some coordinate chart) with $m = 1$, then we get that $u \in H^3_{loc}(U)$. Applying this theorem for $m = 3$ again to some neighborhood $V \subset\subset U$ for which $u, f \in H^3(V)$, we get that u and f are in $H^5_{loc}(V)$. Repeating this, we'll get a sequence of open subsets V_i each compactly contained in each other on which u and f are in $H^{2k+1}(V_k)$. Because $H^{2k+1} \implies C^k$ by choosing continuous representatives for the kth derivative, this will give us smoothness on a neighborhood of x.

There's more to this: continuous representatives don't exist unless we're in $\mathbb{R}^1$, so we use Morrey's inequality/sobolev embedding theorem, which says that if

$$\frac{1}{p} - \frac{k}{n} = -\frac{r+\alpha}{n} \implies W^{k,p}(U) \hookrightarrow C^{r,\alpha}(U), \qquad U \subseteq \mathbb{R}^n \quad \text{open and bounded}$$

in particular

$$||u||_{C^{r,\alpha}} \leq C(r, n, \alpha, U)||u||_{k,p}$$

The point is that because $u \in L^\infty$ (this is given in the set up before the problem) and because it suffices to prove the result in some small, precompact neighborhood about each point, i.e. $U_x \ni x$ with $\mu(U_x) < \infty$, then we can make p arbitrarily large in the sobolev inequality. A priori, we don't have an L^p bound on $|\nabla u|$, however, we can use the interior regularity estimates, with the set up of $L = -\Delta$ and $f = u(u^2 - 1)$ and $Lu = f$.

In particular, L^p elliptic regularity tells us that if $-\Delta u = f$ and $u \in W$, then we have the **Calderon-Zygmud estimate**

$$\int_{B_1} |D^2 u|^p \leq C \left(\int_{B_2} |f|^p + \int_{B_2} |u|^p \right)$$

we can apply this in our setting by making our neighborhoods, $U_{x,i}$, diffeomorphic to balls via rescaling and choosing our neighborhoods appropriately after using the chart map to send $U_{x,i}$ to an open set in $\mathbb{R}^n$. With this, because f and u are bounded, we have that $|D^2 u| \in L^p$ for any p. How do I get higher k though? Via elliptic regularity results for which I'm having difficulty finding references for (see p. 7 here for an example), the $|D^2 u| \in L^p$, comes with $u \in W^{2,p}$ for all p, now setting $\epsilon = n/p$, we have

$$k - \epsilon = r + \alpha$$

immediately, we get that $u \in C^{1,\alpha}$ $(k = 2)$ for any $\alpha < 1$, and so now by restricting to a smaller neighborhood about x, we look at

$$Lu = -\Delta u = u(u^2 - 1) = f$$

and then note that $f \in W^{1,p}$ for all p, because $f'(x)$ exists and is continuous and hence bounded on our small open set. Elliptic regularity then allows us to upgrade u to $W^{3,p}$ for all p. Now we see that we've gotten a higher value of k and so we can repeat this process on a smaller and smaller neighborhood and get taht $u \in W^{k,p}$ for arbitrary k and p, which by the sobolev inequality will tell us that u is smooth at x. ending the proof.

See the sources for references on L^p-elliptic regularity: here and here p. 10 and 11 here. This one points to a lot of resources

(b) Let $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\rho(x) = \begin{cases} 0 & x < 0 - \delta \\ x & x > 0 + \delta \end{cases}$$

i.e. just smooth the corner. Consider the set $\{|u| > 1\} = \{u > 1\} \cup \{u < -1\}$. WLOG, assume that the former is non-empty and hence has positive measure by nature of being open (remember that u is smooth and so continuous). Now consider

$$\varphi(x) = \begin{cases} u(x)\rho(u(x)^2 - 1) & u(x) > -1/2 \\ 0 & \text{else} \end{cases}$$

Just try

$$\varphi(x) = u(x)\rho(u(x)^2 - 1)$$

instead then this is a smooth function which is supported on $u \geq \sqrt{1-\delta}$. Moreover

$$\nabla_g \varphi = (\nabla_g u)\rho(u(x)^2 - 1) + u\rho'(u^2 - 1) \cdot 2u\nabla_g(u) = (\nabla_g u)[\rho(u^2 - 1) + 2u^2\rho'(u^2 - 1)]$$

we know that $0 \leq \rho'(u^2 - 1) \leq 1$ everywhere. For $u^2 - 1 > \delta$, we have

$$\rho(u^2 - 1) + 2u^2\rho'(u^2 - 1) = (u^2 - 1) + 2u^2 > 2 + 3\delta$$

and so by making δ as small as needed, we have

$$\int_\Omega \epsilon g(\nabla_g u, \nabla_g \varphi) = \epsilon \int_{|u^2 - 1| \leq \delta} (\cdots) + \epsilon \int_{|u^2-1|>\delta} |\nabla u|^2 [3u^2 - 1] > 0$$

I conclude strict positivity because $3u^2 - 1 \geq 2$ and $\mu(|u^2 - 1| > \delta)$ is bounded from below by some positive number as $\delta \to 0$ (this is the assumption that $\mu\{u^2 - 1 > 0\} > 0$). For the integral over $\{|u^2 - 1| < \delta\}$, the integrand is

$$\epsilon|\nabla u|^2(3u^2 - 1) \leq 3\epsilon|\nabla u|^2$$

but the measure of the $\{|u^2 - 1| < \delta\}$ decreases as $\delta \to 0$ so the integral tends to 0. Here $\Omega = supp(\varphi)$.

Similarly

$$\frac{1}{\epsilon}W'(u)\varphi = \frac{1}{\epsilon}u^2(x)\rho(u^2 - 1)^2 \qquad \forall x \text{ s.t. } u(x) > -1/2$$

but $u(x) > -1/2$ holds for all $x \in \Omega$. Thus the derivative of the Allen-Cahn energy is strictly positive with this set up, and so we conclude that $|u(x)|^2 \leq 1$ for all x in order for u to be a critical point

1.4 Exercise 2.2

(a) What does "solution" mean? Clearly, solutions exist for small time by picard iterate arguments. There could be a solution which exists for finite time but has infinite energy even on the time domain it is specified on. However, Let's consider $u \equiv 0$ with $\lambda = -1/2$, then we have

$$u'(t)^2 = 0 = 2\frac{1}{4}(1 - 0^2)^2 - \frac{1}{2} = 0$$

However, the energy of this solution is

$$\int_{-\infty}^{\infty} \frac{1}{2}u'(t)^2 + W(u(t)) = \int_{-\infty}^{\infty} 0 + \frac{1}{4}dt = \infty$$

b) If $\lambda > 0$, then I can show finite time blow up by making the bound

$$k_1 u^4 \geq u'(t)^2 \geq k_2 u^4$$

for some large time t. This holds because we have $u'(t) > \sqrt{\lambda}$ or $u'(t) < -\sqrt{\lambda}$ which tells us that u will grow at least linearly for all t. In fact, this contradicts the assumption of $u(t) \in [-1, 1]$ for all t, and contradicts it in finite time.

For $\lambda < 0$, first suppose that $u(0) \geq 0$. Then because $u'(t) > 0$ for all t, we have that $u(t) > 0$ for all $t > 0$. If we have that $\inf_t u'(t) = c > 0$, then we get at least linear growth and can contradict $u \in [-1, 1]$ in finite time. Thus we must have $\inf_t u'(t) = 0$. Let $\{t_k\}$ be a sequence such that $u'(t_k) \to 0$. Then we have that

$$\lim_{k\to\infty} 2W(u(t_k)) + \lambda = 0 \implies u^2(t_k) \to 1 - \sqrt{-2\lambda}$$

in particular, this tells us that $-1/2 \leq \lambda < 0$. Note that $u(t_k) \to 0$ if and only if $\lambda = -1/2$, which because $2W(u(t)) \leq 1/2$, tells us that $u'(t)^2 \leq 0$ for all t, a contradiction. Similarly, $u(t_k) \to 1$ if and only if $\lambda = 0$, a contradiction. Thus $u(t_k) \to c = 1 - \sqrt{-2\lambda}$ with $0 < c < 1$. Looking at the graph of

$$u''(t) = u(t)^3 - u(t) = u(t)(u(t) - 1)(u(t) + 1) = f(u(t))$$

for $f(x) = x(x - 1)(x + 1)$. Then if $u(t_k)$ converges to a value away from the roots of $u''(u)$ and $u(t_k) > 0$, then we see that locally about the value $u = c$, we can get a bound on $u(t_k)$ as

$$\exists K > 0 \text{ s.t. } \forall k > K, \qquad |u(t_k) - c| < \delta$$

For any $\epsilon > 0$, we choose δ sufficiently small, this tells us that

$$|u''(t_k) - f(c)| < \epsilon$$

in particular, setting $\epsilon = |f(c)|/2$ where $f(c) < 0$, we get the uniform bound of

$$u''(t_k) < f(c)/2 \implies u'(t) < u'(t_k) + (t - t_k)\frac{f(c)}{2}, \qquad \forall t \text{ s.t. } |u(s) - c| < \delta \qquad \forall \text{ s.t. } t_k < s \leq t$$

The idea is that for k large, we get $u(t_k)$ close enough to c so that for t close to t_k, we can apply this linear approximation bound, assuming that $u(t)$ stays close to c and doesn't leave the δ-neighborhood of c. This is why we require $|u(s) - c| < \delta$ for all $t_k < s \leq t$, because as soon as $u(s)$ leaves the δ neighborhood, the linear approximation on the first derivative no longer becomes valid

With this, we can choose k large so that $|u(t_k) - c| < \delta/2$ and also $u'(t_k)| < \frac{\delta|f(c)|}{8}$. Now we know that

$$0 < u(t) - u(t_k) < ||u'(t)||_{L^\infty}(t - t_k) < t - t_k \qquad \forall t > t_k$$

because $|u'(t)| < 1/2$for all t. With this, we have that setting $T = t_k + \delta/2$, we get by the triangle inequality that

$$|u(s) - c| < \delta, \qquad \forall t_k < s \leq T = t_k + \delta/2$$

and so

$$u'(t) < |u'(t_k)| + (t - t_k)\frac{f(c)}{2} < \frac{\delta|f(c)|}{8} + (t - t_k)\frac{f(c)}{2} = \frac{f(c)}{2}((t - t_k) - \delta/4)$$

remember that $f(c) < 0$. Setting $t = t_k + \delta/4$, we get that $u'(t) < 0$ a contradiction.

Similarly, we handle the case when $u(0) < 0$. Again we must have a sequence $\{t_k\}$ such that $u'(t_k) \downarrow 0$. But note that when $u(t) < 0$ then $f(u(t)) = u''(t) > 0$. So in order for $u'(t_k)$ (or some subsequence) to be decreasing, we must have that $u(T) \geq 0$ for some T, and hence because $u'(t) > 0$, we have that $u(t) > 0$ for all $t > T$. We can now restart our sequence once $t_k > T$ and repeat the above proof.

With this, we get that $\lambda = 0$ is the only possibility and so we're in the same case that the notes proved, meaning that the solution must have finite energy.

1.5 Exercise 2.3

Yes, this is just a u-sub and using the fact that $H'(t)^2 = 2W(H(t))$ because we're in the $\lambda = 0$ case. The integral is pretty easy, not going to do it.

1.6 Exercise 2.4

We do this for a square first: we have

$$\lim_{\epsilon\to 0} E_\epsilon(u_\epsilon; S_1(0)) = \int_{-1}^{1}\int_{-1}^{1} \frac{\epsilon}{2}|\nabla u_\epsilon|^2 + \frac{1}{\epsilon}W(u_\epsilon)$$

we have that

$$\nabla u_\epsilon = \nabla H(\langle a, x/\epsilon\rangle - b) = \frac{H'(\langle a, x\rangle - b)}{\epsilon}(a_1, a_2) \implies |\nabla u_\epsilon(x)|^2 = \frac{(H')^2}{\epsilon^2}$$

recall that $H'(t)^2 = 2W(t)$ so that

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{\epsilon}{2}|\nabla u_\epsilon|^2 + \frac{1}{\epsilon}W(u_\epsilon) = \int_{-1}^{1}\int_{-1}^{1} \frac{2}{\epsilon}W(u_\epsilon(x,y))dxdy$$

making the change of variables $(x,y)/\epsilon = (s,t)$ gives

$$I = 2\epsilon \int_{-\epsilon^{-1}}^{\epsilon^{-1}}\int_{-\epsilon^{-1}}^{\epsilon^{-1}} W(u(x,y))dxdy$$

Now note that

$$W(u(x,y)) = W(H(\langle a, (x,y)\rangle - b))$$

and so we can make the change of variables so that $\langle a, (x,y)\rangle = x$ via a rotation (which doesn't affect the measure) because $|a| = 1$. With this, we see that

$$I = 2\epsilon \int_{\epsilon^{-1}}^{\epsilon^{-1}}\int_{\epsilon^{-1}}^{\epsilon^{-1}} W(H(x-b))dxdy = 4\int_{\epsilon^{-1}}^{\epsilon^{-1}} W(u(x))dx \to 2\int_{-\infty}^{\infty} 2W(u(x))dx$$

so this is expressable in terms of the integral answer from the previous question. Moreover, to see that integrating over the square and the circle are the same in the limit, we have that the difference is

$$I_d = \int_{x=-1}^{x=1}\int_{y\in V} \frac{2}{\epsilon}W(u_\epsilon(x,y))dxdy$$

where $V = (-1, -(1-\sqrt{1-x^2})) \cup (1-\sqrt{1-x^2}, 1)$. Having adopted the same basis so that $u_\epsilon(x,y) = H(x/\epsilon - b)$, we get that the y integral just contributes a factor of $2(1-\sqrt{1-x^2})$, and so

$$I_d = \frac{4}{\epsilon}\int_{x=-1}^{1} (1-\sqrt{1-x^2})W(H(x/\epsilon - b))dx$$

Note that on this domain, $1-\sqrt{1-x^2} < x^2$ because $1-x^2 < \sqrt{1-x^2}$. With this and the change of variables $x/\epsilon \to u$, we get

$$I_d = 4\epsilon^2 \int_{-\epsilon^{-1}}^{\epsilon^{-1}} u^2 W(H(u-b))du$$

I now show that the integrand is a bounded function, which will imply that $I_d \to 0$ as $\epsilon \to 0$ because it tends like ϵ. To see this, first, we can make a change of variables

$$I_d = 4\epsilon^2 \int_{-\epsilon^{-1}-b}^{\epsilon^{-1}-b} (u+b)^2 W(H(u))du$$

Now note that

$$\tanh(u/\sqrt{2}) = \frac{e^{\sqrt{2}u}-1}{e^{\sqrt{2}u}+1} \implies \tanh(u/\sqrt{2})^2 = \frac{e^{2\sqrt{2}u} - 2e^{\sqrt{2}u}+1}{e^{2\sqrt{2}u}+2e^{\sqrt{2}u}+1} \implies 1-\tanh(u/\sqrt{2})^2 = \frac{4e^{\sqrt{2}u}}{e^{2\sqrt{2}u}+2e^{\sqrt{2}u}+1}$$

when u is large and positive, the above tends like $e^{-\sqrt{2}u}$ via the largest term power term in the denominator cancelling the numerator power, so that $u^2e^{-\sqrt{2}u}$ remains bounded. When u is negative and large, we simply bound the denominator below by 1 and use the numerator to get $e^{\sqrt{2}u}$ so that $u^2e^{\sqrt{2}u} \to 0$ as $u \to -\infty$. With this, we see that

$$I_d \le 4\epsilon^2 \int_{-\epsilon^{-1}-b}^{\epsilon^{-1}-b} C = 4C\epsilon \to 0$$

Ok so retroactively, the above is not totally correct because the bounds of integration over the standard square transform nontrivially under the change $(x,y) \to M_\theta(x,y)$, HOWEVER, we didn't have to integrate over the standard square to begin, we could have integrated over the rotated square so that $\langle a, (x,y)\rangle = x$ in the coordinate frame from the start. Then we would get the same result (with the correct bounds now) and the difference manipulations are still valid.

1.7 Exercise 2.5

(a) To get existence, we consider

$$I_\epsilon(w) = \int_\Omega L(Dw(x), w(x), x)dx = \int_{\Omega_R} \frac{\epsilon}{2}|\nabla_g u|^2 + \frac{1}{\epsilon}W(u)$$

We see that because $||x||^2$ is convex, then L is convex in the p-variable, where we denote L as a function of $L(p, z, x)$. We also have that L is coercive in the p argument, because the W term is always positive, thus

$$L(Dw(x), w(x), x) \geq \frac{\epsilon}{2}|\nabla_g w|^2 \geq \frac{\epsilon}{2}C|Dw(x)|^2$$

In our specific case of working on Ω_R, we can take the standard euclidean metric, in which case $\nabla_g w = Dw$ and so $C = 1$. With this, we have that I_ϵ is lower semicontinuous and so a minimum of I exists by taking a sequence of u_k such that $I(u_k)$ approaches the minimum value

$$c = \min\{I(f) \mid f \in H_0^1(\Omega_R), \qquad |f| \leq 1\}$$

Note that this collection of functions over which we take the minimum is nonempty because $f \equiv 0$ works (though it's not a minimizer of energy). See section 8.2 in Evans for the theorems used

I claim that the minimizer, u_ϵ, is either identically 0 or u does not switch signs and $0 < |u| < 1$ on the interior of Ω_R. The lack of switching signs is easy: note that if we decompose $u = \max(u, 0) + \min(u, 0) = u_+ + u_-$ then from a standard result in weak derivatives

$$\nabla u_+(x) = \begin{cases} 0 & u \leq 0 \\ \nabla u & u > 0 \end{cases} \qquad a.e.$$

and the same for $u_-(x)$, thus $\nabla u = \nabla u_+ + \nabla u_-$ holds a.e. and the supports of ∇u_+ and ∇u_- are disjoint up to a set of measure 0. In particular, this tells us that

$$||\nabla u||_2^2 = ||\nabla u_+||_2^2 + ||\nabla u_-||_2^2$$

in particular, this expression and $W(u)$ are not changed if we consider $u_+ - u_-$ instead. Thus, we may as well take $u = u_+ - u_- \geq 0$ as a minimizer and can apply elliptic regularity to this function to show that it is actually smooth (this was done in a previous problem). I now apply the maximum principle to the operator

$$L = -\Delta u$$

Then by The Allen Cahn equation, we get $-\Delta u = -u(u^2 - 1)$. WLOG, assume that $u > 0$, then $-\Delta u \geq 0$, and so the strong maximum principle (see Evans p. 349/350) says that if u attains it's minimum on the interior, then it is constant. The minimum is 0, and so either $u \equiv 0$ or $u\Big|_{\partial\Omega_R} = 0$ and $u > 0$ on the interior. Now let's consider $w = u - 1$, which is a non-positive function on our domain

$$Lu = -\sum_{i,j} a_{ij}(x)w_{x_i x_j} + \sum_i b_i(x)u_{x_i} + c(x)u = -\Delta w + [u(u+1)]w = 0$$

here, we have that $c(x) \geq 0$ and $Lw = 0$ on our domain, so the strong maximum principle tells us that w cannot attain a maximum or minimum on the interior. See p. 350 in Evans. Thus $u \neq 1$ on the interior.

(b) The idea is that u_R is an energy minimizer and so it suffices to find another function vanishing on the boundary which has at most linear energy. I'll do this for $R > 1$, as the point is to take $R \to \infty$.

$$f(\theta, r) = \eta_R(r)$$

where $\eta(r)$ is a function which goes from $0 \to 1$ on the interval $1/\epsilon$, stays constant at 1 on $[\epsilon, R - \epsilon]$ and then goes from $1 \to 0$ on $[R - \epsilon, R]$. Noting that the gradient in polar coordinates is $e_r\partial_r + \frac{1}{r}e_\theta\partial_\theta$ we have

$$|\nabla f(\theta, r)|^2 = \eta_R'(r)^2$$

then

$$E_1(f) = (\pi/4)(\int_0^\epsilon + \int_{R-\epsilon}^R)(\eta_R'(r)^2 + W(f))rdr$$

we can choose η_R so that the derivative is bounded by $2/\epsilon$. Moreover $W(f) \leq 1$, and so we have the bound

$$\leq \frac{\pi}{4}\left[\epsilon^2 \frac{4}{\epsilon^2} + \frac{\epsilon}{\epsilon^2} R\right] + \frac{\pi}{16}\left[\epsilon + R\epsilon\right] \leq CR$$

where $C = \pi + \pi/(4\epsilon) + (\epsilon + 1)$ is independent of R, and so because u_R is a minimizer, we see that $E_1(u_R; \Omega_R) \leq CR$. With this, we see that $u_R \not\equiv 0$, else the energy would grow quadratically.

(c) From Otis: "Consider the set up of $\Delta u = f$ (Weakly) in $B_1 \setminus \{0\}$, that is, for any $\varphi \in C_0^\infty(B_1 \setminus \{0\})$, we have

$$\int \nabla u \cdot \nabla \varphi + f\varphi = 0$$

Then, interior estimates imply that $u \in C^\infty_{loc}(B_1 \setminus \{0\})$ and u solves the PDE in the classical sense at any point besides 0. Now, we want to extend things across 0 (we need some further assumptions on u, f, otherwise $\Delta \log|x| = 0$ is a counterexample). Let's take $\varphi \in C_0^\infty(B_1)$ and choose a sequence of cutoff functions χ_i with $\operatorname{supp}\chi_i \Subset B_1 \setminus \{0\}$ but $\chi_i \to 1$ near 0 point wise. Then we can plug $\chi_i \cdot \varphi$ into the weak formulation to get

$$\int \chi_i(\nabla u \cdot \nabla \varphi) + \varphi \nabla u \cdot \nabla \chi_i + f\chi_i \varphi.$$

So, e.g. if $u \in H^1(B_1)$, then we would have that the first term tends to $\nabla u \cdot \nabla \varphi$ (by dominated convergence) and the second term tends to zero (by Holder), i.e. it satisfies

$$\leq ||\varphi||_\infty (\int |\nabla u|^2)^{1/2} \cdot (\int |\nabla \chi_i|^2)^{1/2}$$

As long as we can ensure that $\int |\nabla \chi_i|^2 \to 0$. This is possible by a common function known as the log-cutoff trick (taking χ_i cutting off $\sim$ linearly just barely fails since we get a gradient of $\frac{1}{r}$ integrated over a ball of radius $\sim r$, i.e. we get $O(1)$ instead of $o(1)$). Take instead

$$\chi_r(x) = 2 - (\log|x|)/\log r$$

For $|x|$ in (r^2, r) (And 0 for $|x| < r$ and 1 for $|x| > r$). I'll leave it for you to check that the Dirichlet energy goes to zero as $r \to 0$. Thus, as long as u $\in H^1$ and f in L^2, say, we get that u is a weak solution across 0. Then, we can use elliptic regularity (assuming, say $f \in C^\infty$) to get that u is smooth and solves the PDE across 0. Of course, in the Allen—Cahn setting, you have to be a bit more careful, since it's non-linear. But I think the proof should go reasonably along these lines."

I'll verify a few parts of this. So first we reflect $u \in \Omega_R$ to get an H^1 solution on $B_R \cap \{y \geq 0\}$. The reason that this reflection lies in H_0^1 is because of the bounds of the reflection by $2||u||_{H^1(\Omega_R)}$. See Brezis p. 273, Lemma 9.2 in the second on extension operators. This worked via a reflection across the y-axis and the new solution still solves our PDE weakly as the odd reflection across the y-axis allows us to preserve the derivative in the x-direction. Now we reflect the new solution across the x-axis, to get an $H_0^1(B_R)$. We now show that this solves the PDE everywhere except for on $\{xy = 0\}$, which we must verify. **Note**: even though this set is measure 0, it's important to verify that the reflect solution solves the PDE on it, as solutions of the PDE a.e. aren't necessarily minimizers (consider the standard energy functional with $\Delta u = 0$ and the function $u(x) = |x|$, which works a.e. but doesn't have the right energy)

To see that allen cahn still holds across the axis we proceed as follows: consider a cover of the upper half plane disk, D_R^+, given by $U_L = \{x < 0\} \cap D_R^+$, $U_M = \{-\epsilon < x < \epsilon\} \cap D_R^+$, $U_R = \{x > 0\} \cap D_R^+$. Then we take a partition of unity subordinate to the cover with three function ρ_L, ρ_M, ρ_R, where I choose ρ_M to be symmetric in x. Then we have that

$$\int_{D_R^+} \nabla u \cdot \nabla v + u(u^2 - 1)v = \int \nabla u \nabla(\rho_L v + \rho_R v + \rho_M v) + u(u^2 - 1)(\rho_L + \rho_M + \rho_R)$$

we can decompose this into

$$\int \nabla u \nabla(\rho_L v) + u(u^2-1)(\rho_L v) + \int \nabla u \nabla(\rho_R v) + u(u^2-1)(\rho_R v) + \int u(u^2-1)(\rho_M v) + \int \nabla u \nabla(\rho_M v)$$

The first two integrals are 0 for all $\epsilon > 0$ by nature of u being a minimizer on both the left half and the right half individually (note that $\rho_L v$ has compact support within the left half and the same for $\rho_R v$). As we send $\epsilon \to 0$ the middle integral vanishes, and so it suffices to show that the fourth integral vanishes as we send $\epsilon \to 0$. This will use that ∇u is odd about the y-axis and that ρ_M is symmetric. If we write

$$\nabla \rho_M = (\partial_x \rho_M, \partial_y \rho_M)$$

then we can assume that $\partial_y \rho_M$ is uniformily bounded for ϵ small on U_M by the symmetry (in fact we can make ρ_M just to be constant w.r.t to y except for some decay at the top/bottom of the domain independent of ϵ). Moreove $\partial_x u(x,y) = \partial_x u(-x,y)$ and $\partial_y u(x,y) = -\partial_y u(-x,y)$ (by the odd reflection) and finally $\partial_x \rho_M(x,y) = -\partial_x \rho_M(-x,y)$ and we can bound this uniformly in size by $2/\epsilon$. Now we write

$$\int \nabla u \nabla(\rho_M v) = \int_{U_M} (\nabla u)(\nabla \rho_M) \cdot v + \int_{U_M} (\rho_M)(\nabla u) \cdot \nabla v$$

the second integral also vanishes as $\epsilon \to 0$ because the integrand is some bounded function. With the first integral, we write

$$\int_{U_M} (\nabla u)(\nabla \rho_M) \cdot v = \int_{U_M} (\nabla u)(\nabla \rho_M) \cdot (v(0,y) + (x,0) \cdot \vec{h}(x,y))$$

where we've used the taylor theorem in a very shallow manner. The point is that

$$\int_{U_M} (\nabla u)(\nabla \rho_M) \cdot (v(0,y)) = 0$$

because the integrand is odd about x. For the second integral

$$\left| \int_{U_M} (\nabla u)(\nabla \rho_M) \cdot [(x,0) \cdot \vec{h}(x,y)] \right| \le ||\nabla u||_2 \frac{2}{\epsilon} 2\epsilon ||\vec{h}||_\infty \mu(U_M)^{1/2} \to 0$$

The idea with this last one is that $\nabla \rho_M$ is at most $2/\epsilon$, while $|x| < \epsilon$ and so these terms cancel, and also bound $||\vec{h}||$ uniformly because v is smooth. We can then bound the remaining integral as $C \int_{U_M} |\nabla u|$, to which we apply holder's inequality, and note that

$$||\nabla u||_{L^2(D_R^+)} \le 2||\nabla u||_{\Omega_R}, \qquad \mu(U_M) \approx 2R\epsilon$$

and so their product goes to 0 as $\epsilon \to 0$.

With this, we can perform two reflections to get that u_R satisfies allen cahn everywhere except potentially at the origin. At the origin, we verify the steps in the clue, which are all straightforward once we set $f = 0$ and $c(x) = u^2(x) - 1$. Then we have that our operator is

$$L = -\Delta + c(x) \ \text{ s.t. } \ Lu = 0$$

$c(x)$ is first just bounded, and so we get $u \in H^2_{loc}$ on $B_1 - \{0\}$. I now show that $c(x) = (u-1)(u+1)$ is also in H^2_{loc}, via the following:

$$\partial_x^2 c(x,y) = (\partial_x^2 u)(u+1) + 2(\partial_x u)(\partial_x u) + (u-1)\partial_x^2(u+1)$$

integrating this over an open set, V, compactly contained in Ω_R, we can bound the first and last terms by $2|\partial_x^2 u|$, and we can use holder's inequality on the middle term to get that $||\partial_x^2 c||_{L^2(V)} < \infty$. The same argument is repeated everywhere locally, see exercise 2.1 I think I have an argument for this, but it confuses me about the elliptic regularity argument in 2.1. We actually do the set up where $L = \Delta$ and $f = u(u-1)(u+1)$, then we get that $u \in H^2_{loc}$. In particular, it's in H^2 about some a neighborhood, U, of a fixed point $x_0 \in \Omega_R$. Now via the leibniz rule and L^∞ bound on u, we get that $f \in H^1$. Then by the same elliptic regularity argument we get that $f \in H^3_{loc}(U)$, and we can find a smaller neighborhood $V \subset\subset U$ such that $V \ni x$ such that $f \in H^3(V)$. NOW, we can apply morrey's inequality to get that $u \in C^1$: Morrey's inequality/sobolev embedding theorem, which says that if

$$\frac{1}{p} - \frac{k}{n} = -\frac{r+\alpha}{n} \implies W^{k,p}(U) \hookrightarrow C^{r,\alpha}(U), \qquad U \subseteq \mathbb{R}^n \quad \text{open and bounded}$$

in particular

$$||u||_{C^{r,\alpha}} \le C(r,n,\alpha,U)||u||_{k,p}$$

in particular we set $p = 2$, $k = 3$, $n = 2$ so that

$$\frac{1}{2} - \frac{3}{2} = -1 = \frac{-(r+\alpha)}{2} \implies r + \alpha = 2$$

so choosing $r = 1$, $\alpha = 1$, we get C^1 and in particular, that $||\nabla||_{L^\infty(V_2)} < \infty$ for $V_2 \subset\subset V$. Now on V_2, we can show that $f \in H^3$ by using the product rule, holder's inequality, and also the uniform bound on the $|u|$ and $|\nabla u|$. This is because the terms in the derivative via leibniz rule look like

$$\partial_x^2 u \cdot \partial_y(u+1) \cdot (u-1), \qquad \partial_x^3 u \cdot (u^2-1), \qquad \partial_x u \cdot \partial_y(u-1) \cdot \partial_x(u+1)$$

the first term is bound via holder's after uniformly bounding $u-1$ by 2. The second term is bound via $|u^2-1| \leq 2$ and then using the H^3 norm of u. The last term is bound using the L^∞ norm of u and $u \pm 1$ on this smaller set compactly contained in V. We can repeat this back and forth between elliptic regularity and morrey's inequality, while also taking smaller and smaller neighborhoods about x_0 to get infinite regularity at x_0.

Now it suffices to show that otis' cutoff function has decreasing Allen-Cahn energy. First, note that $\chi_r(x)$ is a radial function, so if we label the radial coordinate as ρ, then

$$\nabla \chi_r = e_\rho \partial_\rho \chi_r$$

We can smoothen $\chi_r(x)$ at the points $|x| = r$ and $|x| = r^2$ so that the derivative is bounded by $2/(r \log r)$ and $2/(r^2 \log r)$ around those points. Moreover we can make the smoothening occur in an arbitrarily small annulus about these radii and so

$$||\nabla \chi_r||_2^2 \leq \frac{1}{\log(r)^2} \frac{\pi}{4} \int_{r^2}^{r} \frac{1}{y\rho^2}(\rho d\rho) + \epsilon = \frac{1}{\log(r)^2}[\log(y)]_{r^2}^{r} + \epsilon = \frac{-1}{\log(r)} + \epsilon$$

as $r \to 0$, the denominator tends to ∞ and so we can bound this above by ϵ for all $\epsilon > 0$, meaning that $||\nabla \chi_r||_2^2 \to 0$. For the Potential term, note that

$$\frac{\pi}{4} \int_0^R W(\chi_r) = \frac{\pi}{4} \int_{r^2}^{r} (1 - \chi_r(\rho)^2)^2 \rho d\rho + \epsilon \leq Cr(r - r^2) + \epsilon$$

where we bound the integrand by $1 \cdot r$. This also tends to 0, and so the claim is verified.

(d) The proof sketch is as follows

1. Use schauder estimates to get a uniform bound on $||u_R||_{C^{2,\alpha}}$ for $\alpha > 0$ The Schauder estimates requires some uniform bound in the holder norms of c and f, and these are u_R and hence R dependent, so a prior we don't have this. However, Otis gave me the following bound

$$||u||_{C^{1,\alpha}} \leq ||\Delta u||_{C^0}$$

   we know that the right hand side is bounded by 2 and so in particular, we have $||u||_{C^{0,\alpha}} \leq ||u||_{C^{1,\alpha}} \leq 2$. Now the original schauder estimate tells us that

$$||u||^*_{C^{2,\alpha}} \leq C(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega}$$

   the point is that we have bounded holder norms for f (even with the odd weightings) because we have bounded holder norms for u (holder spaces are closed under multiplication). OK even though the * denotes some distance based weighting, we can get nice $C^{2,\alpha}$ bounds for u_R on a set compactly contained inside Ω_R, which is all that we really need.

2. With the above bound, we can uniformily bound $||u_R||_{C^1}$ for all R. (actually can do this with the laplacian bound)

3. Arzela ascoli then gives us a subsequence of $\{u_{R_i,k}\}$ which converge uniformily on each B_{ρ_k}. Here i is variable when k is fixed. This is where the compactness comes in, we didn't need to bound the C^1 norm on the whole space, just on some compact set contained in the support of u_R, in which case we can convert from $||u||^*_{C^{2,\alpha}} \to ||u||_{C^{2,\alpha}}$.

4. Using a diagonal argument, we can get a subsequence which converges uniformily on each compact set. Formally, we take the sequence which converges on B_{ρ_1} and apply Arzela ascoli to that to get uniform convergence on B_{ρ_2}, and then repeat this for B_{ρ_3} and so on. Now taking a diagonal sequence of these collections of sequences gives a sequence which converges uniformly on each compact set.

5. Once we have the uniform bound in $||u||_{C^{2,\alpha}}$, we can actually apply arzela ascoli to $\partial_x u_{R_i}$ and $\partial_y u_{R_i}$ to get a sequence of functions for which these converge uniformly. First, choose a sequence so that $\{\partial_x u_i\}$ converges uniformily on each compact set via a diagonal sequence argument. Using this diagonal sequence as a starting sequence, repeat this to get a subsequence so that $\partial_y u_i$ also converges uniformily on each compact set. Now with this sequence, repeat again to get a further subsequence so that $\{u_{R_i}\}$ converge uniformily as well. Let $\tilde{u}$ be the limiting function. Then for each φ compactly supported in $\mathbb{R}^2$, there exists an R so that $B_R(0) \supseteq \text{supp}(\varphi)$. We can now choose i sufficiently large so that $\text{supp}(u_i) \supseteq B_R(0)$, so that

$$\int_{\mathbb{R}^2} \nabla\tilde{u}\cdot\nabla\varphi - \tilde{u}(\tilde{u}^2-1)\varphi = \int_{B_R(0)} \nabla\tilde{u}\cdot\nabla\varphi - \tilde{u}(\tilde{u}^2-1)\varphi$$

$$= \int_{B_R(0)} (\nabla\tilde{u} - \nabla u_i)\cdot\nabla\varphi - [\tilde{u}(\tilde{u}^2-1) - u_i(u_i^2-1)]\varphi + \int_{B_R(0)} \nabla u_i\cdot\nabla\varphi - u_i(u_i-1)\varphi$$

The latter integral will vanish because u_i is a minimizer on some B_{ρ_i} (we can extend this integral from $B_R(0)$ to B_{ρ_i} because $\rho_i > R$) and if we choose i sufficiently large then $||\nabla\tilde{u} - \nabla u||_{L^\infty(B_R)} < \epsilon/R^2$ by uniform convergence of the derivatives. Similarly, by uniform convergence of the functions, we can force

$$\left|\left|\tilde{u}(\tilde{u}^2-1) - u_i(u_i^2-1)\right|\right|_{L^\infty(B_R(0))} < \frac{\epsilon}{R^2}$$

making these integrals arbitrarily small. The point is that i is chosen after R is. This shows that $\tilde{u}$ solves the PDE locally everywhere. To see that $\tilde{u}(x,y) \neq 0$ for all (x,y) not on $\{xy = 0\}$, we can apply the strong minimum principle again, which says the set on which $\tilde{u}$ is 0 (note that $\tilde{u} > 0$ because we choose our $\{\tilde{u}_{R_i}\}$ to all be non-negative) is open. But by continuity this set must be closed as well. Applying this to each quadrant, and noting that each quadrant forms a connected set, we get that either $\{\tilde{u} = 0\}$ on all of a single quadrant or just at the nodal lines. Note that by the odd reflection construction of all of these soltuions, $\tilde{u}_R$ being 0 on one quadrant implies it for every other quadrant. However, this contradicts the linear growth in energy. I'm not sure about this: uniform convergence on compact sets tells us that for fixed R, $u_i \to 0$ on B_R as $i \to \infty$, which doesn't contradict the linear growth in energy, as if u_i is supported on B_{ρ_i} then ρ_i may be huge so that $CR^2 \leq K\rho_i$, i.e. it doesn't seem like there's a way to compare R to ρ_i.

I've spent too much time on this problem, but for posterity, here's Otis' work around: "There's a technical issue with applying monotonicity that I am glad to explain, but I don't think it works directly (I made this mistake since for minimal surfaces, this would be OK, but there's a funny feature with the Allen—Cahn monotonicity that's not present for minimal surfaces and causes a headache).

Here is an alternative approach to finishing this part that does the trick, I think:

(a) The functions u_R have the property that for any ball B contained in the first quadrant and if R is large enough so that $B \subset \Omega_R$, then u_R minimize on B. Namely, if w has the same value on ∂B then $E(w; B) \geq E(u_R; B)$. This is clear, since replacing u_R by w inside of B does not change the boundary conditions and would be admissible in the minimization problem.

(b) This property passes to the limit: for any ball B in the first quadrant, the limit of u_R, u, minimizes $E(\cdot)$ on B.

(c) The function $u = 0$ does not minimize $E(\cdot)$ on large B.

You already have given the argument for 3 (just cut off from 0 on ∂B to -1 on a smaller ball; this has less energy as long as B is large). The fact 2 seems obvious until you start to prove it, and then it seems wrong. But it's actually true. The reason it will seem wrong is that if u was not minimizing on B, there is some function w so that $w = u$ on ∂B and $E(w; B) \leq E(u; B) - \delta$ for some $\delta > 0$.

Now you want to glue w into u_R and decrease the energy. But, u_R and w don't exactly agree on ∂B (they're just close). However, you can still make it work because "w drops energy by a definite (independent of R amount)". Namely, if you choose a bump function ϕ that's supported in B, then

$$(1-\phi)u_R + \phi w$$

Is smooth. Moreover, as $R \to \infty$, u_R limits to u, so this will limit to $(1-\phi)u + \phi w$. You can check that as ϕ limits to χ_B, this function limits in H^1 to the function that's u outside of B and w inside of B (this uses the fact that the values of the functions agree at ∂B; otherwise this function could not be in H^1). So, by balancing ϕ and R carefully, we can arrange that

$$E((1-\phi)u_R + \phi w; \Omega_R) = E(u_R; \Omega_R \setminus B) + E(w; B) + o(1)$$

As $R \to \infty$. (This takes some checking and its a bit annoying but it's worth trying if you don't see why it's true). But, then you get

$$\begin{aligned}
E(u_R; \Omega_R) &\leq E((1-\phi)u_R + \phi w; \Omega_R) \qquad \text{since } u_R \text{ minimizes} \\
&= E(u_R; \Omega_R \setminus B) + E(w; B) + o(1) \qquad \text{(by the above construction/claim)} \\
&\leq E(u_R; \Omega_R \setminus B) + E(u; B) - \delta + o(1) \qquad \text{(by the property assumed about w)} \\
&= E(u_R; \Omega_R \setminus B) + E(u_R; B) - \delta + o(1) \qquad \text{(since } u_R \to u \text{ in } C^\infty \text{ on } B). \\
&= E(u_R; \Omega_R) - \delta + o(1) \qquad \text{(putting the pieces back together).}
\end{aligned}$$

This is then a contradiction since $\delta \leq o(1)$ can't hold. Sorry about my confusion! I guess strictly speaking, we did not really need to prove the $E \leq CR$ bound proved in the earlier part of the problem, since we have to use the minimizing property again after passing $R \to \infty$ anyways...

1.8 Exercise 3.2

The idea should be that any perturbation of the minimizer will decrease the energy. Now just need to balance the L^1's vs. the H^1's. Yes this works out: Suppose that u_ϵ is a δ-minimizer, (Not sure what the distinction between strict and non-strict minimizer is - Otis define's one but asks about the other). Then consider

$$E_\epsilon(u_\epsilon + t\varphi; \Omega)$$

where φ is some smooth compactly supported function. For fixed φ, we know that $||t\varphi|| < \delta$ for t sufficiently small, and thus by the minimizer assumption, we have that

$$E_\epsilon(u_\epsilon) \leq E_\epsilon(u_\epsilon + t\varphi; \Omega) \qquad \forall t \text{ sufficiently small}$$

noting that the energy functional is smooth in t, we immediately get that the derivative must equal 0, as a non-zero derivative would allow us to decrease the energy by taking t small and positive/negative, a contradiction. Now because it minimizes energy locally, it must solve the Allen Cahn equation

1.9 Exercise 3.3

Let r be tbd, and define v_ϵ to be a minimizer of E_ϵ over $B_r(\ell)$ ($\ell := \chi_E - \chi_{M \setminus E}$) where the ball is taken w.r.t. the L^1 norm. Note that such a minimizer exists for each individual $\epsilon > 0$ because of the same argument from exercise 2.5/Evans Chapter 8.2. The key thing to notice is that a sequence of of $v_{k,\epsilon}$ such that $E_\epsilon(v_{k,\epsilon}) \to M_\epsilon = \min_{f \in B_r(\ell)} E_\epsilon(f)$ is necessarily bounded uniformly in H^1, which is a hilbert space, and so we get a weakly convergent subsequence and the rest is history.

Let $\{v_\epsilon\}$ be our sequence of minimizers, then we know that for $\epsilon < \epsilon_r$ fixed we can find a family of $\{u_\epsilon\}$ in accordance with proposition 3.6 such that $u_\epsilon \to \ell$ and $\lim_\epsilon E_\epsilon(u_\epsilon; M) = \sigma P(E; \Omega)$. In particular, this forces $E_\epsilon(v_\epsilon) \leq E_\epsilon(u_\epsilon)$ for $\epsilon < \epsilon_r$ and so

$$\limsup_{\epsilon \to 0} E_\epsilon(v_\epsilon) \leq \sigma P(E; M)$$

In particular, this tells us that $E_\epsilon(v_\epsilon) \leq C$ independent of ϵ for ϵ sufficiently small. Now apply proposition 3.5 to get a subsequence which converges to v_0 which is ± 1 a.e. and

$$\liminf_{k \to \infty} E_{\epsilon_k}(v_{\epsilon_k}; \Omega') \geq \sigma P(\{v_0 = 1\}; \Omega')$$

for any $\Omega' \subset\subseteq M = \Omega$. Now using the definition of strict local minimizer, we have that for $\Omega' = \Omega = M$ I can do this because compact containment doesn't imply strict containment, we have that if $\{v_0 = 1\} \Delta E$ is positive measure, then

$$P(E; M) < P(\{v_0 = 1\}; M)$$

if we choose $r = \delta$ in accordance with the definition of strict local minimizer (see definition 3.9). However, we know that

$$\limsup_{\epsilon \to 0} E_\epsilon(v_\epsilon) \leq P(E; M)$$

so the lim inf bound cannot occur, meaning that $\{v_0 = 1\}\Delta E$ is 0 measure. Thus, $\{v_{\epsilon_k}\}$ converges to $\chi_E - \chi_{M\setminus E}$ in L^1. In particular, for k sufficiently large, we know that v_{ϵ_k} lies in the interior of $B_\delta(\ell)$, so that it is necessarily a local minimizer, and hence a solution of Allen Cahn (exercise 3.2)

From here, we just need to pass from the subsequence to the overall continuum of functions $\{u_\epsilon\}$.

I will show that $\{v_\epsilon\} \to \chi_E - \chi_{M\setminus E}$, which will imply that for all ϵ sufficiently small, we have that v_ϵ solve Allen Cahn as they are local minimizers. Suppose not, then we get $\gamma > 0$ and a subsequence, v_{ϵ_k} such that $\epsilon_k \to 0$ and $||v_{\epsilon_k} - \ell|| \geq \gamma$. From here, we can extract a subsequence in accordance with 3.5 such that (after relabelling the subsequence as the sequence) $u_{\epsilon_k} \to u_0$ and

$$\sigma P(\{v_0 = 1\}; M) \leq \liminf_{k\to\infty} E_{\epsilon_k}(v_{\epsilon_k}; M)$$

but if we write $J = \chi_{v_0=1} - \chi_{M\setminus\{v_0=1\}}$ then because $||v_{\epsilon_k} - \ell|| \geq \delta > 0$ then we'll have $\delta \geq ||J - \ell||_1 \geq \gamma > 0$ so that by strict minimization we have

$$\sigma P(E; M) < \sigma P(\{v_0 = 1\}; M) \leq \liminf_{k\to\infty} E_{\epsilon_k}(v_{\epsilon_k}; M) \leq \liminf_{k\to\infty} E_{\epsilon_k}(u_{\epsilon_k}; M) = \sigma P(E; \Omega)$$

where $\{u_\epsilon\}$ is the continuum of functions we constructed with 3.6. This gives a contradiction, and really I don't think I needed the previous paragraph of arguments.

1.10 Exercise 4.1

1. One class of functions for which this works is constant functions. Note that $Ric_g(\nu,\nu) = (n-1)g(\nu,\nu) = (n-1)$ because ν will always equal e_n and so if the metric is that inherited from $\mathbb{R}^{n+1}$, then this works. Note that $|A_\Sigma|^2 = |A_{S^{n-2}}|^2$ is some positive constant in terms of π and gamma functions. Thus for $\varphi \equiv c$, we get that $J\varphi = -(|A_\Sigma|^2 + Ric_g(\nu,\nu))c$ which implies that $\varphi J\varphi M0$ at every point. To get that this is the only subspace
2. Potentially useful source? here
3. Also here for Ricci tensor of sphere blah

Via a computation, the second fundamental form is actually 0 and so the mean curvature, which is the trace of the second fundamental form, is also zero. Moreover, the Ricci curvature tensor of an n sphere is just $n-1$ times the metric, and so

$$J\varphi = -\Delta_\Sigma\varphi - (|A_\Sigma|^2 + (n-1)\cdot 1)\varphi = (-\Delta_\Sigma - (n-1))\varphi = L\varphi$$

because $g(N,N) = 1$. Now we note that the rayleigh quotient is the largest eigenvalues of this operator. Moreover, there is a one to one correspondence between eigenvalues and vectors of L and $\Delta_{S^{n-1}}$. See here for the characterization but the values are

$$\lambda_k = k(k+n-1) \qquad k \in \mathbb{Z}^+ \cup \{0\}$$

In particular, the smallest eigenvalue of $-\Delta_{S^{n-1}}$ is 0, meaning that the smallest the largest eigenvalue of $\Delta_\Sigma + (n-1)$ is $n-1$, and the second largest is $-n + (n-1) = -1$.

One can check that the first eigenvalue for both L and $-\Delta_{S^{n-1}}$ is attained by constant functions. Now note that

$$\lambda_2 = \inf_{\phi\perp 1} \frac{Q_\Sigma(\varphi,\varphi)}{||\varphi||_2}$$

This can be seen via a variational argument, e.g.

$$\frac{d}{dt}\frac{Q_\Sigma(\varphi + t\phi, \varphi + t\phi)}{||\varphi + t\phi||_2} = 0 \qquad \phi \perp 1$$

Also here, $\varphi \perp 1$ in the sense that $\langle\varphi, 1\rangle_{S^{n-1}} = 0$, which means that the average of the function restricted to S^{n-1} is 0.

With this set up at hand, suppose that $Q_\Sigma(\varphi,\varphi) < 0$ for some non-constant φ. After subtracting off its projection onto constant functions, we see that $\varphi - c \perp 1$ and so in particular

$$\frac{Q_\Sigma(\varphi - c, \varphi - c)}{||\varphi - c||_2} > \lambda_2 \implies \int (-\Delta(\varphi - c))(\varphi - c) - (n-1)(\varphi - c)^2 > \lambda_2 \int (\varphi - c)^2$$

If the second eigenvalue for $-\Delta$ is n, then the second eigenvalue for $-\Delta-(n-1)$ is 1, and so

$$\int(-\Delta(\varphi-c))(\varphi-c)-(n-1)(\varphi-c)^2>\lambda_2\int(\varphi-c)^2=\int(\varphi-c)^2\geq 0$$

meaning that Q_Σ is not negative definite on the space spanned by $\{\varphi,1\}$. Thus the morse index is 1.

After talking with Otis, this proof isn't technically correct because the questions asks to show that we can't have a subspace of dimension ≥ 2 on which the form $Q_\Sigma(\cdot,\cdot)$ is negative definite. Note that if we choose any φ non constant for which $Q_\Sigma(\varphi,\varphi)<0$, then the same will hold for any non-zero rescaling. The point is that 1 doesn't have to be in our subspace. HOWEVER, the proof from above shows that if $\langle\varphi,1\rangle=c\int_\Sigma\varphi=0$, then the form cannot be negative. Thus, suppose that our space contains φ and ϕ linearly independent, i.e. one is not a scalar multiple of each other. Then we have one of the following cases: $\int\varphi\neq 0$ and $\int\phi\neq 0$, OR one of these integrals is non-zero. The latter case gives a contradiction as above. In the former case, set $c=\int\varphi$ and $d=\int\phi$, then consider $f=-d\varphi+c\phi$ which will have zero integral. Thus the problem statement is still true

1.11 Exercise 4.2

I think we're supposed to consider $Q_{u_\epsilon}(\psi\sqrt{|\nabla u_\epsilon|^2+\delta^2})$, which gives

$$Q_{u_\epsilon}(\psi\sqrt{|\nabla u_\epsilon|^2+\delta^2})=\int_M\epsilon|\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2}\psi)|^2+\frac{1}{\epsilon}W''(u_\epsilon)\psi^2(|\nabla u_\epsilon|^2+\delta^2)$$

We know that $\sqrt{|\nabla u_\epsilon|^2+\delta^2}$ is smooth because any derivative of it will involve derivatives of u_ϵ and have a denominator of $(|\nabla u_\epsilon|^2+\delta^2)^{k/2}$ which will be non-zero. With this, we expand

$$\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2}\psi)=\psi\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2})+(\sqrt{|\nabla u_\epsilon|^2+\delta^2})\nabla\psi$$

and so noting that $\nabla f^2=2f\nabla f$ and factoring in the cross terms, we get

$$\begin{aligned}|\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2}\psi)|^2&=|\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2})|^2\psi^2+\frac{1}{2}g(\nabla\psi^2,\nabla(|\nabla u_\epsilon|^2+\delta^2))+|\nabla\psi|^2(|\nabla u_\epsilon|^2+\delta^2)\\&=|\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2})|^2\psi^2+\frac{1}{2}g(\nabla\psi^2,\nabla(|\nabla u_\epsilon|^2))+|\nabla\psi|^2(|\nabla u_\epsilon|^2+\delta^2)\\&=|\nabla(\sqrt{|\nabla u_\epsilon|^2+\delta^2})|^2\psi^2-\frac{1}{2}\psi^2(\Delta_g(|\nabla u_\epsilon|^2)+|\nabla\psi|^2(|\nabla u_\epsilon|^2+\delta^2)\end{aligned}$$

note that though $|\nabla u_\epsilon|$ may not be smooth $|\nabla u_\epsilon|^2=\nabla u_\epsilon\cdot\nabla u_\epsilon$ is so we can pull this integration by parts. Adding back in the scaling and applying the Bochner formula again to get

$$\begin{aligned}&\epsilon\int_M|\nabla\psi|^2(|\nabla u_\epsilon|^2+\delta^2)-((|D^2u_\epsilon|^2-|\nabla\sqrt{|\nabla u_\epsilon|^2+\delta^2}|^2)+\mathrm{Ric}_g(\nabla u_\epsilon,\nabla u_\epsilon)+\frac{1}{\epsilon}W''(u_\epsilon)\delta)\psi^2d\mu_g\geq 0\\ \implies I:=&\int_M|\nabla\psi|^2|\nabla u_\epsilon|^2-((|D^2u_\epsilon|^2-|\nabla\sqrt{|\nabla u_\epsilon|^2+\delta^2}|^2)+\mathrm{Ric}_g(\nabla u_\epsilon,\nabla u_\epsilon))\psi^2d\mu_g\geq\int\epsilon\delta^2|\nabla\psi|^2-W''(u_\epsilon)\delta\psi^2\end{aligned}$$

noting that

$$\nabla\sqrt{|\nabla u_\epsilon|^2+\delta^2}|^2=\frac{\nabla(|\nabla u_\epsilon|^2+\delta^2)}{2\sqrt{|\nabla u_\epsilon|^2+\delta^2}}=\frac{\nabla(|\nabla u_\epsilon|^2)}{2\sqrt{|\nabla u_\epsilon|^2+\delta^2}}$$

setting $\vec{v}=\nabla u_\epsilon$, we have

$$\partial_i(|\nabla u_\epsilon|^2)=\partial_i g(\vec{v},\vec{v})=\partial_i(g_{jk}v_jv_k)=(\partial_i g_{jk})v_jv_k+2g(\partial_i\vec{v},\vec{v})$$

Note that if $\vec{v}=\vec{0}$, then all of these terms vanish. Moreover, because $|D^2u_\epsilon|^2>0$, we have

$$\int_{\nabla u_\epsilon\neq 0}-|D^2u_\epsilon|^2\geq\int_M-|D^2u_\epsilon|^2$$

Thus we have

$$\int_{\nabla u_\epsilon\neq 0}|\nabla\psi|^2|\nabla u_\epsilon|^2-((|D^2u_\epsilon|^2-|\nabla\sqrt{|\nabla u_\epsilon|^2+\delta^2}|^2)+\mathrm{Ric}_g(\nabla u_\epsilon,\nabla u_\epsilon))\psi^2d\mu_g\geq I\geq\int_m\epsilon\delta^2|\nabla\psi|^2-W''(u_\epsilon)\delta\psi^2$$

sending $\delta \to 0$ on the right gives a lower bound of 0. Note that when $\nabla u_\epsilon \neq 0$, we have

$$\lim_{\delta \to 0} \frac{\nabla(|\nabla u_\epsilon|^2)}{2\sqrt{|\nabla u_\epsilon|^2 + \delta^2}} = \frac{\nabla(|\nabla u_\epsilon|^2)}{2|\nabla u_\epsilon|} = \frac{2\nabla(|\nabla u_\epsilon|) \cdot |\nabla u_\epsilon|}{2|\nabla u_\epsilon|} = \nabla(|\nabla u_\epsilon|)$$

To get this in an integral formula, we have

$$|D^2 u_\epsilon|^2 = |D^2(u_\epsilon + \delta)|^2 \geq f_\delta = |D^2(u_\epsilon + \delta)|^2 - |\nabla \sqrt{|\nabla u_\epsilon|^2 + \delta^2}|^2 \geq 0$$

by the remark after the exercise. We know u_ϵ is smooth so if M is compact, we've bound f_δ between two integrable things and we can applied dominated convergence to get that

$$-\int |D^2(u_\epsilon + \delta)|^2 - |\nabla \sqrt{|\nabla u_\epsilon|^2 + \delta^2}|^2 \to -|D^2(u_\epsilon + \delta)|^2 - |\nabla |\nabla u_\epsilon||^2$$

If M is not compact, then we can use Fatou's lemma.

1.12 Exercise 4.3

(a) Yes, stability will give us that $Q_\Sigma(\varphi, \varphi) \geq 0$ for all $\varphi \in C^\infty(\Sigma)$. Note that

$$Q_\sigma(\varphi, \varphi) = \int \varphi(-\Delta_\Sigma \varphi - (|A_\Sigma|^2 + \mathrm{Ric}_g(\nu, \nu))\varphi) \geq 0 \implies \int g(\nabla \varphi, \nabla \varphi) \geq \int \mathrm{Ric}_g(\nu, \nu)\varphi^2$$

In particular, we know that $\mathrm{Ric}_g(\nu, \nu) > 0$ for all ν non-zero because the tensor is positive. However, the above is not true for $\varphi \equiv 1$ as $\int g(\nabla \varphi, \nabla \varphi) = 0$, but the right hand side is some positive value. Thus there can be no stable solutions.

(b) Similar idea, apply proposition 4.1, and note that this is equivalent to

$$\int_{\nabla u_\epsilon \neq 0} |\nabla \psi|^2 |\nabla u_\epsilon|^2 \geq \int ((|D^2 u_\epsilon|^2 - |\nabla |\nabla u_\epsilon||^2) + \mathrm{Ric}_g(\nabla u_\epsilon, \nabla u_\epsilon))\psi^2$$

In particular the right hand side is non-zero and strictly positive if we choose $\psi \equiv 1$, however the left hand side will always be 0 because $\nabla \psi = 0$.

1.13 Exercise 4.4

Look here and here to start

`http://mathonline.wikidot.com/every-bounded-sequence-in-a-hilbert-space-has-a-weakly-conve`
`http://mathonline.wikidot.com/helley-s-theorem`

So we know that by nature of $H^1(M)$ being a hilbert space, we can find weak limits. In particular, we can take $\{u_k\}$ and replace it with a subsequence which converges weakly in H^1 to some u. However, we can also take $\{u_k\} \subseteq L^2$ and consider a weak limit, $f \in L^2$. We can then show that f must be weakly differentiable and so $f \in H^1$, which then implies that $f = u$. The point of doing this is that weak convergence in H^1 gives us convergence w.r.t. $\langle a, b \rangle_{H^1} = \langle a, b \rangle_{L^2} + \langle \nabla a, \nabla b \rangle_{L^2}$, but this doesn't tell us if a act the same as elements of $(L^2)^*$. The point of passing to a weak L^2-limit and then showing that it matches with the weak H^1 limit gives us that individually: $\langle u, \varphi \rangle_{L^2} = \lim_{k\to\infty} \langle u_k, \varphi \rangle_{L^2}$ as well as $\langle \nabla u, \nabla \varphi \rangle_{L^2} = \lim_{k\to\infty} \langle \nabla u, \nabla \varphi \rangle_{L^2}$. This is important, because the energy functionals deal with weighted versions of each. See here

`https://math.stackexchange.com/questions/24776/weak-convergence-in-sobolev-spaces`

With this, we have that

$$\lim_{k\to\infty} \int \frac{\epsilon}{2} \langle \nabla u_k, \nabla \varphi \rangle + \frac{1}{\epsilon} W'(u_k)\varphi \xrightarrow{k\to\infty} \int \frac{\epsilon}{2} \langle \nabla u, \nabla \varphi \rangle + \frac{1}{\epsilon} W'(u)\varphi$$

splitting up this integral into two, the fact that the first integral converges is straight from $\nabla u_k \rightharpoonup \nabla u$. The second integral converges because we have $|u_k| \leq 1$ everywhere and so $|u| \leq 1$ a.e. meaning that both are bounded. We're also on a compact set which has finite measure, and so applying Rellich-Kondrachov, we get that $W^{1,2} \hookrightarrow W^{0,2} = L^2$, where this embedding is **compact**. In particular, this tells us that our weakly convergent sequence, which is necessarily bounded in $||\cdot||_{H^1}$ is then sent to a sequence in L^2, which has a strongly convergent subsequence. As showed previously, the limit of this strongly convergent subsubsequence in L^2 must actually be u. With this, we've gained that $u_k \to_{L^2} u$, and so because our space is finite measure, we can use Holder's inequality and boundedness of $\{u_k\}$ to get that $W'(u_K) \xrightarrow{L^2} W'(u)$ meaning that the second integral will converge! Now we can use the fact that $DE\Big|_{u_k} \to 0$ to get that u satisfies Allen Cahn weakly, and hence by problem 1 (recall that u is bounded), it will be a strong solution. To see this, I interpret

$$DE\Big|_{u_k} = I'_\epsilon(u_k) = v_k \text{ s.t. } \forall w, \qquad I_\epsilon(w) = I_\epsilon(u_k) + \langle v_k, w - u_k\rangle + o(||w - u_k||)$$

in particular, if we set $w = u_k + t\varphi$ for some φ smooth, then we get that

$$I(u_k + t\varphi) = I(u_k) + t\langle v_k, \varphi\rangle + o(t||\varphi||)$$

taking the derivative of this expression gives

$$\frac{d}{dt}I(u_k + t\varphi)\Big|_{t=0} = \langle v_k, \varphi\rangle$$

Now consider

$$\frac{d}{dt}I(u + t\varphi)\Big|_{t=0} = \int_\Omega \epsilon\langle\nabla u, \nabla\varphi\rangle + \frac{1}{\epsilon}W'(u)\varphi$$

note that u is the weak limit in H^1 of our subsequence and the strong L^2 limit of our sequence, so each of the integrals above converge individually. In particular

$$\int_\Omega \epsilon\langle\nabla u, \nabla\varphi\rangle + \frac{1}{\epsilon}W'(u)\varphi = \epsilon\lim_{k\to\infty}\int_\Omega\langle\nabla u_k, \nabla\varphi\rangle + \frac{1}{\epsilon}\lim_{k\to\infty}\int_\Omega W'(u_k)\varphi = \lim_{k\to\infty}\langle v_k, \varphi\rangle = 0$$

and so $I'(u) = 0$ by our assumption on $DE\Big|_{u_k}$. Thus u is a solution to Allen Cahn weakly because the left hand side will give

$$\frac{d}{dt}I(u + t\varphi)\Big|_{t=0} = \int \epsilon\langle\nabla u, \nabla\varphi\rangle + \frac{1}{\epsilon}W'(u)\varphi$$

and so the above being 0 for all φ tells us that u is a weak solution of Allen Cahn. From what we cited, it is then a strong solution.

To get strong convergence in H^1, we already have strong convergence in L^2. We have up to first order in $||u_k - u||_{L^2}$ that

$$\frac{\epsilon}{2}\langle\nabla(u_k - u), \nabla(u_k - u)\rangle \equiv \frac{\epsilon}{2}\langle\nabla u_k, u_k - u\rangle - \frac{\epsilon}{2}\langle\nabla u, u_k - u\rangle$$
$$= \langle v_k, u_k - u\rangle - \frac{1}{\epsilon}[\int_M [W'(u_k) - W'(u)](u_k - u)] - \frac{\epsilon}{2}\langle\nabla u, \nabla(u_k - u)\rangle - \int_M \frac{1}{\epsilon}W'(u)(u_k - u)$$
$$= \langle v_k, u_k - u\rangle - \frac{1}{\epsilon}[\int_M [W'(u_k) - W'(u)](u_k - u)]$$

the point is that we can eliminate the last two terms in the second line because u is a solution to Allen-Cahn. The first term tends to 0 as $k \to \infty$ because $v_k \to 0$ and $u_k - u \to 0$ in L^2 This double convergence feels too easy It is easy, but Otis says it's right. However, by holder's inequality and boundedness of u and u', the second term as decreases to 0 because $||u_k - u||_{L^2} \to 0$. With this we have that our sequence actually converges strongly in H^1 to u. Ask Otis about the notation of $DE_\epsilon\Big|_{u_k}$ Yes, the notation is as I've described

1.14 Exercise 5.1 (Mostly done)

(a) Define

$$g^j(x) = \epsilon^{-2}f^*_{\epsilon_j,x_j}g = \sum_{i,k} g_{ik}(\exp_{x_j}(\epsilon_j x)dx^i \otimes dx^k, \qquad g_{ik} : B_K(0) \to \mathbb{R}, \qquad f_{\epsilon_j,x_j} : B_K(0) \to B_{K\epsilon_j}(x_j)$$

the point is that we've pulled the metric back by the map f_{ϵ_j, x_j} which is a chart map sending $B_{K\epsilon_j}(x_j) \to B_K(0)$, but we want to rescale by ϵ^{-2}, as pulling back gives two extra powers of ϵ, and sending $\epsilon \to 0$ would give the 0 metric without the rescaling. We don't need to rescale the function though, so similarly, define

$$\tilde{u}_j = u_{\epsilon_j}(\exp_{x_j}(\epsilon_j x)) = f^*_{\epsilon_j, x_j}(u)(x)$$

(b) To see convergence of the metric, note that for any derivative $\partial_x^\alpha g^j(x) = \epsilon_j^{|\alpha|}(\partial_x^\alpha g \circ \exp_{x_j})(\epsilon_j x)$. Take a subsequence of the $\{x_j\}$ so that these points converge to some $x_0 \in M$. We know that higher derivatives of the original metric (assuming it is smooth itself) are continuous. In particular, as we send $j \to \infty$, $(\partial_x^\alpha g \circ \exp_{x_j})(\epsilon_j x)$ tends to $(\partial_x^\alpha g \circ \exp_{x_0})(0)$ and the factor of $\epsilon_j^{|\alpha|}$ makes it so that

$$\forall |\alpha| \geq 1, \;\; \forall x \in B_K(0), \qquad \lim_{j\to\infty} \partial_x^\alpha g^j(x) = \lim_{j\to\infty} \epsilon_j^{|\alpha|}(\partial_x^\alpha g \circ \exp_{x_j})(\epsilon_j x) = 0$$

Moreover, this convergence is uniform because we have control of $(\partial_x^\alpha g \circ \exp_{x_j})$ (really we have control of the metric everywhere, because we're on a compact domain) and in fact we have a uniform bound on this taking x over all $B_K(0)$ assuming that x_j is sufficiently close to x_0. With this, note that for $\alpha = 0$, we get that

$$\forall x \in B_K(0), \qquad \lim_{j\to\infty} g_j(x) = g(x_0)$$

and this is similarly a uniform convergence once we take j sufficiently large and note continuity of the metric. So the fact that the metric converges is fine because yes we have control of it on the a compact domain, which is all we need as $\{B_{K\epsilon_j}(x_j)\}$ will be contained in some U in our manifold for j sufficiently large (this is assuming we have the subsequence so that $\{x_j\} \to x_0$). However, something we can't take for granted is $\exp_{x_j} \to \exp_{x_0}$ however this is true because the exponential map is the time 1-value of the geodesic flow, which is a map $\Phi^t : TM \to TM$. We can solve this locally in coordinates and think of it as a map on $\mathbb{R}^{2n}$ smooth in both the first n coordinates (the base point) and the second n coordinates (the initial velocity vector). The flow is then smooth and so we get $\exp_{x_j} = \Phi^1(x_j, \cdot) \to \Phi^1(x_0, \cdot)$. See here for details

For our function $\tilde{u}_j$, we know that each is a solution to Allen-Cahn with $\epsilon = 1$, as

$$\Delta \tilde{u}_j = \epsilon_j^2 (\Delta u_{\epsilon_j} \circ \exp_{x_j})(\epsilon_j x) = W'(\tilde{u}_j)$$

Now each $\tilde{u}_j$ is bounded in magnitude by ± 1, so because $\epsilon = 1$ for all j, we can relate the Allen-Cahn energy functional to the H^1 norm on $B_K(0)$. We already have a nice $L^2(B_K(0))$ norm by nature of $\tilde{u}_j$ being L^∞. Now we just need to get a bound on $||\nabla \tilde{u}_j||$. The idea for this is to use the fact that $\tilde{u}_j$ and u_{ϵ_j} are related closely, despite their domains of definition being different. We have

$$\int_{B_K(0)} |\nabla \tilde{u}_j|^2 = \int_{B_K(0)} \epsilon_j^2 |(\nabla u_{\epsilon_j} \circ \exp_{x_j})(\epsilon_j x)|^2 dx$$

$$= \epsilon^{2-n} \int_{B_{K\epsilon_j}(0)} |(\nabla u_{\epsilon_j} \circ \exp_{x_j})(y)|^2 dy \leq \epsilon_j^{2-n} ||\mathbb{1}_{B_{K\epsilon}(0)}||_p \;\; || \, |\nabla (u_{\epsilon_j} \circ \exp_{x_j})| \, ||_q$$

here, set $p = n/(n-2)$ and $q = 2/n$, so that

$$\epsilon_j^{2-n} ||\mathbb{1}_{B_{K\epsilon}(0)}||_p \leq \epsilon_j^{2-n} (CK\epsilon_j^n)^{(n-2)/n} = \tilde{C}(K)$$

where $\tilde{C}$ is independent of ϵ_j because we have the right powers cancelling out. For the other integral, note that

$$||u||_{C^{1,\alpha},\Omega} \leq C_1 ||u||_{C^0,\Omega} \qquad \Omega \subseteq \mathbb{R}^n$$

this is some schauder estimate in our specialized case see Otis' emails. But the point is that $||u_{\epsilon_j}||_{L^\infty} \leq 1$ and all j, and so $||\nabla u_{\epsilon_j}|| \leq ||u_{\epsilon_j}||_{C^1} \leq C_1$ is bounded. We can do this for any chart on M, and then take finitely many charts to cover M, giving a global bound for $|\nabla u_{\epsilon_j}|$ and hence any of its higher power integrands. With this, we get H^1 boundedness of $\{u_{\epsilon_j}\}$.

Once boundedness is resolved, take a weak subsequence of $\{\tilde{u}_j\}$ which will strongly converge to some u in L^2 and weakly to it in H^1. From this, we can define weak derivatives of u as (set $\{u_k\} = \{\tilde{u_j}\}$)

$$\int (\partial_x^\alpha u)\varphi = (-1)^{|\alpha|} \int u \partial_x^\alpha \varphi = (-1)^{|\alpha|} \lim_{k\to\infty} \int u_k \partial_x^\alpha \varphi = \lim_{k\to\infty} \int (\partial_x^\alpha u_k)\varphi$$

here, of course we take φ with support compactly contained in our domain of integration, $B_K(0)$. We now know that u is a weak solution of allen cahn as

$$\int_{B_k(0)} (\Delta u + W'(u))\varphi = \lim_{k\to\infty} \int_{B_K(0)} \Delta u_k \varphi + \lim_{k\to\infty} \int_{B_K(0)} W'(u_k)\varphi = \lim_{k} \int_{B_K(0)} (\Delta u_k + W'(u_k))\varphi = 0$$

because the integral vanishes for each k. Now we can use the fact that weak solutions of Allen Cahn lying in H^1 are strong, smooth solutions of it. Note that u is in H^1 a priori because norms don't increase under weak limits. Moreover the derivatives of u_k convergence pointwise to the derivatives of u via the weak formulation two equations above, i.e. we get convergence of the weak derivatives easily - then noting that u and $\{u_k\}$ are all smooth, then their strong derivatives must also agree. Moreover, this convergence is locally uniform in k, as if we restrict to some $U \subseteq B_K(0)$, then if we take φ with supp$\varphi \subseteq U$, we get

$$\int_{B_K(0)} (\partial_x^\alpha u_k)\varphi = (-1)^{|\alpha|} \int_U u_k D_x^\alpha \varphi \implies \Big| \int_U (u_k - u_j) D_x^\alpha \varphi \Big| \leq ||\varphi||_{C^{|\alpha|}} \mu(U)^{1/2} ||u_k - u_j||_2$$

and we know that the sequence converges strongly in L^2. This gives a uniform local convergence depending only on $||u_k - u_j||$. Not sure what "smoothly converges" means. Also I want the above computation to communicate some sort of uniformity, but there is dependence on the derivatives of φ, which is something I morally should be able to divide out by. I think converges smoothly means convergence in the C^k seminorms for all k as $j \to \infty$. The seminorms don't have to all go to 0 at the same rate, just need to guarantee that this convergence does happen Not sure how to get smooth convergence actually. Weak convergence in all derivatives doesn't work if we consider a smoothed out version of $\mathbb{1}_{[0,1/n]}$ on say the circle For smooth convergence: idea is to apply schauder estimates and use that $\tilde{u}_j \in [-1,1]$ to get that $||\tilde{u}_j||_{C^{1,\alpha}} \leq C$ uniform in j. Now apply arzela ascoli. Interesting to think of schauder estimates as a means to bootstrap To flesh this out more: we have a sequence of $\{u_j\}$ from which we've extracted C^1 convergence on compact sets. If we look at $\{\partial_k u_j\}$ which are solutions to

$$\Delta \partial_k u = \partial_k(u(u^2-1)) = (3u^2 - 1)\partial_k u = -c(x)\partial_k u$$

this can be formulated as a semilinear elliptic PDE which u_j satisfies (the coefficients $c(x)$, will be defined in terms of u as before), from which we can get schauder estimates. In particular, using $||u||_{C^{2,\alpha}} \leq ||u||_{C^\alpha} + ||f||_{C^\alpha}$ (we take $f \equiv 0$) and then $||u||_{C^{1,\alpha}} \leq ||u||_{C^0}$, we can replace u with $\partial_j u$ and get bounds everywhere. Now we have a sequence $\{\partial_k u_j\}$ with uniformly bounded derivatives, giving us a subsequence which converges uniformily. This tells us that our subsubsequence converges in the C^0 norm AND in the C^1 norm by looking at the subsequence of derivatives. We can repeat this with higher derivatives as well, e.g.

$$\Delta(\partial_j \partial_k u) = \partial_j \partial_k [u(u^2-1)] = 6u\partial_j u \partial_k u + (3u^2 - 1)\partial_j \partial_k u = c(x)\partial_j\partial_k u + f(x)$$

so the same $C^{2,\alpha}$ and $C^{1,\alpha}$ schauder estimates (with $f = 6u\partial_j u \partial_k u$ which still has bounded C^α norms on compact sets because of the $C^{1,\alpha}$ norm bound) apply and we can get another subsequence which gives convergence of the second derivatives with respect to the C^0 norm on compact sets..

(c) Having established smooth convergence of $\{\tilde{u}_j\}$ to some u, then note that

$$Q_u(\psi,\psi) = \int_{B_K(0)} |\nabla \psi|^2 + W''(u)\psi^2 d\mu_g = \lim_{j\to\infty} \int_{B_K(0)} |\nabla\psi|^2 + W''(\tilde{u}_j)\psi^2 d\mu_g$$

Make the change of variables $v = \epsilon_j x$ so that the above integral becomes

$$\epsilon_j^{-n} \int_{B_{K\epsilon_j}(0)} |\nabla\varphi|^2(v)\epsilon_j^2 + W''(u_{\epsilon_j} \circ \exp_{x_j}(v))\varphi(v)^2 d\mu_g(v)$$

where $\varphi(x) = \psi(x\epsilon_j^{-1})$. The point here is that even though we have this scaling factor out front, we get that

$$\epsilon_j^{-n} \int_{B_{K\epsilon_j}(0)} |\nabla\varphi|^2(v)\epsilon_j^2 + W''(u_{\epsilon_j} \circ \exp_{x_j}(v))\varphi(v)^2 d\mu_g(v) = \epsilon_j^{-n+1} \int_{B_{K\epsilon_j}(0)} |\nabla\varphi|^2(v)\epsilon_j + \frac{1}{\epsilon_j} W''(u_{\epsilon_j} \circ \exp_{x_j}(v))\varphi(v)^2 d\mu_g(v) \geq 0$$

because the integral is non-negative by assumption of u_{ϵ_j} being stable, $\epsilon_j^{-n+1} > 0$, and $\exp_{x_j}(B_{K\epsilon_j}(0)) \subseteq B_\rho(x_j)$ for ϵ_j sufficiently small. Thus, the equality must hold in the limit as well.

(d) The same argument will hold here: if $Q_{u_0}(\psi, \psi) < 0$, then $Q_{\tilde{u}_j}(\psi, \psi) < 0$ for all j sufficiently large. By undoing the integration by parts above, this translates into a collection of linearly independent functions for which non-stability holds for $\{u_{\epsilon_j}\}$. Thus

$$\liminf_j \text{index}(u_{\epsilon_j}, B_\rho(x_j)) \geq \text{index}(u_0, B_K(0))$$

If we have a constant upper bound on the quantity on the left, then we get the same result for index$(u_0, B_K(0))$.

(e) Really not sure - over what domain are we interpreting $\{u_{\epsilon_j}\}$ to be a δ-locally minimizer? Should be working entirely on $B_K(0)$

Here's an idea, I claim that

$$||\tilde{u} - v||_{L^1(B_K(0))} < \delta \implies E_1(v) > E_1(\tilde{u})$$

The point is that if we have the former then we know that by smooth convergence on compact sets that

$$\exists J_0 \text{ s.t. } \forall j > J_0, \qquad ||\tilde{u}_j - v||_{L^1(B_K(0))} < \delta$$

If we had non-strict inequality, then we might have hit a situation where $||\tilde{u} - v|| = \delta$ and $||\tilde{u}_j - v|| > \delta$.

When the first inequality does hold, compute

$$E_1(v) = \int_{B_K(0)} \frac{1}{2}|\nabla v|(x)^2 + W(v)(x) = \epsilon^{-n} \int_{B_{K\epsilon_j}(0)} \frac{1}{2}|\nabla v|(\epsilon_j^{-1}x)^2 + W(v)(\epsilon_j^{-1}x)$$

Now if we set $p_j(x) = v(\epsilon_j^{-1}x)$, then we get

$$E_1(v) = \epsilon^{1-n} \int_{B_{K\epsilon_j}(0)} \frac{\epsilon_j}{2}|\nabla p_j|(x)^2 + \frac{1}{\epsilon_j}W(v)(\epsilon_j^{-1}x)$$

Now, if we have that u_{ϵ_j} is a local minimizer on $B_{K\epsilon_j}(0)$ (Note: it's not sufficient to have it be a minimizer on a larger set, as $H^1(\Omega_1) \not\leftrightarrow H^1(\Omega_2)$ when $\Omega_1 \subset \Omega_2$.Ask), then we can conclude that

$$E_1(v) \geq \epsilon^{1-n} E_{\epsilon_j}(u) = E_1(\tilde{u}_j)$$

this holds for all j sufficiently large, so

$$E_1(v) \geq E_1(\tilde{u})$$

ending the proof. So we have $\delta - \epsilon$-minimization for any $\epsilon > 0$. Want our perturbations to lie in H_0^1. Otis says that via rescaling the δ in $\delta > ||v - \tilde{u}||$ doesn't matter anymore, so I need to do the integration by parts and check this

(f) We can use the same arguments in exercise 2.5 to get a convergent subsequence out of the $\tilde{u}_K$'s such that this sequence of functions converges to a solution on Allen-Cahn on all of $\mathbb{R}^n$, with the convergence being smooth on compact sets not sure why it would still be smooth if we're citing 2.5. See my comment in part b)

Once we have smooth convergence, if we use the definitions of minimzing and stabilizing as in section 5.1 (bottom of p. 16), i.e being stable on compact subsets of $\mathbb{R}^n$ and minimizing on compact sets, then we have this automatically. If we asked for stability under general, compactly supported perturbations, $\varphi \in C_c^\infty(\mathbb{R}^n)$, we would still get the result: just take the compact set on which we converge smoothly to contain supp(φ). The same holds for minimizing if the comparative function, v, is compactly supported not sure what happens when support is allowed to be infinite.

In general, we expect minimizers to converge to minimizers, stable solutions to converge to stable solutions (think about matrices)

The hope is that $\tilde{u}$ is stable on all of $\mathbb{R}^n$ and a δ-local minimizer.

1.15 Exercise 5.2

(a) The idea is that we can actually take the infinum over all functions satisfying the boundary condition and lying in $H^1 \cap L^4$. Then the minimizer exists by functional analysis arguments (see Ch 8.2 of Evans again), and we know that a minimizer must be smooth on B_R by the first problem in this set of notes. Hence, this minimizer will also be a minimizer taken over the smaller set of smooth functions satisfying the boundary condition.

(b) The point is that τ exists because we have that $\lim_{\tau\to\infty} u(x', x^n + \tau) = 1$ uniformly in (x', x_n). In particular, because this is uniform convergence, we have that

$$\forall \epsilon > 0, \exists \mathrm{T} \text{ s.t. } \tau > \mathrm{T} \implies u_\tau(x', x^n) > 1 - \epsilon, \ \forall (x', x_n) \in B_R(0)$$

Now let $\ell = \max_{\vec{x} \in \overline{B_R(0)}} |u(\vec{x})| < 1$ because the function is < 1 on the interior and the boundary and it's a compact set, so u would have to achieve it's maximum on $\overline{B_R(0)}$. Now set $\epsilon = (1-\ell)/2$ to see that

$$\{\tau \,| u_\tau \geq v \qquad \forall (x', x_n) \in B_R(0)$$

is non-empty. Now take the smallest, non-negative such τ in the above set, which must exist by the well-ordering principle. Note that $\tau = 0$ gives $u_0\Big|_{\partial B_R} = v\Big|_{\partial B_R}$, so it is not the case that $u_\tau > v$ holds for all τ non-negative, i.e. strict inequality everywhere. With this, if γ is the minimal τ value, then we cannot have $u_\gamma > v$ everywhere, else by uniform continuity on $\overline{B_R}$ (b/c of continuity on a compact set) we would have that $u_{\gamma-\epsilon} > v$ everywhere, as strict inequality is an open condition. This contradicts minimality of γ, and so $\exists \hat{x} \in \overline{B_R}$ such that $u_\tau(\hat{x}) = v(\hat{x})$.

1.16 Exercise 5.3 (One question)

Does Otis want me to think of $u_\pm$ as a function on $\mathbb{R}^{n-1}$ and hence a solution to the $n-1$-dimensional Allen Cahn? I might have trouble then... So it actually solves it on both $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$

So pointwise, we know that $u_\pm(x')$ exists by monotonicity. We now ask if a monotone sequence of functions satisfying allen-cahn do actually solve allen-cahn, and this is indeed true via the domainated convergence theorem. To see this, first define

$$v_\pm(x', x_n) = u_\pm(x') = \lim_{\tau\to\infty} u_\tau(x', x_n)$$

Now I drop $\pm$ in favor of $+$ as the work for $-$ is analogous. Take φ with compact support in R^n, then

$$\int \langle \nabla v, \nabla \varphi \rangle + W'(v)\varphi = \int -v\Delta\varphi + W'(v)\varphi = \int_{\mathrm{supp}(\varphi)} -v\Delta\varphi + W'(v)\varphi$$

because supp(φ) is compact, and hence finite in measure, we can use the dominated convergence theorem to get that

$$\int_{\mathrm{supp}(\varphi)} v(-\Delta\varphi) + (W'(v)\varphi) = \lim_{k\to\infty} \int_{\mathrm{supp}(\varphi)} u_k(-\Delta\varphi) + (W'(u_k)\varphi)$$

Now we note that each $u_k = u(\cdot, \cdot + k)$ is a solution to Allen-Cahn, and so this integral is 0. Thus u is a solution to Allen-Cahn.

We now want to run through the argument in proposition 5.8 with Savin's condition replaced by $u_-(x') \leq v(x', x^n) \leq u_+(x')$, where $u_\pm(x')$ are replacing ± 1.

Let

$$J = \inf\{E_1(v) \ : \ v \in C^\infty(B_R), \ \ u_-(x') \leq v(x', x^n) \leq u_+(x') \qquad \forall (x', x^n) \in B_R\} = \inf\{E_1(v) \mid v \in S\}$$

If we can show that a minimizer of the above set exists and satisfies Allen-Cahn, then the same steps will show that the minimizer is not equal to $u_\pm$ at any point and so

$$u_-(x') < v(x', x^n) < u_+(x'), \qquad \forall (x', x^n)$$

Now we can pull the same u_τ argument, again noting uniform convergence on our compact set, and use the same arguments to show that $u = v$ everywhere, meaning that u is a minimizer.

Ok, suppose that of E_1 over S exists. Then if we show that $u_-(x') < v(x', x^n) < u_+(x'), \qquad \forall (x', x^n)$, we can take the minimum of the absolute value of $v - u_\pm$ and take the infinum over all of $\overline{B_R}$. This must be a non-zero quantity, so for t small enough, we'll have that for any smooth, compactly supported perturbation, we get $v + t\varphi$ lies in S. Thus v will be a solution to Allen-Cahn by nature of being a local minimizer! So we just need to show that this minimizer exists. If we want this to be a solution to Allen Cahn, we can just stipulate $u \in H^1(B_R)$. Now we can truly apply the machinery of section 8.2 in Evans, i.e. a minimizer of E_1 taken over the set

$$\{v \in H^1(B_R), \ \text{ s.t. } \ u_-(x') \leq v(x', x^n) \leq u_+(x')\}$$

will exist. I want something that says that v satisfies strict inequality on the interior so that any pertubration, $+t\varphi$, will keep it in the set for t sufficiently small. Otis said he'll get back to me on this Once I have this v will be a bonafide solution to Allen-Cahn

1.17 Exercise 5.4

Log cutoff? Should work just like exercise 2.5

1.18 Exercise 5.5

(a) If $\nabla u = 0$ everywhere, then u is constant, and $\Delta u = u(u^2-1) = 0$, which implies that $u = 0$. I guess 0 is a solution everywhere to allen-cahn.
(b) Is this not the content of the hint in 4.2?

(c) Consider the (k,j)th entry of $D\left(\frac{\nabla u}{|\nabla u|}\right)$, it is

$$\partial_k \frac{u_j}{|\nabla u|} = \frac{u_{jk}|\nabla u| - u_j \partial_k |\nabla u|}{|\nabla u|^2} = \frac{u_{jk}|\nabla u| - u_j \frac{\sum_l u_{kl}u_l}{|\nabla u|}}{|\nabla u|^2} = \frac{1}{|\nabla u|^3}\left(u_{jk}|\nabla u|^2 - u_j \frac{1}{2}\partial_k |\nabla u|^2\right)$$

and so the sum of squares of each term gives

$$\frac{1}{|\nabla u|^6}\sum_{j,k} u_{jk}^2 |\nabla u|^4 - u_{jk}u_j|\nabla u|^2 \partial_k |\nabla u|^2 + \frac{1}{4}u_j^2(\partial_k|\nabla u|^2)^2$$

up to the prefactor, this is

$$|\nabla u|^4|D^2u|^2 - \frac{1}{2}|\nabla u|^2|\nabla|\nabla u|^2|^2 + \frac{1}{4}|\nabla u|^2|\nabla|\nabla u|^2|^2 = |\nabla u|^4|D^2u|^2 - \frac{1}{4}|\nabla u|^2|\nabla|\nabla u|^2|^2 = 0$$

having used that $|D^2u|^2 = |\nabla|\nabla u||^2$ and the fact that

$$|\nabla|\nabla u|^2| = 2||\nabla u|\nabla|\nabla u|| \implies |\nabla u|^2|\nabla|\nabla u|^2|^2 = 4|\nabla u|^4|\nabla|\nabla u||^2$$

With this, we're able to conclude that

$$\frac{\nabla u}{|\nabla u|} = \vec{c} \implies \nabla u = |\nabla u|\vec{c}$$

Thus

$$D\left(\frac{\nabla u}{|\nabla u|}\right)\left(\frac{\nabla u}{|\nabla u|}\right) = 0 = \nabla_{\frac{\nabla u}{|\nabla u|}}\frac{\nabla u}{|\nabla u|} = 0$$

(d) From the previous part, we actually see that $D(\nabla u/|\nabla u|) = 0$ and so $\nabla u = \vec{c}|\nabla u|$ for some $\vec{c} \in \mathbb{R}^n$ constant, **when** $\nabla u \neq 0$. This is an open set, so if we have $x_0 \in \{\nabla u \neq 0\rangle$, then on this set we have that

$$u(\vec{x}) = f(\langle \vec{c}, \vec{x} - \vec{x}_0\rangle)$$

This is because u only changes in the $\vec{c}$ direction, so the change between $u(\vec{x})$ and $u(\vec{x_0})$ only depends on where the projection of $\vec{x} - \vec{x_0}$ is on $\mathbb{R}\vec{c}$.

Consider $\{\nabla u = 0\}$. If this set has any interior, then unique continuation would tell us that $u \equiv 0$, which is fine, but we want to consider non-constant solutions. Thus $\{\nabla u = 0\rangle$ has no interior. Let $y \in \{\nabla u = 0\}$, then we know that

$$\exists \{z_n\} \subseteq \{\nabla u \neq 0\rangle \text{ s.t. } \lim_{n\to\infty} z_n = y$$

and so by continuity of u (which we have already without assuming that u is one-dimensional), we get

$$u(y) = \lim_{n\to\infty} u(z_n) = \lim_{n\to\infty} f(\langle \vec{c}, \vec{z}_n - \vec{x}_0\rangle)$$

If f is assumed to be continuous on all of $\mathbb{R}$, then we can move the limit inside and get the result. We can guarantee this if $\{\nabla u = 0\rangle$ does not contain any lint perpendicular to $\mathbb{R}\vec{c}$, as then we would have f being evaluated at every $r \in \mathbb{R}$ via some $\vec{x} \in \{\nabla u \neq 0\}$ such that $r = \langle \vec{c}, \vec{x} - \vec{x}_0\rangle$. Almost done, just need to get continuity of f everywhere This is true though because we know that u is smooth so f must be smooth as well, and hence continuous

(e) Without the stability condition, consider the constant functions $u = 0$, $u = 1$, and $u = -1$, which all solve Allen Cahn. Do there exist non-constant solutions of Allen-Cahn of this form? We didn't classify all solutions to Allen-Cahn in 1-D, just the ones that have finite energy Note that if there exists a positive component of $\vec{c}$ then we would get the de giorgi monotonicity condition in some direction and understand stabilitiy well. However, if $c_i > 0$, then we have constant growth of u in one direction, meaning that $u > 1$ in finite time via integration by parts, a contradiction..

If we could show that u has finite energy, then this would imply that f has finite energy and so by our work with one dimensional solutions, we would get that $f = \mathbb{H}$. By rotational symmetry, assume that $\vec{a} = ae_1$, then the scaling of $|a| = 1$ should come from

$$\Delta u = \partial_{x_1}^2 f(\langle ae_1, x - x_0\rangle) = a^2 f'' = u(u^2 - 1) = f(f^2 - 1)$$

and so f is a solution to Allen Cahn as well iff $|a| = 1$. Thus, if we have $u = \mathbb{H}(\langle a, x - x_0\rangle)$, then we must have $|a| = 1$.

If stability is replaced by $E_1(u; B_r) \leq CR^{n-1}$, I'm not sure what happens. Replace B_R with a rotated square, $S_{R,\vec{c}}$, i.e a square with perpendicular bisector parallel to $\vec{c}$ and having side length R. Note that if we can show the claim for squares instead of balls, then we still get the claim for balls as the integrand is positive and $B_R(0) \subseteq S_{R,\vec{c}} \subseteq B_{2R}(0)$. We get that

$$\int_{S_{R,\vec{c}}} \frac{1}{2}||\nabla u||^2 + W(u) = (2R)^{n-1} \int_{-R}^{R} \frac{1}{2}||\nabla u(s\vec{c})||^2 + W(u(s\vec{c}))ds$$

i.e. we pick up a factor of CR^{n-1} for free by nature of the solution being one dimensional, and so we need that the one dimensional solution obeys a constant bound in integral, i.e.

$$E_1(f) \leq C$$

where the energy is being calculated over $\mathbb{R}$. If we have that $|a| = 1$, then for $u(x) = f(\langle \vec{c}, x - x_0\rangle)$, we know that f is a solution to Allen-Cahen, and so we can apply our results from the first section: any solution to one dimensional allen cahn with bounded energy is necessary $f = \mathbb{H}$. If $|a| \neq 1$, I'm not sure - maybe we can do another rescaling argument if $|a| \neq 1$? Note that $u(x) = \mathbb{H}(\langle a, x - x_0\rangle)$ is stable precisely because it satisfies the de giorgi monotonicity condition It's already a solution to allen-cahn, so this concern about $|c| = 1$ is not needed. Or rather, if $|c| \neq 1$, we can write $f(\langle c, x - x_0\rangle) = \tilde{f}(\langle \tilde{c}, x - x_0\rangle)$.

(f) No idea why this is the case.

1.19 Exercise 5.6

Use the $||u||_{C^{1,\alpha}} \leq ||f||_{C^0}$ bound (see 67/68 of Gillibarg-trudinger, acutally see lemma 6.35 on p. 135 in the appendix about global interpolation inequalities). I think this is it

1.20 Exercise 5.7 (Not done)

Why doesn't poincare inequality give us a lower bound of $4\pi R^2$? Maybe this is isoperimetric inequality or sobolev inequality, but I feel like my reasoning should work, unless we're talking about all components of the gradient. Poincare inequality should still work: it gives a lower bound via two components of the gradient, so adding the third is icing on top and still boundable from below Google poincare eigenvalue to get a sharp constant

So let's do the following: the poincare inequality gives

$$||u||_{L^2}^2 \leq C||\nabla u||_{L^2}^2 \implies \frac{1}{C} \leq \frac{||\nabla u||_{L^2}^2}{||u||_{L^2}^2}$$

If we can minimize the right hand side over some reasonable set, then setting this equal to $1/C$ gives us the optimal value for C. Note that this looks like a Rayleigh quotient problem, except we have the added constraint that $u \equiv 1$ on $B_R(0)$.

Let's say that φ is supported on W, then we have that

$$||\varphi||_{L^2}^2 \leq C_W||\nabla\varphi||_{L^2}^2$$

Then we have

$$C_W = \frac{1}{\lambda_{1,W}} \qquad \lambda_{1,W} = \inf_{f\in S}\{||\nabla f||_2^2\} \qquad S = \{f \in H_0^1(W) \mid ||f||_{L^2} = 1\}$$

Note that $W \supseteq B_R(0)$ implies that $\lambda_{1,W} \leq \lambda_{1,B_R(0)}$ and so $C_W \geq C_{B_R(0)}$. This is problematic because we get the opposite direction of bound we want, i.e. ideally, we'd have

$$\lambda_{1,B_R(0)} \leq \lambda_{1,W} \leq \frac{||\nabla u||_{L^2}^2}{||u||_{L^2}^2}$$

because then we could use our knowledge of $\lambda_{1,B_R(0)}$ and $||u||_{L^2}^2 \geq \frac{4}{3}\pi R^3$ to get the result. Note that even if we had the correct direction in the bound, we have

$$\lambda_{1,B_R(0)} = \frac{c}{R^2}$$

where $c \approx 2.4$ (see one of the sources below), which is bad because then we don't get the $4\pi R^2$ bound. Instead, we get something like $\frac{c}{3}4\pi R^2$.

see here:

`https://math.stackexchange.com/questions/1088051/the-best-constant-of-poincare-inequality-can-be-determined`
`http://php.math.unifi.it/users/cime/Courses/2017/01/201712-Notes.pdf    p. 5`
`https://mathworld.wolfram.com/BesselFunctionZeros.html`

See Otis' notes. It's a "capacity" argument

1.21 Exercise 5.8

Just as in problem 5.3, we know that $u^{\pm\infty}(x_1, x_2)$ is a solution to Allen Cahn in both 3 and 2 dimensions (because $\partial_3^2 u^{\pm\infty} = 0$). If u is a stable solution on $\mathbb{R}^3$, then we want to show that $u^{\pm\infty}$ is a stable solution on $\mathbb{R}^2$. Let's write

$$\int_{\mathbb{R}^2} |\nabla\psi|^2 + W''(u^\pm)\psi^2 d\mu_g = \lim_{t\to 0}\int_{\mathbb{R}^3} |\nabla f_t|^2 + W''(u^\pm) f_t^2 d\mu_g$$

define

$$f_t(x,y,z) = \psi(x,y)\psi_t(z) \qquad \psi_t(z) = \begin{cases} t/2 & |z| < 1/t \\ t/2 - (\pm z \mp 1/t) & 1/t < |z| < 1/t + t \\ 0 & |z| > 1/t + t \end{cases}$$

Then we see that $\lim_{t\to\infty} ||\psi_t||_{L^2(\mathbb{R}^1)}^2 = 1$ and also

$$\int_{\mathbb{R}} \psi_t'(z)^2 = 2\int_{1/t}^{1/t+t} \left(\frac{t/2}{t}\right)^2 = \frac{1}{2}t$$

which tends to 0. From this, we note that

$$\nabla f_t = (\psi_t \nabla\psi, \psi\psi_t') \implies |\nabla f_t|^2 = |\nabla\psi(x,y)|^2|\psi_t|^2 + |\psi(x,y)|^2|\psi_t'|^2$$

And so taking the limit to 0, we get that

$$\lim_{t\to 0}\int |\nabla f_t|^2 = ||\nabla\psi||_{L^2}^2$$

and also

$$\lim_{t\to 0}\int W''(u^\pm(x,y)) f_t^2 = \int_{\mathbb{R}^2} W''(u^\pm(x,y))\psi(x,y)^2$$

because the z dependence just cancels out and yields an integral of 1. Thus, it suffices to show that

$$Q_{u^\pm}(f_t, f_t) = \int_{\mathbb{R}^3} |\nabla f_t|^2 + W''(u^\pm)|f_t|^2 \geq 0$$

for all t, as then the limit will be positive as well. To show the above, it suffices to cite the fact that $u^\pm$ is smoothly approximated by $u^s(x)$ for s large, on fixed compact sets. This works, because for a fixed t, the support of ∇f_t and f_t are compact in $\mathbb{R}^3$. Thus

$$\forall s, t, \qquad Q_{u^s}(f_t, f_t) \geq 0 \implies \forall t, \qquad Q_{u^\pm}(f_t, f_t) \geq 0$$

and so $u^\pm(x,y)$ is stable on $\mathbb{R}^2$.

Now from theorem 5.9, aka exercise 5.5, we know that $u^{\pm} = \mathbb{H}(\langle a, x - x_0 \rangle)$ for some a and x_0. The fact that we have

$$E_1(u^{\pm}; B_R \subseteq \mathbb{R}^3) \leq CR^2$$

is then immediate (see exercise 5.5) - replace B_R with a square of radius R oriented so that one of the perpendicular bisectors is parallel to a. Then the integrations in the directions orthogonal to a give a factor of R^2, and the integration in the a direction gives a constant bound because we know that $E_1(\mathbb{H}(t)) < C$, i.e. the energy is finite when integrating over all of $\mathbb{R}$, and hence bounded when integrating over an interval $[-R, R]$.

Now because $u^t \to u^{\pm}$ smoothly on compact sets (including B_R for R fixed), we get that

$$\lim_{t \to \pm\infty} E_1(u^; B_R \subseteq \mathbb{R}^3) \leq CR^2$$

1.22 Exercise 6.1

For $n = 1$ consider

$$\varphi_R(x) = \begin{cases} 1 & |x| \leq R \\ 2 - \frac{\log|x|}{\log(R)} & R < |x| < R^2 \\ 0 & |x| \geq R^2 \end{cases}$$

then as Otis does on p. 20, we know that even though this is lipschitz, it can be suitably approximated by C^1_c functions. We then have that

$$\int_{\mathbb{R}} |\varphi'(x)|^2 = 2\int_R^{R^2} \frac{1}{x^2 \log(R)^2} = \frac{2}{\log(R)^2}\left[\frac{1}{R} - \frac{1}{R^2}\right]$$

clearly, this tends to 0 as $R \to \infty$ but we also have that

$$\int e^{-u}\varphi_R^2 \to \int e^{-u} > 0$$

i.e. this quantity is positive unless $u = \infty$ everywhere, which is impossible.

For $n = 2$, it's essentially the same computation except the measure is angle-symmetric measure $2\pi r dr$ and so the integral becomes

$$\int_{\mathbb{R}^2} |\varphi'(x)|^2 = 2\int_{r=R}^{r=R^2} \frac{1}{x^2 \log(R)^2} r dr = \frac{2}{\log(R)^2}\left[\log(R^2) - \log(R)\right] = \frac{2}{\log(R)} \to 0$$

and so the same argument works.

1.23 Problem 1 (almost done, details on (e) and (f))

a) Oh, if the infinum is non-zero, then just do the fundamental theorem of calculus and note that we'll have $u > 1$ in finite time, i.e. solve for a path $\gamma(t)$ such that $\gamma'(t) = \nabla u$, because then if $\gamma(0) = 0$, we get

$$u(\gamma(t)) - u(\gamma(0)) = \int_0^t \nabla u \cdot \gamma'(t) = \int_0^t |\nabla u|^2 \geq C^2 t$$

such a gradient flow exists because u is smooth.

b) This holds via computation. Note that

$$\Delta P = \Delta|\nabla u|^2 - 2\Delta W(u)$$

we break this down as follows and use that $\Delta u = W'(u)$

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 &= |D^2u|^2 + \langle \nabla \Delta u, \nabla u\rangle = |D^2 u|^2 + W''(u)|\nabla u|^2 \\
\Delta W(u) &= \nabla \cdot (W'(u)\nabla u) = W''(u)|\nabla u|^2 + W'(u)\Delta u = W''(u)|\nabla u|^2 + W'(u)^2 \\
\implies \Delta P &= 2|D^2u|^2 - 2W'(u)^2.
\end{aligned}$$

We separately have that

$$\begin{aligned}
\nabla P &= 2(D^2u \cdot \nabla u) - 2W'(u)\nabla u \\
\implies \frac{1}{2}|\nabla P|^2 &= 2|D^2u \cdot \nabla u|^2 - 4[(D^2u \cdot \nabla u) \cdot \nabla u]W'(u) + 2W'(u)^2|\nabla u|^2 \\
\nabla u \cdot \nabla P &= 2(D^2u \cdot \nabla u) \cdot \nabla u - 2W'(u)|\nabla u|^2 \\
\implies \frac{1}{2}|\nabla P|^2 + 2W'(u)\nabla u \cdot \nabla P &= 2|D^2u \cdot \nabla u|^2 - 2W'(u)^2|\nabla u|^2
\end{aligned}$$

With this, we see that

$$\Delta P|\nabla u|^2 - \frac{1}{2}|\nabla P|^2 - 2W'(u)\nabla P \cdot \nabla u = 2[|D^2u|^2|\nabla u|^2 - |D^2u \cdot \nabla u|^2] \geq 0$$

ending the proof.

(c) We have that the laplacian is ≤ 0 at $x = 0$ if it is an absolute maximum, but it's a maximum so $\nabla P = 0$ at this point. If $\nabla u = 0$, then by definition, $P \leq 0$, a contradiction. Thus, assume $\nabla u \neq 0$, and consider the differential operator

$$L(P) := -|\nabla u|^2\Delta P + \frac{1}{2}|\nabla P|^2 - 2W'(u)\nabla u \cdot \nabla P \leq 0$$

by the weak maximum principle (see here), we get that the maximum of P on a neighborhood of 0 on which $|\nabla u|$ is bounded away from 0, is equal to the maximum on the boundary of said region. This implies that P is locally constant about 0. Being locally constant and $\nabla u \neq 0$ are both open conditions, so there exists a maximal connected region on which $P \equiv c$ and $\nabla u \neq 0$. If $\nabla u \neq 0$ everywhere, then P is constant everywhere, but this contradicts the fact in (a), that $\inf |\nabla u| = 0$, meaning that there exists a sequence of points for which

$$P(x_i) = |\nabla u|(x_i)^2 - 2W(u(x_i)) \quad \downarrow$$

The first term on the right is tending to 0 and the second term is nonpositive, so $\inf P \leq 0$, a contradiction to $c > 0$. If $\nabla u = 0$ at some point, then there must exist a point x_0, on the boundary of the region on which $P \equiv c$ and $\nabla u \neq 0$. By continuity, we'll have that $P = c$ at x_0 and $\nabla u = 0$, so that $P(x_0) = c = -2W(u) \leq 0$, a contradiction to $c > 0$. Thus, we get that $P \leq 0$ when the supremum is attained somewhere.

Might need to do a refined maximum principle using the fact that if $|\nabla u| \neq 0$, then we would have a uniformly elliptic operator in a neighborhood of 0 and then we can get better bounds.

(d) The idea is that $u_i(x) = u(x - x_i)$ is a bounded sequence in H^1 of solutions to Allen-Cahn with $\epsilon = 1$. Using Schauder estimates and Arzela-Ascoli as in exercise 2.5, we get a subsequence of solutions which converge uniformly on compact subsets of $\mathbb{R}^n$. Really, we want to perform the subsequence construction on $\partial_{xx}u_i$, $\partial_{xy}u_i$, $\partial_{yy}u_i$, i.e. extract a subsequence for which the first one converges, and then a subsubsequence for which the second one converges, and so on. Then repeat this for $\partial_x u_i$, $\partial_y u_i$, and $\{u_i\}$ for further subsequences. With this, we'll get a sequence which converges in C^2 on compact sets and hence in C^2_{loc}, to some $\tilde{u}$

From here, we want to make a limiting argument as follows: define $\tilde{P} = |\nabla \tilde{u}|^2 - 2W(u)$ so that $\sup_{x \in \mathbb{R}^n} \tilde{P} = \tilde{P}(0) > 0$. From part c), this is a contradiction because $\tilde{P}$ is still a solution to Allen-Cahn, and so $P \leq 0$ must have been the case from the start.

(e) First I want to show that if $P = 0$ at a point, then $P \equiv 0$ everywhere. We have the same uniform elliptic trick from the previous parts, assuming that $|\nabla u| \neq 0$ at all points of interest. Let's start with $P(x_0) = 0$. If $|\nabla u| = 0$, then we have that

$$P(x_0) = 0 = -2W(u)$$

I claim that $|u| \neq 1$, which means that $-2W(u) < 0$, a contradiction. To see this, apply Lemma 5.7 from the notes to u and $v = \pm 1$, which shows that if $u = \pm 1$ at a point, then $u \equiv \pm 1$ everywhere, for which the statement doesn't apply but we don't want to constant case anyway.

With this, we see that $|\nabla u|$ must be non-zero when $P = 0$, and so P will be locally 0 about x_0 by the maximum principle. This argument shows that being 0 is an open condition, but it's also a closed condition, so $P \equiv 0$ everywhere.

With this, we define

$$\varphi = \mathbb{H}^{-1}(u) \;\text{ s.t. }\; u = \mathbb{H}(\varphi)$$

we compute that

$$\nabla u = \mathbb{H}'(\varphi)\nabla\varphi = \frac{1}{\sqrt{2}}(1-\mathbb{H}^2)\nabla\varphi$$

$$|\nabla u|^2 = 2W(u) \iff \frac{1}{2}(1-\mathbb{H}(\varphi)^2)^2|\nabla\varphi|^2 = \frac{2}{4}(1-\mathbb{H}(\varphi)^2)^2 \implies |\nabla\varphi|^2 = 1$$

because $1-\mathbb{H}(\varphi) \neq 0$ for any value of φ. From this and the fact that φ is smooth, we can conclude that φ must be linear. See here, which I don't totally understand. Another thing to note is that $\Delta u = u(u^2-1)$ will tell us that $\Delta\varphi = 0$, which is nice information, though superfluous. A better way to do this is note that $\Delta u = 0$ tells us that this is a harmonic function and in particular $\Delta\partial_i u = 0$, and $||\partial_i u|| \leq 1$, so each of the derivatives are bounded harmonic functions

(f) Yes this is true, and is evident if we rewrite

$$E_R = R^{1-n}\cdot f(R) \implies \frac{E_R}{dR} = (1-n)R^{-n}f(R) + R^{1-n}f'(R)$$

where

$$f'(R) = \lim_{h\to 0}\frac{1}{h}\int_{r=R}^{R+th}\left(\frac{1}{2}|\nabla u|^2 + W(u)\right) = \int_{\partial B_R(0)}\left(\frac{1}{2}|\nabla u|^2 + W(u)\right)$$

We can then rewrite

$$R^{1-n}f'(R) = R^{-n}\int_{\partial B_R(0)}\left(\frac{1}{2}|\nabla u|^2 + W(u)\right)\langle x,\nu\rangle d\mu$$

because $\langle x,\nu\rangle = R$. Maybe this is the green formula for integration by parts?

We write

$$\int_{\partial B_R}\frac{1}{2}|\nabla u|^2\langle x,\nu\rangle = \int_{B_R}\frac{1}{2}|\nabla u|^2 n + \langle (D^2u)(\nabla u), x\rangle$$

$$\int_{\partial B_R} W(u)\langle x,\nu\rangle = \int_{B_R} W'(u)\langle\nabla u, x\rangle + W(u)n$$

having considered $\varphi = \frac{1}{2}||x||^2$. Note that

$$\int_{\partial B_R}(\partial_\nu u)^2 = R\int_{\partial B_R}\langle\nabla u, x\rangle(\nabla u\cdot\nu) = R\int\Delta u\langle\nabla u, x\rangle + \langle (D^2u)(\nabla u), x\rangle + |\nabla|^2$$

having written $\psi = \langle\nabla u, x\rangle$ and $\varphi = u$. With this, we get that

$$(1-n)R^{-1}E_R + R^{-n}\int_{\partial B_R}[\frac{1}{2}|\nabla u|^2 + W(u)]\langle x,\nu\rangle$$

$$= (1-n)R^{-n}\int_{B_R}\frac{1}{2}|\nabla u|^2 + W(u) + R^{-n}\int_{B_R} W'(u)\langle\nabla u, x\rangle + n(\frac{1}{2}|\nabla u|^2 + W(u)) + \langle (D^2u)(\nabla u), x\rangle$$

$$= \int_{B_R}\frac{1}{2}|\nabla u|^2 + W(u) + R^{1-n}\int_{\partial B_R}(\partial_\nu u)^2 - \int|\nabla u|^2 = -2\int_{B_R}P + \int(\partial_\nu u)^2$$

which is what I think we wanted

(g) Yes, I can at least do this part. For the limits, we replace the balls with the squares of side length $2R$ and $\sqrt{2}R$ and show that the limits are the same. We compute

$$F_R := R^{1-n}\int_{S_R}\frac{1}{2}|\nabla u|^2 + W(u) = R^{1-n}\int_{S_R}2W(u) = \int_{-R}^{R}2W(u)dx = 4\int_{-R}^{R}W(u)$$

having used that $|\nabla u|^2 = 2W(u)$. Now we write

$$\int_{-R}^{R}W(u) = \int_{-R}^{R}(1-\tanh(x/\sqrt{2})^2)^2 = \sqrt{2}\int_{-R/\sqrt{2}}^{R/\sqrt{2}}(1-\tanh(x)^2)^2dx = \sqrt{2}\int_{-R}^{R/\sqrt{2}}\frac{64e^4x}{(e^{2x}+1)^4}dx = 64\sqrt{2}\int_a^b\frac{(u-1)}{u^4}du$$

where $b = e^{\sqrt{2}R}+1$ and $a = e^{-\sqrt{2}R}+1$. We have

$$\int_a^b\frac{(u-1)}{u^4}du = \frac{2-3b}{6b^3} - \frac{2-3a}{6a^3} = \frac{-1-3e^{\sqrt{2}R}}{6(e^{\sqrt{2}R}+1)^3} - \frac{-1-3e^{-\sqrt{2}R}}{6(e^{-\sqrt{2}R}+1)^3}$$

as $R \to \infty$ the first term dies off, and the second term persists, and we get $1/6$. As $R \to 0$, both terms contribute, and we get 0. Note that if we replaced R by $R/\sqrt{2}$ in our bounds of integration, then these limiting values would be the same. Thus, we can conclude that

$$\lim_{R\to\infty} E_R = \lim_{R\to\infty} E_R = \frac{32\sqrt{2}}{3}$$

in the limit that $R \to 0$

$$\lim_{R\to 0} E_R = \lim_{R\to 0} E_R = 0$$

Not sure what the monotoncity formula is? Ask otis but also see here. The statement is something like

$$\frac{area(\Sigma \cap B_r)}{r^n}$$

is increasing as $r \to 0$ for $\Sigma^n \subseteq \mathbb{R}^{n+k}$.

Extra reading on why monotoncity formulas are useful: here. The idea is that a monotonicity formula prevents the energy from accumulating in small regions. More philosophically, monotonicity allows us transfer information about energy accumulation at zoomed in scales (for example when zooming in). (h) We calculate

$$\Delta P = 2\epsilon[|D^2u|^2 + \mathrm{Ric}_g(\nabla u, \nabla u) - \epsilon^{-4}W'(u)^2]$$
$$\frac{1}{2}|\nabla P|^2 = 2[\epsilon^2|D^2u \cdot \nabla u|^2 - 2W'(u)(D^2u\cdot\nabla u)\cdot\nabla u + \epsilon^{-2}W'(u)^2|\nabla u|^2]$$
$$W'(u)\nabla u \cdot \nabla P = 2[\epsilon W'(u)(D^2u\cdot\nabla u)\cdot\nabla u - \epsilon^{-1}W'(u)^2|\nabla u|^2]$$

so a candidate for a good inequality would be

$$A - B - C := \Delta P|\nabla u|^2 - \epsilon^{-1}\left(\frac{1}{2}|\nabla P|^2\right) - \epsilon^{-2}(2W'(u)\nabla u\cdot\nabla P)$$
$$= 2\epsilon[|D^2u|^2|\nabla u|^2 - |D^2u\cdot\nabla u|^2 + \mathrm{Ric}_g(\nabla u,\nabla u)|\nabla u|^2] \geq 2\epsilon\mathrm{Ric}_g(\nabla u,\nabla u)|\nabla u|^2$$

(i) If $\mathrm{Ric}_g \geq 0$ then the idea is to mimic the proof in parts (b) through (d). We already have an analogous inequality for (b), namely

$$\Delta P|\nabla u|^2 \geq \epsilon^{-1}\left(\frac{1}{2}|\nabla P|^2\right) + \epsilon^{-2}(2W'(u)\nabla u\cdot\nabla P)$$

If the supremum is positive and obtained somewhere in $\mathbb{R}^n$, WLOG at $x = 0$, then $\Delta P = 0$ and $\nabla P = \vec{0}$ at those points. If $\nabla u = 0$ at that point, then $P(0) = -2\epsilon^{-1}W(u) \leq 0$, a contradiction. If $\nabla u \neq 0$, then we can run the same argument of the maximum principle with

$$L(P) = \Delta P|\nabla u|^2 - \epsilon^{-1}\left(\frac{1}{2}|\nabla P|^2\right) - \epsilon^{-2}(2W'(u)\nabla u\cdot\nabla P)$$

so that the leading order term is elliptic in some neighborhood where $|\nabla u|$ is bounded away from 0. For fixed ϵ, the same argument in (d) works as well.

1.24 Problem 6

(a) So the notation ∇u is a little mysterious when u is complex valued, but if $u(x) = a(x) + ib(x)$, then I'll write

$$\nabla u(x) = \nabla a(x) + i\nabla b(x)$$

and so

$$E_\epsilon(u+t\varphi) = \int \frac{1}{2}(|\nabla u|^2 + t\overline{\nabla u}\cdot\nabla\varphi + t\nabla u\cdot\overline{\nabla\varphi}) + t^2|\nabla\varphi|^2) + \frac{1}{2\epsilon^2}(1-|u|^2 - t(\overline{u}\varphi + u\overline{\varphi}) - t^2|\varphi|^2)^2$$

the derivative at $t = 0$, is then

$$\int \frac{1}{2}(\overline{\nabla u}\cdot\nabla\varphi + \nabla u\cdot\overline{\nabla\varphi}) - \frac{1}{\epsilon^2}(1-|u|^2)(\overline{u}\varphi + u\overline{\varphi}) = \int(\frac{-1}{2}\Delta\overline{u} - \frac{1}{\epsilon^2}(1-|u|^2)\overline{u})\varphi + (\frac{-1}{2}\Delta u - \frac{1}{\epsilon^2}(1-|u|^2)u)\overline{\varphi} = 0$$

having used integration by parts. If we choose φ real, then these two terms add, and we get

$$\Delta Re(u) = \frac{2}{\epsilon^2} Re(u)(1 - |u|^2)$$

And if φ is chosen to be purely imaginary, then we get the same statement for $Im(u)$, so a necessary condition is that

$$\Delta u = \Delta Re(u) + i\Delta Im(u) = 2\epsilon^{-2} u(1 - |u|^2)$$

which is the analogous PDE for Allen Cahn. Moreover this is clearly sufficient if we look at the previous integral.

(b) Not sure. I want to do log-cutoff again because we're in two dimensions, but getting $-\log(\epsilon)$ is hard. I think one idea is to have $u \equiv 1$ for $r < 1 - f(\epsilon)$ and then twist u from 1 to $e^{i\theta}$ via some homotopy like construction using a partition of unity. I tried both a linear and log cut off and wasn't able to get it to work. Otis doesn't know the construction so I'm just going to leave this

(c) This is the same argument as in Evans chapter 8, i.e. theorem 2 on p. 470. The only thing needed to show is that

$$L(p, z, x) \geq \alpha |p|^q - \beta$$

for some $\alpha > 0$ and $1 < q < \infty$, where

$$L(p, z, x) = \frac{1}{2}|p|^2 + \frac{1}{2\epsilon}^2 (1 - |z|^2)^2 \geq \frac{1}{2}|p|^2$$

so this holds for $q = 2$, $\alpha = \frac{1}{2}$, and hence a minimizer exists.

(d) Need to balance a.e. pointwise convergence vs. L^1 convergence vs. convergence in measure... Can I construct a subsequence which converges to a function? Might need a finite energy bound, which doesn't make sense.

I need to use the fact that $\{u_\epsilon\}$ represent a sequence of minimizers. I think the idea is that for ϵ very small, if $||u_\epsilon(x)| - 1| > \delta$, then this condition must hold on some neighborhood and so bumping the function closer to ± 1 would decrease the potential term at a finite cost of the $|\nabla u|^2$ term. The problem is we want a uniform value for δ and a uniform bound from below on the size of the neighborhood of x, else the decrease in the potential function may not be comparable to the increase in the $|\nabla u|^2$ integral. So it should be some geometric measure theory decomposition where I let S be the set of points where $|u_\epsilon| \not\to 1$, and then decompose it as

$$S = \cup_k \{x \in S \mid \exists \{\epsilon_{j,x}\} \text{ s.t. } \Big| |u_{\epsilon_{j,x}}(x)| - 1 \Big| > 1/k\}$$

but I'm concerned about this. So Otis told me that actually we only care about pointwise convergence a.e. up to a subsequence. The way we get this is by noting that we get L^1 convergence because

$$\int (1 - |u_\epsilon|^2)^2 \leq \epsilon^2 E_\epsilon(u_\epsilon) \leq \epsilon^2 \alpha(1/\epsilon) + \epsilon^2 C$$

As $\epsilon \to 0$, this bound tends to 0, and so by some theorem this has a subsequence which converges almost everywhere. By the value of the integral, we get that this subsequence converges to ± 1 a.e.

(e) Suppose α_ϵ did not tend to infinity, i.e. there exist a subsequence of $\epsilon_j \to 0$ with $\alpha_{\epsilon_j} < C$ for C uniform in j. This gives a uniform bound on the H^1 norm of this sequence of functions, and so we can extract a subsequence which converges weakly in H^1 and strongly in L^2 See exercise 4.4. This limiting function u, will have bounded H^1 norm, be equal to 1 a.e. and satisfy $u(x)\Big|_{\partial B_1} = x$, a contradiction to the fact.

2 Problems to look into

2.1 Minimizers as a function of ϵ

The idea is to write

$$u_\epsilon(x) = u(\epsilon, x)$$

and then see if we can glean any information from this double parameterization. In particular, I think one can show that

$$\partial_\epsilon u(\epsilon, x)^2 \leq 0$$

i.e. the absolute value increases at any fixed point as $\epsilon \to 0$

Here are the computations I have

$$v(\epsilon, x) := u(\epsilon, \epsilon x)$$
$$\partial_\epsilon v = (\partial_0 u)(\epsilon, \epsilon x) + \sum_{j=1} \epsilon (\partial_i u)(\epsilon, \epsilon x)$$
$$\partial_\epsilon^2 v = (\partial_\epsilon^2 u)(\epsilon, \epsilon x) + 2\epsilon \sum_{j=1}^n (\partial_j \partial_0 u)(\epsilon, \epsilon x) + \sum_{i=1}^n (\partial_i u)(\epsilon, \epsilon x) + \epsilon^2 \sum_{k,\ell=1}^n (\partial_k \partial_\ell u)(\epsilon, \epsilon x)$$
$$(\partial_i \partial_\epsilon v) = \epsilon(\partial_i \partial_0 u)(\epsilon, \epsilon x) + \epsilon^2 \sum_{j=1}^n (\partial_i \partial_j u)(\epsilon, \epsilon x)$$
$$\partial_i^2 v = \epsilon^2 (\partial_i^2 u)(\epsilon, \epsilon x)$$

In retrospect, I don't think defining this function $v(\epsilon, x)$ is useful. The whole point of defining a v function is to get rid of ϵ dependency, so it's best to just investigate u on it's own. These look somewhat interesting, though dubious in quality

1. Paper on parameter dependency for parameters
2. Continuation on the above
3. Stack exchange post

After asking Otis about this, he doesn't think it's plausible and there are some example of $\mathbb{H}(\epsilon^{-1}\langle a, x - x_0\rangle + c(\epsilon))$ which he thinks would contradict a monotonicity property. It seems like the best way to do things is get monotonicity up to a subsequence.

3 Transformed Allen-Cahn

Otis says to consider

$$\varphi = \mathbb{H}^{-1}(u)$$

for $\epsilon = 1$. There's some interesting things that end up happening with this because $\mathbb{H}$ is a nice function

4 Pigati-Stern

Investigate solutions of abelian higgs renormalized to $\epsilon = 1$ so that we can do a local argument and take a solution on a chart to a solution on R^n. Would have to figure out how to renormalize in the first place and then see what could be extrapolated from global solutions

4.1 Set up

Recall the set up is as follows: M is a closed, oriented riemannin manifold with a hermitian structure $\langle \cdot, \cdot \rangle$, which may be denoted as g. We're investigating a complex line bundle

$$\begin{array}{c} L \\ \downarrow \\ M \end{array}$$

i.e. the fiber is $\mathbb{C}$. The equations we're interested in are the following

$$\nabla^* \nabla u = \frac{1}{2\epsilon^2}(1 - |u|^2)u$$
$$\epsilon^2 d^* \omega = \langle \nabla u, iu \rangle$$

In the first equation, we have that ∇^* is defined via the following equation: for $s \in \Gamma(L)$ (i.e $\Gamma(L) = C^\infty(M, L)$) and $V \in TM$, we have

$$\nabla_V^* s + \nabla_V s = -div(v)s$$

where

$$div(V) = trace(\nabla v) = \sum_i \langle \nabla_{e_i} v, f_i \rangle$$

where $\{e_i\}$ and $\{f_i\}$ are dual bases under the inner product. The symbol $\nabla^*\nabla := \sum_i \nabla_i^* \nabla_i$ is the Bochner laplacian and is required to be taken with respect to an orthonormal basis - I'm now abbreviating $\nabla_i := \nabla_{e_i}$ for some specified basis $\{e_i\}$. Choose $\{e_i\}$ to be geodesic normal coordinates (GNC), such that $\nabla_i e_j = 0$. See here and here for questions of existence (These definitions require a complex structure J it seems. Do we get this for free if we have $\langle \cdot, \cdot \rangle$ and g? Both of which we're given, but I'm unsure if the scope is the same as the paper). In this choice of basis, we have the particular nice identity

$$div(e^i)\Big|_x = 0 \implies \nabla_i^*\Big|_x = -\nabla_i\Big|_x$$

at our distinguished point. Thus

$$\nabla^*\nabla u = -\sum_i \nabla_i \nabla_i (u_k e^k) = -\sum_i \nabla_i \left(e^i(u_k) e^k + u_k \nabla_i e^k \right) = -\sum_i \nabla_i (e^i(u_k) e^k) = -\sum_i [e^i(e^i(u_k))] e^k$$

The definition of a Bochner-laplacian is independent of the orthonormal frame we choose. Actually this last line is not true because $\nabla_i \nabla_i e^k$ may not vanish! I don't end up using this later, so it's okay.

Overall, there are three things we can change: the section/function u, the connection ∇, and the metric/hermitian inner product $\langle \cdot, \cdot \rangle / g$

4.2 Working it out

Suppose we have a neighborhood $V \subseteq M$ and on the manifold and a chart $(\exp, U)$ such that $U \subseteq \mathbb{R}^n$ and $\exp : U \to V$ is a diffeomorphism for exp the exponential map in geodesic normal coordinates. Suppose $0 \in U$ and choose a $B_K(0) \subseteq U$. Define

$$f = \exp : B_K(0) \to V$$

Now consider $f^*\nabla = \tilde{\nabla}$ for ∇ the restriction of our connection to V. Then we have that for X a vector field on $B_K(0)$ the following

$$\tilde{\nabla}_X \tilde{u} = f^*(\nabla_{df(X)} u)$$

(see here) where we think of $\tilde{u}$ as a section of $f^*L\Big|_V \cong U \times \mathbb{C}$, i.e in the chart of sufficiently small size, the pullback of the bundle is trivial. This allows us to work in local coordinates.

Dropping the $\sim$'s, we can also consider

$$f_\epsilon = \exp_\epsilon : B_K(0) \to V \;\text{ s.t. }\; f_\epsilon(x) = \exp_\epsilon(x) = \exp(\epsilon x)$$
$$\tilde{u}_\epsilon = f_\epsilon^* u = u(\exp(\epsilon x))$$

Working in $B_K(0) \subseteq \mathbb{R}^n$ for simplicity, we ask what the connection of potential of $f_\epsilon^*\nabla = {}^\epsilon\nabla$ is. Through the defining property of pullback connection, one can shown that in local coordinates

$${}^\epsilon\nabla\Big|_x = d + \epsilon A\Big|_{\epsilon x}$$

This is the right thing needed to show that

$${}^\epsilon\nabla_i {}^\epsilon\nabla_i u_\epsilon \overset{?}{=} (1 - |u_\epsilon|^2) u_\epsilon$$

we calculate as follows

$$
\begin{aligned}
{}^\epsilon\nabla_i u_\epsilon\Big|_x &= du_\epsilon(e^i) + \epsilon A\Big|_{\epsilon x}(e^i) u_\epsilon = \epsilon \cdot e^i(u)(\epsilon x) + \epsilon A\Big|_{\epsilon x}(e^i) u_\epsilon(x) \\
{}^\epsilon\nabla_i {}^\epsilon\nabla_i u_\epsilon\Big|_x &= d\left(\epsilon \cdot e^i(u)(\epsilon x) + \epsilon A\Big|_{\epsilon x}(e^i) u_\epsilon(x) \right)(e^i) + \epsilon A\Big|_{\epsilon x}(e^i) \left(\epsilon e^i(u)(\epsilon x) + \epsilon A\Big|_{\epsilon x}(e^i) u_\epsilon(x) \right) \\
&= \epsilon^2 (\nabla_i \nabla_i u)(\epsilon x)
\end{aligned}
$$

the scaling is clear in the second term, and this is also clear in the first term if we write

$$A(e^i)(x) = a_i(x)$$

Here, we've specified a frame on $B_K(0)$. A is usually a matrix in coordinates, taking in a tangent vector and yielding an endomorphism of the section, but here our section is just a complex valued function, so we can think of $A \in \mathrm{Hom}(V, \mathbb{C})$, in which case it makes sense to define $a_i(x)$ with respect to some basis. **Important:** here, we're using $\{e^i\}$ as an orthonormal basis to calculate the bochner laplacian with both in the $B_K(0)$ picture and the pulled back picture (i.e. $\epsilon B_K(0)$). This indicates that we do **not** want to rescale the metric. Compare this to a natural rescaling of $g \to \epsilon^{-2} g$ accomodate $\{e^i\} \to \{df_\epsilon e^i\} = \{\epsilon e^i\Big|_{\epsilon x}\}$. However, this is what happens in exercise 5.1, so I think it's reasonable

An alternative way to do this is just use the property of pullback connections twice

$${}^{\epsilon}\nabla_{e^i} u_\epsilon = (f_\epsilon^* \nabla)_{e^i}(f_\epsilon^* u) = f_\epsilon^*(\nabla_{df_\epsilon e^i} u) = f_\epsilon^*(\nabla_{\epsilon e^i} u) = \ \epsilon f_\epsilon^*(\nabla_{e^i} u)$$

If we do this again, we get another factor of ϵ. To iterate again, we're using $\{e^i\}$ as a preferred orthonormal frame in both the standard and pulled back setting. Note that df_ϵ both scales e^i by ϵ and also changes it's base point to lie in $\epsilon B_K(0)$, which is where the computations are happening even though u is defined on more than just $\epsilon B_K(0)$. I'm not sure if this is analogous to Exercise 5.1.

For the second equation, we want

$$d^* \omega_\epsilon = g_\epsilon({}^{\epsilon}\nabla u_\epsilon, i u_\epsilon)$$

where ω_ϵ is the multiplicative factor from the curvature tensor after the pullback by f_ϵ. I calculate this as follows

$$v_\epsilon(x) = v(\epsilon x)$$
$$[{}^{\epsilon}\nabla_X, {}^{\epsilon}\nabla_Y]\, v_\epsilon = \epsilon^2 f_\epsilon^*([\nabla_X, \nabla_Y] v)$$
$${}^{\epsilon}\nabla_{[X,Y]} v_\epsilon = f_\epsilon^*(\nabla_{df_\epsilon [X,Y]} v) = \epsilon^2 f_\epsilon^*(\nabla_{[X,Y]} v)$$

The reason I write the first line is that any section on $B_K(0)$ can be written as the pull back of a section on $\epsilon B_K(0)$ I'm not sure if this is the right class of functions to be looking at, because while I certainly make this rescaled version for u, I want to record general information about $F_{{}^{\epsilon}\nabla}$ on arbitrary sections/$\mathbb{C}$-valued functions. The second line follows through essentially the same reasoning as with the Bochner-laplacian in the first part. The third line follows from

$$df_\epsilon[X, Y] = [df_\epsilon X, df_\epsilon Y] = [\epsilon X, \epsilon Y] = \epsilon^2 [X, Y]$$

however, if we just think of df_ϵ as ϵId, then we should only get one factor of ϵ? So for one, I need to be more careful about where these vectors are being evaluated and it should be at ϵx. I wasn't vareful about this before with the previous computation or the bochner-laplacian computation, but I don't think the point of evaluation comes into play in the prior work. It is important to think about with what basis are X and Y being expressed, i.e. in some standard basis $\{e^i\}$ before the pushforward or in another basis, $\{df_\epsilon e^i\} = \{\epsilon e^i\Big|_{\epsilon \cdot ()}\}$, i.e. the pushforward changes the point of evaluation and scales by ϵ. This should resolves things because when I write

$$df_\epsilon[X, Y](x) = \epsilon[X, Y](\epsilon x)$$

then here I'm expressing $[X, Y]$ with respect to the $\{\epsilon e^i\}$ basis, i.e. the pushforward basis. However, when I write

$$[df_\epsilon X, df_\epsilon Y] = \epsilon^2 [X, Y]$$

I'm expressing this with respect to the original basis, hence the extra factor of ϵ

With this, we see that $\omega_\epsilon\Big|_x = \epsilon^2 \omega\Big|_{\epsilon x}$, which can be thought of as $f_\epsilon^*(\omega)$ because

$$f_\epsilon^*(\omega)(Y, Z)\Big|_x = \omega\Big|_{\epsilon x}(df_\epsilon X, df_\epsilon Y) = \epsilon^2 \left(\omega(X, Y)\right)\Big|_{\epsilon x}$$

again, where $(df_\epsilon(X))(x) = \epsilon X(\epsilon x)$. Now we compute

$$\begin{aligned}
{}^\epsilon d^*\omega_\epsilon(e^k) &= -\sum_j ({}^\epsilon\nabla_j\omega_\epsilon)(e^j, e^k) = -\sum_j f_\epsilon^*(\nabla_{df_\epsilon e^j}\omega)(e^j, e^k) \\
&= -\epsilon\sum_j (\nabla_{e^j}\omega)\Big|_{\epsilon x}(df_\epsilon e^j, df_\epsilon e^k) = -\epsilon^3\sum_j (\nabla_{e^j}\omega)(e^j, e^k)\Big|_{\epsilon x} \\
&= \epsilon^3 f_\epsilon^*((d^*\omega)(e^k))(x) = \epsilon f_\epsilon^*(g(\nabla_{e^k}u, iu)) = \epsilon f_\epsilon^*(g(\nabla_{\epsilon e^k}u, iu)) \\
&= \sum_{j,k} g_{j,k}(\epsilon x)\,(\nabla_{\epsilon e^k}u)\,(\epsilon x)^j(iu)(\epsilon x)^k = \sum_{j,k} g_{j,k}(\epsilon x)(f_\epsilon^*(\nabla_{df_\epsilon e^k}u))(x)^j(iu_\epsilon)(x)^k \\
&= \sum_{j,k} g_{j,k}(\epsilon x)({}^\epsilon\nabla_{e^k}u_\epsilon))(x)^j(iu_\epsilon)(x)^k = {}^\epsilon\langle{}^\epsilon\nabla_{e^k}u_\epsilon, iu_\epsilon\rangle
\end{aligned}$$

here ${}^\epsilon\langle\cdot,\cdot\rangle = {}^\epsilon g = g\Big|_{\epsilon x}(\cdot,\cdot)$ is notated for consistency. The point is that this is what we want, and so a rescaled version of the ϵ-dependent Yang-Mills-Higgs energy works out.

4.2.1 Previous train of thought

The point is that we don't want to rescale the connection, else the following property:

$$\tilde{\nabla}_i^*\tilde{\nabla}_i\tilde{u} = -\tilde{\nabla}_i\tilde{\nabla}_i\tilde{u} = -f^*(\nabla_i\nabla_i u)$$

would mean that there's no rescaling resulting from differentiation (here $\tilde{u} = f^*u$ and one can see that if we replaced $\tilde{\nabla}$ with $\tilde{\nabla}_\epsilon$ and $\tilde{u}$ with $\tilde{u}_\epsilon$ then all of the differentiation would happen normally). Here, we use $_i$ to denote covariant differentiation with respect to some basis $\{\tilde{e}^i\}$ on $B_K(0)$ when paired with $\tilde{\nabla}$, as well as $\{e^i\} = \{df(\tilde{e}^i)\}$ when paired with ∇ on V. So while this identity is true, things would be different if we used ${}^\epsilon\tilde{\nabla}$ and $\tilde{u}_\epsilon$, because then $\{e^i\} = \{\widetilde{df_\epsilon e^i}\}$ but $df_\epsilon = \epsilon Id$, so the new basis is not orthonormal unless we rescale the connection by a factor of ϵ^{-2}. This factor of ϵ^{-1} from the metric could be useful! I also think that the metric is one extra variable that I haven't messed with, which could help accommodate the scaling. Actually, to any extraneous factor of ϵ we get from computations of ∇u_ϵ, we should be able to make up for these factors by messing with the metric.

We now calculate

$$\begin{aligned}
\tilde{\nabla}_i\tilde{u}_\epsilon(x) &= \tilde{\nabla}_i(\tilde{u}(\epsilon x)) = f^*(\nabla_i u)(\epsilon x)\cdot\epsilon \\
\tilde{\nabla}_i\tilde{\nabla}_i\tilde{u}_\epsilon(x) &= f^*(\nabla_i\nabla_i u)(\epsilon x)\epsilon^2 = f^*(\epsilon^{-2}(1-|u|^2)u)(\epsilon x)\epsilon^2 = (1-|\tilde{u}_\epsilon|^2)\tilde{u}_\epsilon
\end{aligned}$$

because we're working in charts so that our bundles are trivial, we can pass the multiplicative factors of ϵ through the pullback.

For the other equation, we again note that the connection is pulled back but note rescaled. However, the scaling factors are kind of poor for this situation, because we have

$$\langle\tilde{\nabla}\tilde{u}_\epsilon, i\tilde{u}_\epsilon\rangle = \epsilon\langle(\tilde{\nabla}\tilde{u})(\epsilon x), i\tilde{u}(\epsilon x)\rangle = \epsilon^3 d^*\omega\Big|_{\epsilon x}$$

even this last equality doesn't really make sense, as we don't know where to evaluate the form at, and hence, from which tangent space we should input vectors. Note that in the above, we may also want to pull back the hermitian inner product, though this is confusing end previous train of thought

4.3 Potentially useful links

1. pushforward and lie bracket
2. pull back connection
3. ref on differentials/pushforwards
4. Hermitian Structures - good to know when rescaling the hermitian inner product

4.4 Easy Extensions

Going through Pigati-Stern, I note the following equalities which rely on some global bound, i.e. closedness of the manifold

1. (3.7) - though this can be salvaged if we have some uniform bound on the Weitzenbock curavture (notated as $\mathcal{R}_2$)
2. (4.10) - for its reliance on (3.7)
3. P. 12 - Hessian Comparison theorem on bottom of the page. Don't know exactly what this is but I found the following links on stackexchange which might indicate salvagability because the sectional curvature is trivial in euclidean space
4. Anything relying on (4.2)

Something which only relies on local arguments

5 Radially Symmetric solutions to Allen-Cahn

Otis once said that the laplacian has no directional preference, which is true. Neither does the allen-cahn equation on $\mathbb{R}^n$, so maybe it's good to look for solutions this way

Didn't get anything from the PDE (i.e. transforming the Laplacian), but note that if we let $A = H_0^1(B_1)$ then the minimizer must be single signed because of the same reasoning as in exercise 2.5 part a). Now via the result here, which is a standard result in PDE about radially symmetric functions and also Evan's chapter 9 p. 556, we get that the solution to Allen-Cahn with Dirichlet conditions on the ball is radially symmetric. Nice. I think analogously to exercise 2.5, we can construct a radial solution on $\mathbb{R}^2$ - worried that the limiting function would be $\equiv 1$, as this is compatible with the energy growth

Something that would be interesting to show is that the only solution to Allen-Cahn on $\mathbb{R}^n$ **that is radially symmetric** is just constant functions, so in particular 0, ± 1. Would have to show that the operator, which is degenerate elliptic operator, has zero dimensional kernel

Also wondering if the mellin transformation could be good for this. Not sure if $u(u^2-1)$ has a nice mellin transformation, but the laplacian sure does