Solutions

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Question 1: This is easy: if A is a closed, convex set, and $x \in A^c$, then Hahn Banach second geometric form tells us that there exists a hyperplane strictly separating x and A. So A^c is weakly open. $\square$

Question 2: Let K be our compact set. Let $\beta \in [-\infty, \infty), \beta = \inf_{x \in K} f(x)$. Assume towards a contradiction that $\beta \notin f(K)$ i.e. $\beta = -\infty$ or $\beta \in \mathbb{R} \setminus f(K)$. Then take a sequence x_n such that $f(x_n) \downarrow \beta$. Let $V_n = f^{-1}((f(x_n), \infty))$. Since $\beta \notin f(K)$, $\bigcup_n V_n$ is an open cover of $f(K)$ which cannot have a finite subcover. This is clearly a contradiction. $\square$

Question 3: We want to show that $\{f \leq a\}$ is weakly closed for all $a \in \mathbb{R}$. Since f is continuous, we know that $\{f \leq a\}$ is closed. Thanks to the result of Question 1, all we have left to show is that $\{f \leq a\}$ is convex. But this comes immediately from the convexity of f: if $x, y \in f^{-1}((-\infty, a])$, $t \in (0, 1)$, then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \leq ta + (1-t)a = a. \quad \square$$

Question 4: Since f is coercive, we know that

$$\inf_{x \in X^*} f(x) = \inf_{x \in \overline{B_R(0)}} f(x)$$

for some large enough R. Since f is weak-* lower semicontinuous, and $\overline{B_R(0)}$ is weak-* compact (Banach-Alaoglu), Question 2 lets us conclude that f attains its minimum on $\overline{B_R(0)}$. $\square$

Question 5: We will prove a slightly more general result: let $A \subset X$ be a closed, convex, and unbounded set. Let $f : A \to \mathbb{R}$ be continuous, convex, and coercive. Since f is coercive, we know that

$$\inf_{x \in A} f(x) = \inf_{x \in \overline{B_R(0)} \cap A} f(x)$$

for some large enough R. From Question 3 (it clearly applies to f defined on convex, closed subsets of X given the subspace topology inherited from the weak topology in X), we know that f is lower semi continuous with respect to the weak topology. Since X is reflexive, we know that $\overline{B_R(0)}$ is weakly compact (Kakutani's Theorem). Since $\overline{B_R(0)} \cap A$ is weakly closed (Question 1), we conclude that $\overline{B_R(0)} \cap A$ is weakly compact. Our result now follows from Question 2. $\square$

Question 6: For $f \in L^1(\mathbb{R})$ let $\lambda_n(f) = \int_n^\infty f dx + \int_{-\infty}^{-n} f dx$. Note that for all n we have $|\lambda_n(f)| \leq ||f||_{L_1}$. Now define the function $\phi : L^1(\mathbb{R}) \to \mathbb{R}$ by:

$$\phi(f) = ||f||_{L^1} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (\lambda_n(f) - 1)^2 \tag{0.0.1}$$

We will verify that ϕ is (i) continuous, (ii) convex, (iii) coercive, and (iv) does not attain its minimum.
(i) Continuity. It follows from the above that for all f, g, n we have $|\lambda_n(f) - \lambda_n(g)| = |\lambda_n(f-g)| \leq ||f-g||_{L^1}$. Hence, when $||f-g||_{L^1} < \epsilon < 1$ we have $|(\lambda_n(f)-1)^2 - (\lambda_n(g))^2| < 3|\lambda_n(f-g)| \leq 3||f-g||_{L^1}$ and thus:

$$|\phi(f) - \phi(g)| < 4||f-g||_{L^1} \tag{0.0.2}$$

and thus ϕ is continuous.
(ii) Convexity. Let $h = tf + (1-t)g$ for $t \in [0,1]$ and $f, g \in L^1$. Clearly $\lambda_n(h) = t\lambda_n(f) + (1-t)\lambda_n(g)$ for all n. Thus, by the convexity of $q(x) = (x-1)^2$ we have $(\lambda_n(h)-1)^2 \leq t(\lambda_n(f)-1)^2 + (1-t)(\lambda_n(g)-1)^2$. Combining this with the triangle inequality yields the convexity of ϕ:

$$\phi(h) \leq t\phi(f) + (1-t)\phi(g) \tag{0.0.3}$$

(iii) Coercivity. ϕ is bounded below by the norm so is trivially coercive.
(iv) ϕ doesn't attain its minimum. I claim that $\inf_{f \in L^1} \phi(f) = \frac{3}{4}$. First, the sequence of functions $f_n = \frac{1}{2}\chi_{[n,n+1]}$ has $\phi(f_n) = \frac{3}{4}(1 + \frac{1}{2^n}) \to \frac{3}{4}$. Now, all that remains to show is that $\phi(f) > \frac{3}{4}$ for all $f \in L^1$. We do this case-by case:
Case 1: $||f|| > \frac{3}{4}$. Since $\phi(f) \geq ||f||_{L^1}$ for all $f \in L^1(\mathbb{R})$ we clearly have the stated claim.
Case 2: $||f|| \leq \frac{3}{4}$. We clearly have $\lambda_n(f) \leq ||f||$ for all f. Note that this inequality is strict for infinitely many n. Now since $q(x) = (x-1)^2$ is decreasing on the interval $(-\infty, 1)$ we have for $||f|| \leq \frac{3}{4}$:

$$\phi(f) > ||f||_{L^1} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (||f||_{L^1} - 1)^2 = ||f||^2 - ||f|| + 1 \geq \frac{3}{4} \tag{0.0.4}$$

and thus the result is proved. $\square$

Question 7: We will start this problem with a linear algebra result that I verified on the internet.

Proposition: A quadratic form $\langle A\cdot, \cdot\rangle$ on $\mathbb{R}^n$ is convex if and only if $A \geq 0$.

Proof: Diagonalize A with an orthonormal eigenbasis $e_1, \ldots, e_n$. With respect to this basis we have

$$\langle Ax, x\rangle = \sum_{i=1}^{n} \lambda_i x_i^2$$

where λ_i are the corresponding eigenvalues. If $\lambda_j < 0$ for some j, then for any $t \in (0,1)$ we get

$$\langle A(te_j + 0), (te_j + 0)\rangle = t^2\lambda_j > t\lambda_j = t\langle Ae_j, e_j\rangle + (1-t)\langle A0, 0\rangle.$$

For the other direction, if should be sufficiently obvious that our quadratic form is convex when all of the eigenvalues are greater than or equal to zero. $\square$

In this problem, $T : H^1(\Omega) \to L^2(\partial\Omega)$ will denote the trace operator, which is a continuous, surjective linear operator whose kernel is $H_0^1(\Omega)$ (since Ω is a bounded, Lipschitz domain).

We know that the set

$$K = \{u \in H^1(\Omega) : Tu = Tf\} = f + H_0^1(\Omega)$$

is obviously a closed, convex set.

Claim: $J : K \to \mathbb{R}$ is convex, continuous, and coercive.

We will first show that J is coercive. Let $A(x)$ denote the symmetric matrix $\big(a_{ij}(x)\big)$. We compute that for any $u \in K$ we have

$$\begin{aligned} J(u) = \int_\Omega \langle A(x)\nabla u(x), \nabla u(x)\rangle \, dx + \langle g, u\rangle_{L^2} &\geq \int_\Omega \lambda |\nabla u(x)|^2 \, dx + \langle g, u\rangle_{L^2} \\ &= \langle g, u\rangle_{L^2} + \lambda \|\nabla u\|_2^2. \end{aligned}$$

Letting $v = u - f \in H_0^1(\Omega)$, we compute that

$$\begin{aligned} \|\nabla u\|_2^2 &= \|\nabla v\|_2^2 + 2\int \nabla v \cdot \nabla f \, dx + \|\nabla f\|_2^2 \\ &\geq \|\nabla v\|_2^2 - 2\|\nabla v\|_2 \|\nabla f\|_2 + \|\nabla f\|_2^2 = (\|\nabla v\| - \|\nabla f\|)^2. \end{aligned}$$

Recall that Ω is a bounded domain. We know from Poincaré's Inequality that there exists $C > 0$ such that $\|\nabla h\|_2 \geq C\|h\|_{H^1}$ for all $h \in H_0^1(\Omega)$. It follows that for all $u \in K$ with $\|u\|_{H^1} \geq \|f\|_{H^1} + \frac{1}{C}\|\nabla f\|_2$ we get

$$\|\nabla v\|_2 \geq C\|v\|_{H^1} \geq C(\|u\|_{H^1} - \|f\|_{H^1}) \geq \|\nabla f\|_2,$$
$$\Rightarrow \|\nabla u\|_2^2 \geq (\|\nabla v\| - \|\nabla f\|)^2 \geq (C\|u\|_{H^1} - [C\|f\|_{H^1} + \|\nabla f\|_2]\,)^2.$$

It follows that for all $u \in K$ with $\|u\|_{H^1} \geq \|f\|_{H^1} + \frac{1}{C}\|\nabla f\|_2$ we have

$$J(u) \geq \lambda(C\|u\|_{H^1} - [C\|f\|_{H^1} + \|\nabla f\|_2]\,)^2 - \|g\|_2\|u\|_2.$$

So J is coercive.

The continuity of J follows easily from the fact that $\|\cdot\|_{H^1}$ and the norm $\|\cdot\|_2 + \|\nabla\cdot\|_2$ are equivalent. Just look at the expression.

To show convexity, note that $J = \langle g, \cdot\rangle_{L^2} + \int_\Omega \langle A(x)\nabla\cdot, \nabla\cdot\rangle\, dx$, and $\langle g, \cdot\rangle_{L^2}$ is a convex function. Since quadratic forms on $\mathbb{R}^d$ are convex if and only the corresponding matrix is positive semidefinite, and $A(x)$ is positive definite for all $x \in \Omega$, it follows that

$$\int_\Omega \langle A(x)\nabla\cdot, \nabla\cdot\rangle\, dx$$

is a convex function on $H^1(\Omega)$. Therefore, J is the sum of two convex functions and is itself convex.

Claim: J attains its minimum on K.

This follows directly from our result in Question 5.

Claim: Let u be a point in K where J attains its minimum. Then

$$\nabla \cdot A\nabla u = \frac{1}{2}g$$

in the distributional sense.

We can rewrite K as $u + H_0^1(\Omega)$. Then we know that for all $v \in H_0^1(\Omega)$ we have

$$J(u+v) - J(u) = \langle g, v\rangle_{L^2} + 2\int_\Omega \langle A(x)\nabla u(x), \nabla v(x)\rangle\, dx + \int_\Omega \langle A(x)\nabla v(x), \nabla v(x)\rangle\, dx \geq 0.$$

We also know that the function $\phi_v : \mathbb{R} \to \mathbb{R}, \phi_v(t) = J(u + tv)$ always attains it global minimum at $t = 0$. Since we know that

$$\phi_v(t) = t^2\int_\Omega \langle A(x)\nabla v(x), \nabla v(x)\rangle\, dx + t\left[\langle g, v\rangle_{L^2} + 2\int_\Omega \langle A(x)\nabla u(x), \nabla v(x)\rangle\, dx\right],$$

setting $\phi_v'(0) = 0$ yields

$$\langle g, v \rangle_{L^2} = -2 \int_\Omega \langle A(x)\nabla u(x), \nabla v(x) \rangle \, dx.$$

If we let $\big(A(x)\nabla u(x)\big)_i$ denote the i^{th} column of $A(x)\nabla u(x)$, then we compute that

$$\int_\Omega g(x)v(x)\, dx = -2 \sum_{i=1}^d \int_\Omega \big(A(x)\nabla u(x)\big)_i \frac{\partial v}{\partial x_i}\, dx$$

$$= 2 \sum_{i=1}^d \int_\Omega \partial_i \big(A(x)\nabla u(x)\big)_i v(x)\, dx = 2 \int_\Omega \big(\nabla \cdot A(x)\nabla u(x)\big) v(x)\, dx.$$

The integration by parts worked for arbitrary $v \in H_0^1(\Omega)$ because $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. □

Remark: The last observation in the previous proof gives a short proof that $H^1(\Omega) \neq H_0^1(\Omega)$: passing to the limits from functions $v \in C_c^\infty(\Omega)$, we see that

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i} v\, dx \quad \forall\, u \in H^1(\Omega), v \in H_0^1(\Omega),$$

but this obviously cannot hold for all $v \in H^1(\Omega)$, so $H_0^1(\Omega) \neq H^1(\Omega)$. An obvious counterexample when Ω is bounded is to consider the constant functions on Ω.

Question 8: We know that the intersection of two closed, convex sets is itself closed and convex. We know that

$$\{u \in H^1(\Omega) : u - f \in H_0^1(\Omega) \text{ and } u \geq \phi\, a.e. \text{ in } \Omega\} = \big(f + H_0^1(\Omega)\big) \cap \{u \in H_0^1(\Omega) : u \geq \phi\, a.e.\}.$$

$f + H_0^1(\Omega)$ is the translation of a closed linear subspace, so it is closed and convex. $\{u \in H_0^1(\Omega) : u \geq \phi \; a.e.\}$ is closed and convex because L^2 convergence implies a pointwise a.e. convergent subsequence. □

Question 9: Let us first introduce some notation:

$$S = f + H_0^1(\Omega)$$

$$S' = \{u \geq 0 \; a.e.\} \cap S$$

Note that since S and $\{u \geq 0 \; a.e.\}$ are closed and convex, $S' \subset S$ is a closed, convex set. Furthermore, define

$$J : S \to \mathbb{R}, \quad J(u) = \int_\Omega |\nabla u|^2 + u_+ \, dx$$

$$J' : S' \to \mathbb{R}, \quad J'(u) = \int_\Omega |\nabla u|^2 + u \, dx.$$

Since $Tf \geq 0$, $S' \neq \emptyset$. Notice that $J\big|_{S'} = J'$.

Claim: J is coercive.

Let D denote the Dirichlet energy. We know that $J(u) \geq D(u)$ for all $u \in S$, and that D is coercive on S (see Appendix). It follows that J is coercive.

Claim: J is both continuous, and strictly convex, and therefore attains a unique minimum on S.

The continuity of J is obvious.

To show strict convexity, we first note that $\frac{u+v}{2}_+ \leq \frac{u+v_+}{2} \leq \frac{u_+ + v_+}{2}$ for any u, v measurable, simply because $u_+ \geq u$ everywhere. Second, we note that if $x, y \in \mathbb{R}^n$, $x \neq y$, then since $f(x) = x^2$ is a C^2 function whose derivative is positive everywhere, we have

$$\left|\frac{x+y}{2}\right|^2 = \sum_{i=1}^n (\frac{x_i + y_i}{2})^2 < \frac{1}{2} \sum_{i=1}^n x_i^2 + y_i^2.$$

Let $u, v \in f + H_0^1(\Omega)$, $u \neq v$. Since $u - v \in H_0^1(\Omega)$, we know that $u - v$ cannot be a constant function. It follows that $\nabla u \neq \nabla v$ on a set of positive measure. From this, and the above facts, we compute that

$$\begin{aligned} J(\frac{u+v}{2}) &= \int_\Omega \left|\frac{\nabla u + \nabla v}{2}\right|^2 + \frac{u+v}{2}_+ \, dx \\ &\leq \int_\Omega \left|\frac{\nabla u + \nabla v}{2}\right|^2 + \frac{u_+ + v_+}{2} \, dx \\ &< \int_\Omega \frac{1}{2}\big|\nabla u\big|^2 + \frac{1}{2}\big|\nabla v\big|^2 + \frac{u_+ + v_+}{2} \, dx = \frac{J(u) + J(v)}{2}. \end{aligned}$$

It follows (see Appendix) that J attains a unique minimum on S.

Claim J attains is minimum over S in S'.

For all $u \in S$, $u_+ = \min(u, 0) \in H^1(\Omega)$ (this comes from the result of Question 10). Since $f \geq 0$, we know that $Tu = f = Tu_+$ for all $u \in S$. Therefore, for all $u \in S$, there exists

$u_+ \in S'$ such that $J(u_+) \leq J(u)$.

Claim: J and J' attain their minimums at the same function.

J attains its minimum in S', and $J\big|_{S'} = J'$. ☺

Now all that remains is to compute the Euler-Lagrange equation for the minimizer. So, let u^* be the minimizer of J over A_1. Let $\varphi \in C_c^\infty(\mathbb{R})$, and define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = J(u^* + t\varphi)$. Then

$$\begin{aligned} F(t) &= \int_\Omega |\nabla(u^* + t\varphi)|^2 + \max\{u^* + t\varphi, 0\}dx \\ &= J(u^*) + \int_\Omega 2t\nabla u^* \cdot \nabla\varphi + (\max\{u^* + t\varphi, 0\} - u_+^*)dx + t^2 \int_\Omega |\nabla\varphi|^2 dx. \end{aligned} \tag{1.0.30}$$

The most confusing term is $(\max\{u^* + t\varphi, 0\} - u_+^*)$. By direct calculation, we have that

$$(\max\{u^* + t\varphi, 0\} - u_+^*)(x) = \begin{cases} t\varphi(x), & u^*(x) > -t\varphi(x) \\ -u^*(x), & u^*(x) \leq -t\varphi(x) \end{cases}. \tag{1.0.31}$$

Thus since $\varphi \in L^\infty$, dividing by t and taking the limit as $t \to 0$ we get that

$$\frac{(\max\{u^* + t\varphi, 0\} - u_+^*)(x)}{t} \to \begin{cases} \varphi(x), & u^*(x) > 0 \\ 0, & u^*(x) = 0 \end{cases}, \tag{1.0.32}$$

pointwise almost everywhere. Thus using the fact that F has a minimum at $t = 0$ and dominated convergence, it follows that

$$0 = \frac{d}{dt}F(t)\bigg|_{t=0} = \lim_{t\to 0} \frac{F(t) - F(0)}{t} = \int_\Omega 2\nabla u^* \cdot \nabla\varphi + \varphi(x)\chi_{\{u>0\}}dx. \tag{1.0.33}$$

Since this holds for every $\varphi \in C_c^\infty(\Omega)$, we thus have that u^* satisfies

$$\Delta u^* = \frac{1}{2}\chi_{\{u^*>0\}}, \tag{1.0.34}$$

in the sense of distributions.

Question 10: Let $u, v \in H^1(\Omega)$.

Answer of exercise 10

Since $\min\{u, v\} = u - (u - v)_+$, it suffices to prove that $u \in H^1(\Omega) \Rightarrow u_+ \in H^1(\Omega)$.

We know that if $G : \mathbb{R} \to \mathbb{R}$ is C^1 with bounded derivative, then $G(u) \in H^1(\Omega)$ for all $u \in H^1(\Omega)$ with $\nabla(G(u)) = G'(u)\nabla u$ (You should have seen this proof in graduate functional analysis. You can find it in Brezis at least).

With that in mind, for each $\epsilon > 0$ we define G_ϵ by

$$G_\epsilon(t) = \begin{cases} t - \epsilon/2, & t \geq \epsilon \\ t^2/2\epsilon, & 0 \leq t \leq \epsilon \\ 0, & t \leq 0 \end{cases}. \tag{1.0.35}$$

Thus $G_\epsilon(u) \in H^1(\Omega)$ for all $\epsilon > 0$ with $||G_\epsilon(u)||_{H^1} \leq ||u||_{H^1}$.

It's clear that $G_\epsilon(u)(x) \to u_+(x)$ and $\nabla(G_\epsilon(u))(x) = G'_\epsilon(u)\nabla u(x) \to \chi_{\{u>0\}}(x)\nabla u(x)$ pointwise. Since both sequences $\{G_\epsilon(u)\}_{\epsilon>0}$ and $\{G'_\epsilon(u)\nabla u(x)\}_{\epsilon>0}$ are bounded in $L^2(\Omega)$, we know that along some subsequence $\epsilon_i \to 0$, we must have that $G_{\epsilon_i}(u) \rightharpoonup \varphi \in L^2(\Omega)$, $G'_{\epsilon_i}(u)\nabla u \rightharpoonup \Phi \in L^2(\Omega; \mathbb{R}^d)$ for some functions φ, Φ. But since we knew what these sequences already converged pointwise, we thus have that $G_{\epsilon_i}(u) \rightharpoonup u_+$ and $G'_{\epsilon_i}(u)\nabla u \rightharpoonup \chi_{\{u>0\}}\nabla u$.

So, all that remains to do is to prove that $u_+ \in H^1(\Omega)$ is to prove that $\nabla(u_+) = \chi_{\{u>0\}}\nabla u$ in the sense of distributions. So, let $\varphi \in C_c^\infty(\Omega; \mathbb{R}^d)$ be arbitrary. Then by definition of weak convergence, we have that

$$\int_\Omega u_+ \operatorname{div}\varphi dx = \lim_{\epsilon_i \to 0} \int_\Omega G_{\epsilon_i}(u) \operatorname{div}\varphi dx = \lim_{\epsilon_i \to 0} -\int_\Omega G'_{\epsilon_i}(u)\nabla u \cdot \varphi dx = -\int_\Omega \chi_{\{u>0\}}\nabla u \cdot \varphi dx. \tag{1.0.36}$$

Thus $\nabla(u_+) = \chi_{\{u>0\}}\nabla u$, so we're done.

Question 11:

Answer of exercise 11

(a) Let $\rho \in C_c^\infty(B_1)$ be a smooth, symmetric mollifier. That is, $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho = 1$, and $\rho(x) = \rho(-x)$.

For each $\epsilon > 0$, define $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$, $u_\epsilon = u * \rho_\epsilon$, $v_\epsilon = v * \rho_\epsilon$, and $\Omega_\epsilon = \{x \in \Omega | B_\epsilon(x) \subseteq \Omega\}$.

We claim then that u_ϵ, v_ϵ are superharmonic in Ω_ϵ. To see this, note that for any $\varphi \in C_c^\infty(\Omega_\epsilon)$ that

$$\int_{\mathbb{R}^d} u_\epsilon \Delta\varphi dx = \int_{\mathbb{R}^d} (u * \rho_\epsilon)\Delta\varphi dx = \int_{\mathbb{R}^d} u(\rho_\epsilon * \Delta\varphi) dx = \int_{\mathbb{R}^d} u\Delta(\rho_\epsilon * \varphi) dx \leq 0, \tag{1.0.37}$$

and similarly for v_ϵ.

Since u_ϵ, v_ϵ are smooth superharmonic functions, it follows that $\Delta u_\epsilon(x), \Delta v_\epsilon(x) \leq 0$ pointwise everywhere in Ω_ϵ. Now define $U_\epsilon = \{u_\epsilon < v_\epsilon\}$ and $V_\epsilon = \{v_\epsilon < u_\epsilon\}$, and $w_\epsilon = \min\{u_\epsilon, v_\epsilon\}$.

If $U_\epsilon = \emptyset$ (or $V_\epsilon = \emptyset$), then $w_\epsilon = v_\epsilon$ (or u_ϵ resp.) and hence is a supersolution. So, assume $U_\epsilon, V_\epsilon \neq \emptyset$. Since Ω is connected, it then follows that $\Gamma_\epsilon := \{u_\epsilon = v_\epsilon\} \neq \emptyset$ and that $\Omega = U_\epsilon \cup \Gamma_\epsilon \cup V_\epsilon$.

Since u_ϵ, v_ϵ are smooth, by Sard's theorem we have that generically Γ_ϵ is a smooth, hypersurface with $\Omega_\epsilon \cap \partial U_\epsilon = \Omega_\epsilon \cap \partial V_\epsilon = \Gamma_\epsilon$ (Note that if 0 was a critical value of $u_\epsilon - v_\epsilon$, we still have that δ is not for arbitrarily small $\delta > 0$. Thus we can repeat the same argument for any arbitrary sequence of $\delta \to 0$.) Orient Γ_ϵ so that it's "outward" unit normal is $\hat{n} = \frac{\nabla(u_\epsilon - v_\epsilon)(x)}{|\nabla(u_\epsilon - v_\epsilon)(x)|}$.

On the set U_ϵ, we have that $w_\epsilon = u_\epsilon$ is smooth with $\Delta w_\epsilon(x) \leq 0$, and similarly on V_ϵ. Thus using Green's identities, we have that for any $\varphi \in C_c^\infty(\Omega_\epsilon)$ with $\varphi \geq 0$ that

$$\begin{aligned}
\int_{\Omega_\epsilon} w_\epsilon \Delta\varphi dx &= \int_{U_\epsilon} u_\epsilon \Delta\varphi dx + \int_{V_\epsilon} v_\epsilon \Delta\varphi dx \\
&= \left(\int_{U_\epsilon} \Delta u_\epsilon \varphi dx + \int_{\Gamma_\epsilon} u_\epsilon \hat{n} \cdot \nabla\varphi - \varphi\hat{n} \cdot \nabla u_\epsilon dH^{d-1}\right) + \left(\int_{V_\epsilon} \Delta v_\epsilon \varphi dx - \int_{\Gamma_\epsilon} v_\epsilon \hat{n} \cdot \nabla\varphi - \varphi\hat{n} \cdot \nabla v_\epsilon dH^{d-1}\right) \\
&\leq \int_{\Gamma_\epsilon} (u_\epsilon - v_\epsilon)\hat{n} \cdot \nabla\varphi - \varphi\hat{n} \cdot (\nabla u_\epsilon - \nabla v_\epsilon) dH^{d-1} \\
&= -\int_{\Gamma_\epsilon} \varphi |\nabla u_\epsilon - \nabla v_\epsilon| dH^{d-1} \leq 0.
\end{aligned} \tag{1.0.38}$$

Thus since this holds true for all $\varphi \in C_c^\infty(\Omega_\epsilon)$ with $\varphi \geq 0$, we have that w_ϵ is superharmonic.

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As $u_\epsilon, v_\epsilon \to u, v$ in $L^1_{loc}(\Omega)$ as $\epsilon \to 0$, it follows that $w_\epsilon \to \min\{u, v\}$ in L^1_{loc} as well. Thus

$$\int_\Omega \min\{u, v\}\Delta\varphi dx = \lim_{\epsilon \to 0} \int_\Omega w_\epsilon \Delta\varphi dx \leq 0, \tag{1.0.39}$$

for any nonnegative $\varphi \in C_c^\infty(\Omega)$, so $\min\{u, v\}$ is superharmonic.

Answer of exercise 11

(b) Let $w = \min\{u, v\} \in H^1(\Omega)$ by exercise 10. Let $A = \{f \in H^1(\Omega) | f - w \in H_0^1(\Omega), f \geq w\}$ and J be the functional

$$J(f) = \int_\Omega \nabla f \cdot a_{ij}(x) \nabla f dx. \tag{1.0.40}$$

By exercises 7 and 8, we have that there exists a minimizer f^* of J over A. For any $\varphi \in H_0^1(\Omega)$ and $t \geq 0$, $f^* + t\varphi \in A$ and hence $J(f^* + t\varphi) \geq J(f^*)$. It thus follows that

$$0 \leq \lim_{t \to 0+} \frac{J(f^* + t\varphi) - J(f^*)}{t} = \int_\Omega \nabla\varphi \cdot a_{ij}(x) \nabla f^* dx, \tag{1.0.41}$$

so

$$\partial_i a_{ij}(x) \partial_j f^* \leq 0, \tag{1.0.42}$$

in the sense of distributions. Thus it suffices to prove that $f^* = w$.

Since $f^* \geq w$ by construction, we just need to show that $f^* \leq w \iff f^* \leq u$ and $f^* \leq v$.

To start proving this,

Claim: If $\varphi \in H_0^1(\Omega)$ is such that $\varphi(x)\chi_{\{f^*=w\}}(x) \equiv 0$, then

$$\int_\Omega \nabla\varphi \cdot a_{ij}(x) \nabla f^* dx = 0. \tag{1.0.43}$$

Given the claim, consider the function $f^* - \min\{u, f^*\} \in H_0^1(\Omega)$. It's clear that $f^* - \min\{u, f^*\} \geq 0$, and that $(f^* - \min\{u, f^*\})\chi_{\{f^*=w\}} \equiv 0$, since $f^* = w \Rightarrow u \geq f^*$. Thus using the claim and the fact that u is a supersolution, we have that

$$\begin{aligned}
0 &\leq \int_\Omega \nabla(f^* - \min\{u, f^*\}) \cdot a_{ij}(x) \left(\nabla u - \nabla f^*\right) dx \\
&= \int_\Omega \nabla(f^* - \min\{u, f^*\}) \cdot a_{ij}(x) \left(\nabla \min\{u, f^*\} - \nabla f^*\right) \\
&\leq -\lambda ||\nabla(f^* - \min\{u, f^*\})||^2_{L^2(\Omega)} \leq 0.
\end{aligned} \tag{1.0.44}$$

Thus we must have that $f^* - \min\{u, f^*\} = 0$, so $f^* \leq u$. The same argument works for v, so we thus have $f^* \leq w \Rightarrow w = f^*$ is a supersolution.

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Question 12: From the last pset, we were asked us to a superharmonic function in $L^\infty(B_1)$ function which is lower semi-continuous but not continuous in $\mathbb{R}^d$ for $d \geq 3$. Define

$$u_k(x) = \min(1, 2^{-k(d-1)}|x - \frac{1}{k}e_1|^{-(d-2)})$$

Then we have that

$$|x - \frac{1}{k}e_1| \geq 2^{-k} \implies |x - \frac{1}{k}e_1|^{-(d-2)} \leq 2^{k(d-2)}$$

$$\implies 2^{-k(d-1)}|x - \frac{1}{k}e_1|^{-(d-2)} \leq 2^{-k}$$

Now consider $x \in \mathbb{R}^d$ with $d \geq 3$, I claim that there exists at most 1 value of $k \in \mathbb{N}$ such that

$$|x - \frac{1}{k}e_1| < 2^{-k}$$

The proof is that given that there is at least 1 such k, then we show that $1/(k \pm 1)$ is far away enough so that $|x - 1/(k+j))| > 2^{-k}$ for all other values of $j \in \mathbb{N}$ such that $k + j > 0$. Note that

$$|x - \frac{1}{k}e_1| \implies |x - \frac{1}{k \pm 1}e_1| = |x - \frac{1}{k}e_1 + \frac{1}{k}e_1 - \frac{1}{k \pm 1}e_1|$$

$$\geq |\frac{1}{k} - \frac{1}{k \pm 1}| - |x - \frac{1}{k}e_1| \geq |\frac{1}{k} - \frac{1}{k \pm 1}| - 2^{-k}$$

yet

$$\left|\frac{1}{k} - \frac{1}{k \pm 1}\right| = \frac{1}{k(k \pm 1)} > 2^{-k+1} \quad \forall k \geq 7$$

$$\implies |x - \frac{1}{k \pm 1}e_1| > 2^{-k+1} - 2^{-k} = 2^{-k}$$

Note that

$$\left|\frac{1}{k} - \frac{1}{k+j}\right| = \frac{j}{k(k+j)} = |j|\frac{k \pm 1}{k+j}\frac{1}{k(k \pm 1)} > \frac{1}{k(k \pm 1)}$$

because for

$$2 \leq |j| \leq k - 2 \implies |j|\frac{k \pm 1}{k+j} \geq 2\frac{k-1}{2k-2} = 1$$

$$|j| \geq k - 1 \implies |j|\frac{k \pm 1}{k+j} \geq \frac{3j}{k+|j|} \geq 1$$

for $k - 1 \geq 3$ which holds for $k \geq 7$. Thus the distance to other values of $1/(k+j)$ does increase.

So from the previous argument, if we consider integers $k \geq 7$, than any x can only be within 2^{-k} of one such k. Now consider the function

$$u(x) = \sum_{k=7}^{\infty} u_k(x)$$

For $x \in \mathbb{R}^3$, it is either the case that

$$|x - \frac{1}{k}e_1| \geq 2^{-k} \qquad \forall k \geq 7$$

$$\implies u_k(x) \leq \sum_{k=7}^{\infty} 2^{-k} = 2^{-6}$$

or that

$$\exists! k_0 \geq 7 \text{ s.t. } |x - \frac{1}{k_0}e_1| < 2^{-k}, \quad \forall k \neq k_0,\ k \geq 7,\ |x - \frac{1}{k_0}e_1| \geq 2^{-k}$$

$$\implies u_k(x) \leq 1 + \sum_{k=7}^{\infty} 2^{-k} = 1 + 2^{-6}$$

so the function is bounded. Note that the function is defined at 0, for

$$u_k(0) = \min\left(1, 2^{-k}\left[\frac{k}{2^k}\right]^{d-2}\right) = 2^{-k}\left[\frac{k}{2^k}\right]^{d-2}$$

$$\implies 0 < u(0) = \sum_{k=7}^{\infty} 2^{-k}\left[\frac{k}{2^k}\right]^{d+2} < 1$$

but the function will not be continuous at 0, because any neighborhood of 0 will have $x \in \mathbb{R}^3$ such that $u(x) \geq 1$. This function is lower semi-continuous though because it is continuous everywhere except for $x = 0$, **as it converges locally uniformly for every** $x \neq 0$, and at $x = 0$, we have that

$$\liminf_{x \to 0} u(x) \geq \liminf_{x \to 0} \sum_{k=7}^{n} u_k(x) \qquad \forall n \in \mathbb{N}$$

and because the right sum is finite, we have

$$\liminf_{x \to 0} \sum_{k=7}^{n} u_k(x) = \sum_{k=7}^{n} 2^{-k}\left[\frac{k}{2^k}\right]^{d+2}$$

which is less than $u(0)$ but converges to $u(0)$ as $n \to \infty$, thus

$$\liminf_{x \to 0} u(x) \geq u(0)$$

so we indeed have lower semicontinuity. To show that this function is weakly superharmonic, we note the original definition as given in Caffarelli

Definition 0.1. $v \in L^1_{\text{loc}}$ is super harmonic in D if, for any $\psi \in C_c^{1,1}(D)$ with ψ non-negative, we have

$$\int v\Delta\psi \leq 0$$

Note that as per our discussion with Silvestre, we use $C_c^{1,1}(D)$ instead of $C_0^{1,1}(D)$. With this, for each $\psi \in C_c^{1,1}(D)$, we take an open cover of $\text{supp}(\psi)\backslash B_\delta(0)$ for $\delta > 0$ arbitrarily small so that

$$\Big| \int_{B_\delta(0)} v\Delta\psi \Big| \leq \epsilon$$

which necessarily holds by the dominated convergence theorem because both $v, \psi \in L^1$ (ψ being in L^1 is a product of it being bounded on its compact support), so that their product is in L^1.

The cover of $\text{supp}(\psi)\backslash B_\delta(0)$ (which is still compact) consists of sets on which $u(x)$ converges locally uniformly. Then we extract a finite subcover, so that there is a global rate of uniform convergence, which allows us to interchange integration and summation on $\text{supp}(\psi)\backslash B_\delta(0)$, i.e.

$$\int_{\text{supp}(\psi)\backslash B_\delta(0)} u(x)\Delta\psi(x) = \int \sum u_k(x)\Delta\psi(x) = \sum \int u_k(x)\Delta\psi(x) \leq 0$$

$$\implies \int_D u(x)\Delta\psi(x)dx = \int_{\text{supp}(\psi)} u(x)\Delta\psi(x) \leq \epsilon$$

because $\int u_k(x)\Delta\psi(x) \leq 0$ for each k individually. Repeat this for all $\epsilon > 0$, and then repeat the process for all ψ. Therefore

$$\forall \psi \in C_c^{1,1}(D) \quad \int_D u\Delta\psi \leq 0$$

so the sum is indeed weakly superharmonic, bounded in the unit ball, lower semicontinuous, but not continuous.

Question 13: Define $\phi : \mathbb{R}^d \to \mathbb{R}^d$,

$$\phi(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-1}, f(x_1, \ldots, x_{d-1}) + x_d).$$

We will use the ℓ^1 norm on $\mathbb{R}^d$ because it is equivalent. Let L be the Lipschitz constant of f with respect to the ℓ^1 norm on $\mathbb{R}^{d-1}$. Then if we denote $x = (x', x_d), x' \in \mathbb{R}^{d-1}$, we see

$$|\phi(x) - \phi(y)| = \sum_{i=1}^{d-1} |x_i - y_i| + |x_d - y_d + f(x') - f(y')|$$

$$\leq \sum_{i=1}^{d} |x_i - y_i| + |f(x') - f(y')|$$

$$\leq \sum_{i=1}^{d} |x_i - y_i| + L\sum_{i=1}^{d-1} |x_i - y_i|$$

$$\leq (1+L)\sum_{i=1}^{d}|x_i - y_i|.$$

Since the inverse of ϕ is

$$\phi^{-1}(x', x_d) = (x-, x_d - f(x')),$$

an analogous argument shows that ϕ^{-1} is also Lipschitz with a constant less than or equal to $(1+L)$. It should be clear that $\phi^{-1}(S) = \{x_d < 0\}$. $\square$

Question 14: [Jared Solution] Note that $\mathbb{R}$ should instead be a bounded domain in $\Omega \subseteq \mathbb{R}^n$. The norm for $C^{1,1}(\Omega)$ is

$$||f||_{C^{1,1}} = ||f||_{L^\infty(\Omega)} + ||\nabla f||_{C^{0,1}}$$

with

$$||\nabla f||_{C^{0,1}} = \sup_{x\neq y}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|}$$

Now take f to be the function which in $(-1,1)$ looks like $\int_0^x |x|$ and then outside of this neighborhood becomes a C^∞ such that it and its first two derivatives are bounded in norm and tail off to 0. Then for any $g \in C^\infty(\Omega)$, we have

$$||f' - g'||_{C^{0,1}} = \sup_{x\neq y}\frac{|f'(x) - f'(y) - g'(x) + g'(y)|}{|x-y|} = \sup_{x\neq y}\left|\frac{f'(x)-f'(y)}{x-y} - \frac{g'(x)-g'(y)}{x-y}\right|$$

For $y = 0$ and $x \to 0$, we know that

$$\frac{g'(x) - g'(0)}{x - 0} \to g''(0)$$

where as for f, we have that

$$\lim_{x\to 0^-}\frac{f'(x)-f'(0)}{x-0} = -1, \qquad \lim_{x\to 0^+}\frac{f'(x)-f'(0)}{x-0} = 1$$

$$\implies ||f'-g'||_{C^{0,1}} \geq \max\{\lim_{x\to 0^\pm}\left|\frac{f'(x)-f'(y)}{x-y} - \frac{g'(x)-g'(y)}{x-y}\right|\} = \max\{|g''(0)+1|, |g''(0)-1||\} \geq 1$$

so no density can occur with respect to the $C^{1,1}$ norm.

Note that $C^\infty(\Omega) \not\subseteq C^{1,1}(\Omega)$ even for bounded domains. Take $1/x$ on $(0,1)$ which satisfies the conditions of being $C^\infty(\Omega)$, but doesn't have a lipschitz derivative. Thus we should probably consider $\overline{C^\infty(\Omega) \cap C^{1,1}(\Omega)} \subseteq C^{1,1}(\Omega)$ under the given norm.

I claim that such a closure would be $C^2(\Omega)\cap C^{1,1}(\Omega)$. Consider $C^2(\Omega)\cap C^{1,1}(\Omega) \subseteq C^{1,1}(\Omega)$, then for functions in this space, we can approximate all elements of the Hessian by smooth functions (say up to ϵ tolerance) and then solve a system of equations to get a function with that Hessian of smooth functions, up to some linear functions. Note that after adjusting the constants of these linear functions, the smooth function, ϕ, should differ in L^∞ norm by $\epsilon\mu(\Omega)^2$, so that

$$||\phi - f||_{L^\infty} < \epsilon\mu(\Omega)^2 = \delta/2$$

which can be made arbitrarily small because $\mu(\Omega) < \infty$. Having two Hessians that are close should also be able to make $||\nabla[f - \phi]||_{C^{0,1}}$ quite small because

$$\nabla f(x) - \nabla f(y) = H_f(x) \cdot (y - x) + R_f(x, y)$$

where $R_f(x, y)/|x - y|^2 \to 0$ as $y - x \to 0$, and thus

$$||\nabla f - \nabla\phi||_{C^{0,1}} = \sup_{x \neq y} \frac{|[H_f(x) - H_\phi(x)] \cdot (y - x) + [R_f(x, y) - R_\phi(x, y)]|}{|x - y|}$$

boundedness should allow us to conclude that

$$\sup_{x \neq y} \frac{|R_f(x, y) - R_{\phi(x,y)}|}{|x - y|} < \epsilon$$

and by close approximation of the Hessian, we have

$$\sup_{x \neq y} \frac{|[H_f(x) - H_\phi(x)] \cdot (y - x)|}{|y - x|} \leq \sup_x |H_f(x) - H_\phi(x)| < \epsilon$$

so that

$$||f - \phi||_{C^{1,1}} < \epsilon$$

overall. This shows that

$$C^2(\Omega) \cap C^{1,1}(\Omega) \subseteq \overline{C^\infty(\Omega) \cap C^{1,1}}(\Omega)$$

For the other direction, assume that there was a function f without a continuous Hessian everywhere. Then for some (i, j) and $x \in \Omega$, we have that

$$\lim_{y \to x} \frac{\partial^2 f}{\partial x_i \partial x_j}(y) \text{ DNE}$$

so that in particular

$$v_1 = \limsup_{h \to 0} \frac{f_i(x + he_j) - f_i(x)}{h} \neq \liminf_{h \to 0} \frac{f_i(x + he_j) - f_i(x)}{h} = v_2 \quad \text{s.t.} \quad h \in \mathbb{R}$$

from this, we can extract a sequence of $\{h_q\} \to 0$ and $\{h_k\} \to 0$ such that the lim sup is achieved when using the first sequence and the lim inf is achieved when using the latter.

For our fixed aforementioned x, we have that

$$||\nabla f - \nabla\phi||_{C^{0,1}} = \sup_{x \neq y} \frac{|\nabla[f - \phi](x) - \nabla[f - \phi](y)}{|x - y|} \geq \max[A, B]$$

$$A = \lim_{q\to\infty} \frac{|\nabla f(x+h_q e_j) - \nabla f(x) - [\nabla\phi(x+h_q e_j) - \nabla\phi(x)]|}{|h_q|}$$
$$B = \lim_{k\to\infty} \frac{|\nabla f(x+h_k e_j) - \nabla f(x) - [\nabla\phi(x+h_k e_j) - \nabla\phi(x)]|}{|h_k|}$$

because we're using the norm $|x| = \sum_{i=1}^n |x_i|$, we know that

$$A \geq \lim_{q\to\infty} \left| \frac{f_i(x+h_q e_j) - f_i(x)}{h_q} - \frac{\phi_i(x+h_q e_j) - \phi(x)}{h_q} \right|$$

$$B \geq \lim_{k\to\infty} \left| \frac{f_i(x+h_k e_j) - f_i(x)}{h_k} - \frac{\phi_i(x+h_k e_j) - \phi(x)}{h_k} \right|$$

However, we know that because ϕ is smooth, that

$$\lim_{q\to\infty} \frac{\phi_i(x+h_q e_j) - \phi_i(x)]}{h_q} = \lim_{k\to\infty} \frac{\phi_i(x+h_k e_j) - \phi_i(x)]}{h_k} = \phi_{ij}(x)$$

where $\phi_{ij}(x) = \frac{\partial^2\phi}{\partial x_i \partial x_j}$. Thus using our notation from before

$$A \geq |v_1 - \phi_{ij}(x)|, \quad B \geq |v_2 - \phi_{ij}(x)$$

because $v_1 \neq v_2$, for all values of $\phi_{ij}(x)$, we have that

$$\max[|v_1 - \phi_{ij}(x)|, |v_2 - \phi_{ij}(x)|] \geq \frac{|v_2 - v_1|}{2}$$

and thus

$$\forall\phi \in C^\infty(\Omega) \cap C^{1,1}(\Omega), \quad ||\nabla f - \nabla\phi||_{C^{0,1}} \geq \max[A, B] \geq \frac{|v_2 - v_1|}{2} \neq 0$$

so any collection of $\{\phi_i\}$ smooth with lipschitz first derivative won't be dense. This establishes that

$$\overline{C^\infty(\Omega) \cap C^{1,1}(\Omega)} = C^2(\Omega) \cap C^{1,1}(\Omega)$$

□

Question 14:[Isaac Solution] Shout out to Jared for solving this problem. Let Ω be bounded. We know that $C^\infty(\Omega) \setminus C^{1,1}(\Omega) \neq \emptyset$. For example, take $\Omega = (0,1)$ and $f(x) = \frac{1}{x}$. $f \in C^\infty(\Omega) \setminus C^{1,1}(\Omega)$.

It should be clear that $C^2(\Omega) \cap C^{1,1}(\Omega) \neq C^{1,1}(\Omega)$, and that $C^\infty(\Omega) \cap C^{1,1}(\Omega) \subset C^2(\Omega) \cap C^{1,1}(\Omega)$. The following claim will show show that $\overline{C^\infty(\Omega) \cap C^{1,1}(\Omega)} \neq C^{1,1}(\Omega)$.

Claim: $C^2(\Omega) \cap C^{1,1}(\Omega)$ is closed.

Let $f \in \overline{C^2(\Omega) \cap C^{1,1}(\Omega)}$. Fix $x \in \Omega$ and $\epsilon > 0$. Pick $\phi \in C^2(\Omega) \cap C^{1,1}(\Omega)$ such that $\|f - \phi_n\|_{C^{1,1}} < \epsilon$. Then we know that for $\alpha \in (-1,1) \setminus \{0\}$ and each $i,j = 1,\ldots,n$ we have can choose points $y^j_\alpha = x + \alpha e_j$ such that

$$\limsup_{\alpha \to 0} \left| \frac{\partial_i f(y^j_\alpha) - \partial_i f(x)}{|y^j_n - x|} - \partial_{ji}\phi(x) \right| \leq$$

$$\limsup_{\alpha \to 0} \left| \frac{\partial_i f(y^j_\alpha) - \partial_i f(x)}{|y^j_n - x|} - \frac{\partial_i \phi(y^j_\alpha) - \partial_i \phi(x)}{|y^j_\alpha - x|} \right| + \left| \frac{\partial_i \phi(y^j_\alpha) - \partial_i \phi(x)}{|y^j_\alpha - x|} - \partial_{ji}\phi(x) \right|$$

$$\leq \|f - \phi\|_{C^{1,1}} < \epsilon.$$

If we have two functions $\phi_1, \phi_2 \in C^2(\Omega) \cap C^{1,1}(\Omega)$ such that $\|f - \phi_k\|_{C^{1,1}} < \epsilon$ for $k = 1,2$, then we see that

$$|\partial_{ji}\phi_1(x) - \partial_{ji}\phi_2(x)| \leq \limsup_{\alpha \to 0} \left| \frac{\partial_i f(y^j_\alpha) - \partial_i f(x)}{|y^j_n - x|} - \partial_{ji}\phi_1(x) \right| + \left| \frac{\partial_i f(y^j_\alpha) - \partial_i f(x)}{|y^j_n - x|} - \partial_{ji}\phi_2(x) \right|$$

$$< 2\epsilon.$$

Therefore, if we take a sequence $\phi_k \in C^2(\Omega) \cap C^{1,1}(\Omega)$ such that $\phi_k \xrightarrow{C^{1,1}} f$, then we see that $\partial_{ij}\phi_k(x)$ is a Cauchy sequence for $i,j = 1,\ldots,n$ and we deduce that

$$\lim_{\alpha \to 0} \frac{\partial_i f(y^j_\alpha) - \partial_i f(x)}{|y^j_n - x|} = \lim_{k \to \infty} \partial_{ji}\phi_k(x).$$

$$\text{i.e.} \quad \partial_{ji} f(x) = \lim_{k \to \infty} \partial_{ji}\phi_k(x). \quad ☺$$

Claim: If Ω has a C^1 boundary, $\overline{C^\infty(\Omega) \cap C^{1,1}(\Omega)} = C^2(\Omega) \cap C^{1,1}(\Omega)$.

We must prove the density of $C^\infty(\Omega) \cap C^{1,1}(\Omega)$ in $C^2(\Omega) \cap C^{1,1}(\Omega)$ to show that $\overline{C^\infty(\Omega) \cap C^{1,1}(\Omega)}$ is not a proper subset of $C^2(\Omega) \cap C^{1,1}(\Omega)$.

Pick $f \in C^2(\Omega) \cap C^{1,1}(\Omega)$. Ω is bounded, f is bounded, and the first and second derivatives of f are also bounded, so take some bounded neighborhood V of Ω and extend f to a C^2 function $\tilde{f}$ on $\mathbb{R}^n$ whose support lies inside V. Let φ_ϵ be a family of mollifiers. We know that (1) $\varphi_\epsilon * \tilde{f} \in C^\infty_c(\mathbb{R}^n) \subset C^{1,1}(\mathbb{R}^n)$, (2) $\partial_{ij}(\varphi_\epsilon * \tilde{f}) = \varphi_\epsilon * \partial_{ij}\tilde{f}$, and (3) $(\varphi_\epsilon * \tilde{f})$ and $(\varphi_\epsilon * \partial_{ij}\tilde{f})$ converge locally uniformly, and therefore, since there is a compact set containing the support of all of these functions, they converge uniformly.

Let f_ϵ denote $\varphi_\epsilon * \tilde{f}$. It follows from applying the mean value theorem to the second derivatives of the functions that

$$\sup_{x,y \in \Omega x \neq y} \frac{|\partial_i(f(x) - f_\epsilon(x)) - \partial_i(f(y) - f_\epsilon(x))|}{|x - y|} \leq \max_{1 \leq j \leq n} \|\partial_{ji}(f - f_\epsilon)\|_\infty \to 0.$$

It follows that $f_\epsilon\big|_\Omega \xrightarrow{C^{1,1}} f$. $\square$

Remark: This result probably holds for Ω with less well-behaved boundary. Try extending to $\tilde{f}$ continuous instead of C^2, and you can probably work the details out.

Question 15: $f \in L^\infty(\Omega)$, and is therefore locally integrable. We know from the Lebesgue Differentiation Theorem that the limit

$$\lim_{r\to 0^+} \frac{\int_{B_r(x)} f(y)\, dy}{|B_r(x)|}$$

exists and is equal to $f(x)$ almost everywhere. We will now prove something stronger.

Claim: The above limit exists everywhere in Ω.

For all $r > 0$ such that $B_r(x) \subset \Omega$ we have

$$\text{essinf}_{B_r(x)} f \leq \frac{\int_{B_r(x)} f(y)\, dy}{|B_r(x)|} \leq \text{esssup}_{B_r(x)}.$$

We know from the inequality that we were given that

$$\lim_{r\to 0^+} \text{essinf}_{B_r(x)} = \lim_{r\to 0^+} \text{esssup}_{B_r(x)}.$$

Claim: If we let $\tilde{f}(x)$ be equal to the above limit everywhere in Ω, then $\tilde{f}$ is α-Hölder continuous.

It is nontrivial to show that $\tilde{f}$ is continuous: for example, if we had let $\Omega = \mathbb{R}$ and we have made f the Heaviside step function, then $\tilde{f}$ would have existed everywhere, but would have still been discontinuous at $x = 0$. However, if we can show that $\tilde{f}$ is continuous, then since $\tilde{f} = f$ a.e. we will know that

$$\sup_{\Omega\cap B_r(x)} \tilde{f} - \inf_{\Omega\cap B_r(x)} \tilde{f} \leq Cr^\alpha$$

and conclude that $\tilde{f}$ is α-Hölder continuous.

Fix $x, y \in \Omega$ with $|x - y| = R$. Then for all $r > 0$ we have

$$|\tilde{f}(x) - \tilde{f}(y)| \leq \text{esssup}_{B_{R+r}(\frac{x+y}{2})} - \text{essinf}_{B_{R+r}(\frac{x+y}{2})} \leq C(R + r)^\alpha.$$

So $\tilde{f}$ is continuous. $\square$

Question 16: Let $C = \underset{B_2}{\operatorname{osc}} f$. It follows that for all $x \in B_1$ we have $\underset{B_{\frac{1}{2^n}}(x)}{\operatorname{osc}} f \le (1-\delta)^{n+1} C$ for all $n \ge 0$. Pick any $x, y \in B_1$ with $x \neq y$. Then there exists a unique $n \in \mathbb{N}$ such that $\frac{1}{2^n} \le |x-y| \le \frac{1}{2^{n-1}}$, and we have

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \le 2^{n\alpha} \underset{B_{\frac{1}{2^{n-1}}}(x)}{\operatorname{osc}} f \le \left[2^\alpha(1-\delta)\right]^n C \quad \forall\, \alpha \in \mathbb{R}^+.$$

If $\alpha < -\frac{\log(1-\delta)}{\log 2}$, then $2^\alpha(1-\delta) < 1$, and we conclude that

$$\sup_{x,y\in B_1} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \le C. \quad \square$$

Question 17: We will get our first inequality with case work:

- If $\sup f > 0 > \inf f$, then $\operatorname{osc} f > \|f\|_\infty$.
- If $\inf f \ge 0$, then $\|f\|_\infty = \operatorname{osc} f + \inf f \le \operatorname{osc} f + \frac{\|f\|_1}{|B_1|}$.
- If $\sup f \le 0$, then $\|f\|_\infty \le \operatorname{osc} f + \frac{\|f\|_1}{|B_1|}$ by an analogous argument.

So it follows that $\|f\|_\infty \le \operatorname{osc} f + \frac{\|f\|_1}{|B_1|}$ for all $f \in C^\alpha(B_1)$. Furthermore, since $|x-y| < 1$ for all $x, y \in B_1$, we compute that

$$\operatorname{osc} f = \sup_{x,y\in B_1} |f(x)-f(y)| \le \sup_{x,y\in B_1} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

It follows that

$$\|f\|_{C^\alpha} \le \frac{\|f\|_1}{|B_1|} + 2 \sup_{x,y\in B_1} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

so the constant $C = 2$ works. $\square$

Question 18: [NOT DONE] First, assume that the domain is convex (which Stephen said is ok). Also note that the second norm should be

$$||f||_{C^{1,\alpha},2} = ||f||_{L^\infty} + \sup_{x,y\in\Omega} \frac{|f(x)-f(y)-(x-y)\cdot\nabla f(y)|}{|x-y|^{\alpha+1}}$$

David's proof of the first inequality is

$$f(y)-f(x)-(y-x)\cdot\nabla f(x) = [\nabla f(z)-\nabla f(x)]\cdot(y-x) \qquad z \in \{tx+(1-t)y\},\ \text{s.t. } t\in[0,1]$$

$$\implies \frac{|f(x)-f(y)-(x-y)\cdot\nabla f(y)|}{|x-y|^{\alpha+1}} \le \frac{|\nabla f(z)-\nabla f(x)|\,|y-x|}{|x-y|^{1+\alpha}}$$

$$\leq \frac{|\nabla f(z) - \nabla f(x)|\,|y-x|}{|z-x|^\alpha} \leq \sup_{x,y\in\Omega} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}$$

which implies that

$$||f||_{C^{1,\alpha},2} \leq ||f||_{C^{1,\alpha},1}$$

For the other direction, let

$$K = \sup_{x,y\in\Omega} \frac{|f(x) - f(y) - (x-y)\cdot\nabla f(y)|}{|x-y|^{\alpha+1}}$$

fix an x and a y, and choose z_1, z_2 such that $(x - z_1)$ is parallel (and not antiparallel) to $\nabla f(x)$ and that $(z_2 - y)$ is parallel to $\nabla f(y)$, and that $|z_2 - y| = |x - z_1| = |x - y|$. Then we have that

$$\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha} = \frac{|x-y||\nabla f(x) - \nabla f(y)|}{|x-y|^{1+\alpha}} = \frac{|(x-z_1)\cdot\nabla f(x) - (z_2 - y)\cdot\nabla f(y)|}{|x-y|^{1+\alpha}}$$

Now note that the numerator can be written as

$$N = (x-z_1)\cdot\nabla f(x) - (z_2-y)\cdot\nabla f(y)$$

$$= [f(z_1) - f(x) - (z_1 - x)\cdot\nabla f(x)] + [f(z_2) - f(y) - (z_2 - y)\cdot\nabla f(y)] + [f(x) - f(z_1)] + [f(y) - f(z_2)]$$

$$= a + b + c + d$$

clearly

$$|N| \leq |a| + |b| + |c| + |d|$$

$$\frac{|a| + |b|}{|x-y|^{1+\alpha}} \leq 2K$$

because $|x - z_1| = |z_2 - y| = |x - y|$, and when $|x - y| \geq 1$, we have

$$\frac{|c| + |d|}{|x-y|^\alpha} \leq 4||f||_\infty$$

so it suffices to handle the case when $|x - y| < 1$.

Ball Bound

We have

$$K = \sup_{x,y\in\Omega} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|} \geq \lim_{\delta\to 0} \frac{|f_i(w + \delta e_j) - f_i(w)|}{|\delta|} \qquad \delta \in \mathbb{R}$$

and thus each partial derivative is Lipschitz, so that we know it is absolutely continuous and thus ∇f_i exists a.e. in B, so each second partial exists and

$$D^2 f = \{f_{ij}\}$$

exists a.e. in $B \subseteq \mathbb{R}^n$. Moreover $||f_{ij}|| \leq K$, so that

$$||D^2 f||_{L^\infty} \leq n^2 K$$

which gives

$$||f||_{L^\infty} + ||D^2 f||_{L^\infty} \leq n^2 ||f||_{C^{1,1}}$$

For the other direction, note the following

$$\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|} = \frac{\sum_{i=1}^n |f_i(x) - f_i(y)|}{|x-y|} \leq \sum_{i=1}^n \frac{|y-x| \sum_{j=1}^n ||f_{ij}||_{L^\infty}}{|y-x|} = ||D^2 f||_{L^\infty}$$

To show the above, let

$$x - y = \sum_{i=1}^n a_i \vec{e_i}, \qquad |x-y| = \sum_{i=1}^n |a_i|$$

we then get that

$$|f_k(x) - f_k(y)| \leq \sum_{i=1}^n \left| f_k\left(x + \sum_{j=1}^i a_j \vec{e_j}\right) - f_k\left(x + \sum_{j=1}^{i-1} a_j \vec{e_j}\right) \right|$$

$$\left| f_k\left(x + \sum_{j=1}^i a_j \vec{e_j}\right) - f_k\left(x + \sum_{j=1}^{i-1} a_j \vec{e_j}\right) \right| = \left| \int_0^{a_i} f_{ki}\left(x + \sum_{j=1}^{i-1} a_j \vec{e_j} + \vec{e_i} t\right) \right|$$

which follows by absolute continuity (which means that the second partials existing a.e.). Then note

$$\left| \int_0^{a_i} f_{ki}\left(x + \sum_{j=1}^{i-1} a_j \vec{e_j} + \vec{e_i} t\right) \right| \leq \int_0^{|a_i|} \left| f_{ki}\left(x + \sum_{j=1}^{i-1} a_j \vec{e_j} + \vec{e_i} t\right) \right| \leq |a_i| \cdot ||f_{ki}||_\infty$$

repeating this for all k yields

$$|\nabla f(x) - \nabla f(y)| = \sum_{k=1}^n |f_k(x) - f_k(y)| \leq \sum_{k=1} \sum_{i=1}^n \left| f_k\left(x + \sum_{j=1}^i a_j \vec{e}_j\right) - f_k\left(x + \sum_{j=1}^{i-1} a_j \vec{e}_j)\right) \right|$$

$$\leq \sum_{k=1}^n \sum_{i=1}^n |a_i| ||f_{ki}||_{L^\infty} \leq |x-y| ||D^2 f||_{L^\infty}$$

$$\implies \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|} \leq ||D^2 f||_{L^\infty}$$

And thus

$$||f||_{C^{1,1}} = ||f||_{L^\infty} + \sup_{x,y\in\Omega} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|} \leq ||f||_{L^\infty} + \sup_{x,y\in\Omega} \frac{||D^2 f||_{L^\infty} |x-y|}{|x-y|} = ||f||_{L^\infty} + ||D^2 f||_{L^\infty}$$

completing the equivalence. $\square$

Question 19: We will try and prove this for as large a class of domains Ω as we can. Suppose that there exists $r > 0$ such that the set $\Omega_r = \{x \in \Omega : d(x, \partial\Omega) \geq r\}$ has a nonempty interior and for all $y \in \Omega \setminus \Omega_r$ there exists $x \in \Omega_r$ such that the line going through x and y connects x to $\partial\Omega$ with a line segment of length at most βr where $\beta \geq 1$ is some constant. Then our result will work on the domain Ω, as we shall show shortly. This class of domains that I have just described is rather broad, and contains all open star domains, and therefore all open convex sets.

For this problem, let $L = \sup_{x,y\in\Omega} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}$.

Claim: For all $x \in \Omega_R$, $|\nabla f(x)| \leq \frac{2}{R}\|f\|_\infty + \frac{R^\alpha}{1+\alpha}L$.

Let $x \in \Omega_R$ and let $0 < r < R$. If we let $u = r\frac{\nabla f(x)}{|\nabla f(x)|}$, we compute that

$$\begin{aligned}
2\|f\|_\infty \geq |f(x+u) - f(x)| &= \left| \int_0^1 \nabla f(x+tu) \cdot u \, dt \right| \\
&\geq r|\nabla f(x)| - \left| \int_0^1 (\nabla f(x+tu) - \nabla f(x)) \cdot u \, dt \right| \\
&\geq r|\nabla f(x)| - \int_0^1 \big|(\nabla f(x+tu) - \nabla f(x)) \cdot u\big| \, dt \\
&\geq r|\nabla f(x)| - r\int_0^1 \big|(\nabla f(x+tu) - \nabla f(x))\big| \, dt \\
&\geq r|\nabla f(x)| - r^{\alpha+1}\int_0^1 t^\alpha L \, dt \\
&\geq r|\nabla f(x)| - \frac{r^{\alpha+1}}{1+\alpha}L. \\
\Rightarrow |\nabla f(x)| &\leq \frac{2}{r}\|f\|_\infty + \frac{r^\alpha}{1+\alpha}L.
\end{aligned}$$

Letting $r \uparrow R$, we get our result.

Claim: $\|\nabla f\|_\infty \leq \max(\frac{2}{R}, \frac{2+\alpha}{1+\alpha}R^\alpha)\|f\|_{C^{1,\alpha}}$.

We deduce from the specified properties of our set Ω that for all $y \in \Omega \setminus \Omega_R$, we know that there exists $x \in \Omega$ such that

$$|\nabla f(y)| \leq |\nabla f(x)| + R^\alpha L \leq \frac{2}{R}\|f\|_\infty + \frac{2+\alpha}{1+\alpha}R^\alpha L.$$

This bound obviously holds for $y \in \Omega_R$ as well. $\square$

Question 20: Note: the question must be rephrased to be every ball intersecting with Ω not every ball contained in Ω or else there are counterexamples in domains like the slit disk.
For a ball $B_r(x)$ denote the function described in the problem statement as $\ell_{x,r}(y) = a_{x,r} \cdot y + b_{x,r}$. We start by proving that f is differentiable and then bounding its derivative. I claim that if $B_{r_1}(x1) \supset B_{r_2}(x_2) \supset B_{r_3}(x_3)...$ is a decreasing sequence of open balls with $\cap_n B_{r_n}(x_n) = x$ then we have:

$$\nabla f(x) = \lim_{n\to\infty} a_{x_n,r_n} \tag{0.0.5}$$

For this we need a lemma first:
Lemma: If $\ell(x) = a \cdot x + b$ is linear and $|\ell(x)| \leq K$ on a ball of radius r then $|a| \leq \frac{K}{r}$.

Proof. W.l.o.g. assume the ball B is centered at 0. Now choose y with absolute value $r(1-\epsilon)$ oriented such that $|y \cdot a| = |y| \cdot |a|$ (this can be done because in finite dimensions the Cauchy-Schwarz inequality is never strict). Now we calculate:

$$|\ell(y) - \ell(-y)| \leq 2K \tag{0.0.6}$$

but also have:

$$|\ell(y) - \ell(-y)| = 2a \cdot y = 2|a|r(1-\epsilon) \tag{0.0.7}$$

and hence we get:

$$|a| \leq \frac{K}{(1-\epsilon)r} \tag{0.0.8}$$

which gives the result when we let ϵ go to 0. $\square$

Now take any balls $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$ contined in the initial ball with $r_2 \geq \frac{r_1}{2}$. Then we have for any $y \in B_{r_2}(x_2)$:

$$|(a_{x_2,r_2} - a_{x_1,r_1}) \cdot y + (b_{x_2,r_2} - b_{x_1,r_1})| \leq |\ell_{x_1,r_1}(y) - f(y)| + |f(y) - \ell_{x_2,r_2}(y)| \tag{0.0.9}$$

but by our suppositions we have:

$$|\ell_{x_1,r_1}(y) - f(y)| + |f(y) - \ell_{x_2,r_2}(y)| \leq Cr_1^{1+\alpha} + Cr_2^{1+\alpha} \leq 5Cr_2^{\alpha} \tag{0.0.10}$$

Hence by our lemma we thus have

$$|a_{x_1,r_1} - a_{x_2,r_2}| \leq 5Cr_2^{\alpha} \tag{0.0.11}$$

Now by forming a sequence of balls with radii all 1/2 of the previous one we can get that for any balls $B_{r_n}(x_n) \supseteq B_{r_{n-1}}(x_{n-1})$ we have:

$$|a_{x_n,r_n} - a_{x_{n-1},r_{n-1}}| \leq 5Cr_n^{\alpha} \frac{1}{1 - \frac{1}{2^{\alpha}}} \leq Kr_n^{\alpha} \tag{0.0.12}$$

Hence if we take a sequence of balls as described at the beginning of the problem the sequence a_{x_i,r_i} will be Cauchy and thus have a limit, call this limit a. I claim that $a = \nabla f(x)$. To prove this choose $\epsilon > 0$ and find r_1 such that $|a_{x,r} - a| < \epsilon$ whenever $r \leq r_1$. Now for y such that $|x-y| = r_1$ we have:

$$|f(x)-f(y)-a\cdot(x-y)| \leq |f(x)-\ell_{x,r_1}(x)-(f(y)-\ell_{x,r_1}(y))|+\epsilon|x-y| \leq 2Kr_1^{1+\alpha}+\epsilon r_1 \quad (0.0.13)$$

and hence we have:

$$\lim_{|x-y|\to 0} \frac{|f(x)-f(y)-a\cdot(x-y)|}{|x-y|} \leq \lim_{r\to 0} 2Kr^{\alpha}+\epsilon \quad (0.0.14)$$

but as this can be made arbitrarily small it must be 0 and hence a satisfies the definition of $\nabla f(x)$.

All that remains is to prove that $f \in C^{1,\alpha}$. But we calculate for $|x-y| = r$ find a ball of radius r containing x and y. Call this $B_r(z)$. Now we have be construction that:

$$|\nabla f(x) - a_{z,r}| \leq Kr^{\alpha} \quad (0.0.15)$$

and the same is true for $\nabla f(y)$. Hence we calculate by the triangle inequality:

$$|\nabla f(x) - \nabla f(y)| \leq |\nabla f(x) - a_{z,r}| + |\nabla f(y) - a_{z,r}| \leq 2Kr^{\alpha} \quad (0.0.16)$$

which is precisely the bound we needed. □

Question 21: Note that we assume u is non-constant everywhere, else the problem wouldn't be true. Now suppose that the global maximum, which occurs at x_0, belongs outside the support, then apply the mean value property to show that a higher local max is attained in a neighborhood of x_0, so it cannot be a global max (because u is harmonic outside the support of Δu by definition), a contradiction.

Question 22: [Stephen's solution] Pick a ball $B = B(0,r)$ for some $r > 0$. Define the distribution $\mu(\phi) = \int_{\partial B} \phi dS$ and the function $F : \mathbb{R}^n \to \mathbb{R}, F(x) = \int_{\partial B} \Phi(x-y)\, dS_y$.

Claim: $F = \Phi * \mu$.

μ is a distribution with compact support, so we know that this convolution is well-defined. We now compute that for any $\phi \in C_c^{\infty}(\mathbb{R}^n), z \in \mathbb{R}^n$ we have

$$\begin{aligned}
\Phi * (\mu * \phi)(z) &= \int_{\mathbb{R}^n} \Phi(y)(\mu * \phi)(z-y)\, dy \\
&= \int_{\mathbb{R}^n} \Phi(y) \int_{\partial B} \phi(z-y-x)\, dS_x\, dy \\
&= \int_{\partial B} \int_{\mathbb{R}^n} \Phi(y)\phi(z-(y+x))\, dy dS_x \\
&= \int_{\partial B} \int_{\mathbb{R}^n} \Phi(u-x)\phi(z-u)\, du dS_x
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \left(\int_{\partial B} \Phi(u-x)\, dS_x \right) \phi(z-u)\, du$$
$$= \int_{\mathbb{R}^n} F(u)\phi(z-u)\, du$$
$$= (F * \phi)(z).$$

Claim: F is radially symmetric.

This is rather obvious, but we will compute it anyway. Let T be an orthogonal transformation. Then

$$F(Tx) = \int_{\partial B} \Phi(Tx-y)\, dS_y = \int_{\partial B} \Phi(x - T^{-1}y)\, dS_y$$
$$= \int_{T^{-1}\partial B} \Phi(x-y)\, dS_y = \int_{\partial B} \Phi(x-y)\, dS_y = F(x).$$

Claim: F is constant on B.

We know that $-\Delta F = \mu$, so F is weakly harmonic in B, and therefore is harmonic in B. Take any closed ball $\overline{B(0,\rho)} \subset B$. From the maximum principle, $F\big|_{\overline{B(0,\rho)}}$ attains both its maximum and its minimum on $\partial B(0,\rho)$. Since F is radially symmetric, the maximum and minimum over $\overline{B(0,\rho)}$ must coincide and it must be constant on $\overline{B(0,\rho)}$.

Note that the constant must be

$$n = 2 \implies u(0) = \int_{\partial B_1} \log|y| dy = 0$$

$$n \geq 3 \implies u(0) = \int_{\partial B_1} \Phi(y) = \frac{1}{n(n-2)\alpha(n)} \int_{\partial B_1} dy = \frac{\alpha(n)n}{n(n-2)\alpha(n)} = \frac{1}{n-2}$$

and so if

$$u = \Phi \star \mu_{\mathbb{H}^{d-1}}(x)$$
$$\implies u(x) = \begin{cases} 0, \frac{1}{n-2} & |x| < 1 \\ \dots & |x| > 1 \end{cases}$$

Now solving

$$|x| = r > 1, \quad \Delta u(r) = 0 \implies u''(r) + \frac{n-1}{r} u'(r) = 0$$
$$\implies u(r) = \begin{cases} b \log r + c & (n=2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases}$$

but we see that $u(r) \to 0$ as $r \to \infty$ (the integrand goes to zero and then apply dominated convergence). In particular, for large values of $|x|$, we'd have that

$$n = 2 \implies \int_{\partial B_1} \Phi(x-y) \cong \int_{\partial B_1} -\frac{1}{2\pi} \log|x| = -\log|x|$$

$$n \geq 3 \implies \int_{\partial B_1} \Phi(x-y) \cong \int_{\partial B_1} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^n} dy = \frac{1}{n-2} \frac{1}{|x|^n}$$

and so we at least know

$$u(x) = \begin{cases} \left[\frac{1}{n-2}, 0\right] & |x| < 1 \\ \left[\frac{1}{n-2}\frac{1}{r^{n-2}}, -\log(r)\right] & |x| > 1 \end{cases}$$

□

1 Another solution to Question 22

We discussed a clever *guess and check* solution to Question 22. Here is a less clever but more direct strategy using the divergence theorem.

Let

$$A(x,r) = \frac{1}{r}\int_{\partial B_r} \Phi(x-y)\ \mathrm{d}S.$$

Claim: $A(r)$ is a constant provided $x \in B_r$.

By the scaling $x \mapsto rx$, we can easily see that $A(x,r) = A(x/r, 1)$. In particular

$$\lim_{r\to\infty} A(x,r) = A(0,1) = \frac{1}{d-2}.$$

Let us compute $\partial_r A(x,r)$. We get

$$\begin{aligned} \partial_r A(x,r) &= \partial_r \left(r^{d-2} \int_{\partial B_1} \Phi(x-ry)\ \mathrm{d}S(y) \right), \\ &= -(d-2)r^{-1}A(x,r) + \frac{1}{r}\int_{\partial B_r} \Phi_\nu(x-y)\ \mathrm{d}S, \\ &= -(d-2)r^{-1}A(x,r) - \frac{1}{r}\int_{B_r} \Delta\Phi(x,y)\ \mathrm{d}y = \frac{-(d-2)A(x,r)+1}{r}. \end{aligned}$$

Thus, we are left with the ODE $\partial_r A(x,r) = -(d-2)A(x,r)/r + 1/r$, with $A(+\infty) = 1/(d-2)$, whose only solution is the constant function $A \equiv 1/(d-2)$.

Question 23[NOT DONE] Apply problem 22 and modify it

Question 24

Let $u : B_r \to \mathbb{R}$ satisfy the equation

$$\begin{aligned} u \leq 0 &\qquad \text{on } \partial B_r \\ \Delta u \geq -C_0 &\qquad \text{in } B_r \end{aligned}$$

Prove that

$$u \leq \frac{C_0}{2d} r^2 \qquad \text{in } B_r.$$

Proof. Let

$$v = u + \frac{C_0}{2d}|x|^2.$$

Note that

$$\Delta v = \Delta u + C_0 \geq 0.$$

Thus, v is subharmonic, and satisfies the maximum principle. On ∂B_r, we have

$$v \leq \frac{C_0}{2d} r^2,$$

so inside B_r we have the same result. It follow that in B_r, we have

$$u \leq \frac{C_0}{2d}(r^2 - |x|^2) \leq \frac{C_0}{2d} r^2$$

□

Question 25

Prove the following generalization of Harnack's inequality. Let $u : B_{4r} \to \mathbb{R}$ be a nonnegative function that satisfies

$$\Delta u = f \qquad \text{in } B_{4r}.$$

Then

$$\max_{B_r} u \leq C \left(\min_{B_r} u + \|f\|_\infty r^2 \right)$$

Proof. We define

$$v = u - \frac{\|f\|_\infty}{2d}|x|^2$$
$$w = u + \frac{\|f\|_\infty}{2d}|x|^2$$

Notice that

$$\Delta v = f - \|f\|_\infty \leq 0 \qquad \text{a.e.}$$
$$\Delta w = f + \|f\|_\infty \geq 0 \qquad \text{a.e.}$$

so v is superharmonic and w is subharmonic.

We also define

$$K(s) = \int_{B_s} \frac{|x|^2}{d} \, \mathrm{d}x = |\partial B_1| \int_0^s \frac{r^{d+1}}{d} \, \mathrm{d}r = C s^{d+2},$$

where C depends only on dimension, and note that

$$\|f\|_\infty K(s) + \int_{B_s} v = \int_{B_s} w.$$

Now, fix $x, y \in B_r$. Note that $B_r(x) \subset B_{3r}(y) \subset B_{4r}$. It follows that

$$|B_r|w(x) \leq \int_{B_r(x)} w \leq \int_{B_{3r}(y)} w = \|f\|_\infty K(3r) + \int_{B_{3r}(y)} v \leq \|f\|_\infty K(3r) + |B_{3r}|v(y)$$

The first and third inequalities are due to w and v being subharmonic and superharmonic, respectively. The second inequality is due to w being nonnegative. Also, it is clear that $u(x) \leq w(x)$ and $v(y) \leq u(y)$. It follows that

$$|B_r|u(x) \leq |B_{3r}|u(y) + K(3r)\|f\|_\infty.$$

We divide through by $|B_r|$, and our above computation shows that $K(3r)/|B_r| = Cr^2$, so some C which only depends on dimension. Also, $|B_{3r}|/|B_r| = 3^d$, so we obtain

$$u(x) \leq C\left(u(y) + \|f\|_\infty r^2\right)$$

for all $x, y \in B_r$, for some C depending only on dimension. The result follows. □

Question 26 [NOT DONE]

Question 27 The solution to this problem is largely based on filling in details from Caffereli. Let the function $D : H^1(\Omega) \to \mathbb{R}$ denote the Dirichlet energy and let T denote the trace operator. Finally, let K be the closed, convex set

$$K = \{u \geq \varphi\} \cap T^{-1}(f).$$

We won't worry about the uniqueness of the solution to the obstacle problem until after we have done Question 28. Recall that we have already proven that there is a unique solution u_0 to the minimization problem. We will prove that u_0 is a solution to the obstacle problem.

Lemma: If $v \in L^1_{\text{loc}}(\Omega)$ is superharmonic, then v has a pointwise representative

$$v(x_0) = \lim_{r \downarrow 0} \fint_{B_r(x_0)} v(x)\, dx.$$

This representative of v is lower semicontinuous.

Proof: This result is Corollary 1 on page 9 of Caffarelli. We already know (Lebesgue Differentiation) that given any representative of v, the above limit exists and is equal to v almost everywhere. We wish to strengthen this fact by showing that the limit actually exists everywhere.

We know (see the lemma on page 7 of Caffarelli) that the limit is monotone increasing as $r \downarrow 0$. All we have to show is that the limit never blows up to infinity. [INSERT]

Finally, we show that this representative of v is lower semicontinuous. [INSERT]

Claim: u_0 is weakly super harmonic, and therefore has a pointwise defined lower semi-continuous representative.

It is a theorem (see Caffarelli page 9) that any weakly superharmonic function has a lower semicontinuous representative, so all that remains for this claim is to show that u_0 is superharmonic. Indeed, pick $\psi \in C_c^2(\Omega), \psi \geq 0$. It is obvious that $\psi + u_0 \in K$. For arbitrary $\epsilon > 0$ we see that

$$\int |\nabla u_0|^2 \, dx \leq \int (\nabla u_0 + \epsilon \psi)^2$$
$$= \int |\nabla u_0|^2 + 2\epsilon \int \nabla u_0 \nabla \psi + \epsilon^2 \int |\nabla \psi|^2$$
$$\Rightarrow -\frac{\epsilon}{2} \int |\nabla \psi|^2 \, dx \leq \int \nabla u_0 \nabla \psi \, dx.$$

Let $\epsilon \downarrow 0$ we get

$$0 \leq \int \nabla u_0 \nabla \psi \, dx.$$

We know from Bresiz 9.2 that $u_n \in C_c^\infty(\mathbb{R}^n)$ such that $u_n \xrightarrow{L^2(\Omega)} u_0$ and $\nabla u_n \xrightarrow{L^2(\text{ supp}\psi)} \nabla u_0$. We can also choose $u_n \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_n \xrightarrow{W^{1,p}} u_0$ according to Remark 9.5 in Bresiz, but this is more powerful than we need. Anyway, letting $n \to \infty$ we get

$$\left| \int u_0(-\Delta \psi) - \int \nabla u_0 \nabla \psi \right| \leq \left| \int [u_0 - u_n](-\Delta \psi) \right| + \left| \int [\nabla u_0 - \nabla u_n] \nabla \psi \right| \to 0.$$

So

$$\int u_0 \Delta \psi \, dx \leq 0.$$

Claim: $\{u_0 > \varphi\}$ is open.

Let $x_0 \in \{u_0 > \varphi\}$. Then we know that $u_0(x_0) > \varphi(x_0) + 2\epsilon$ for some $\epsilon > 0$. It follows from the continuity of φ and the lower semi-continuity of u_0 that $\{u_0 > \varphi(x_0) + \epsilon\} \cap \{\varphi < \varphi(x_0) + \epsilon\}$ is an open neighborhood of x_0 contained in $\{u_0 > \varphi\}$.

Claim: The support of the distribution Δu_0 is contained in the set $\{u_0 = \varphi\}$.

Let $\psi \in C_c^2(\{u_0 > \varphi\})$. Since lower semicontinuous functions attain their minimums on compact sets, we can choose

$$\epsilon \in \left(0, \frac{\min_{x \in \text{supp } \psi} u_0(x) - \varphi(x)}{\|\psi\|_\infty}\right)$$

such that $u_0 - \epsilon\psi \in K$. It follows that

$$\int |\nabla u_0|^2\, dx \leq \int (\nabla u_0 - \epsilon\psi)^2$$

$$= \int |\nabla u_0|^2 - 2\epsilon \int \nabla u_0 \nabla\psi + \epsilon^2 \int |\nabla\psi|^2$$

$$\Rightarrow \int \nabla u_0 \nabla\psi \leq \frac{\epsilon}{2} \int |\nabla\psi|^2.$$

Letting $\epsilon \downarrow 0$, we get

$$\int \nabla u_0 \nabla\psi \leq 0.$$

But we know from above that

$$0 \leq \int \nabla u_0 \nabla\psi.$$

So

$$\int \nabla u_0 \nabla\psi = 0.$$

Repeating the bounding argument from before, we conclude that

$$\int (\Delta u_0)\psi\, dx = \int u_0 \Delta\psi = 0.$$

Claim: u_0 is continuous.

This follows from Evans theorem on page 10 of Caffarelli.

To recap, u_0 not only is a solution to the obstacle problem but also is continuous. □

Question 28:

We know by Theorem 1 of the Caffarelli notes that u as described must be continuous because it is a) superharmonic and b) continuous in the support of its Laplacian (namely the contact set $u = \varphi$. Hence, by continuity we have that $\{u = \varphi\}$ must be closed. Now on this closed set we have that $v \geq \varphi = u$ and thus $v \geq u$ so we only need to look at the set $\{u > \varphi\}$.
Consider the closure of this set $C = \overline{\{u > \varphi\}}$ and the function $v - u$ on C. Because u is harmonic on C and v is superharmonic we must also have $v - u$ is superharmonic on C. Now by the minimum principle for superharmonic functions we thus know that $v - u$ attains its minimum on ∂C. But $\partial C = \partial\Omega \cup \partial D$ where $D = \{u = \varphi\}$. On $\partial\Omega$ we know $u = f$ and $v \geq f$ so $v - u \geq 0$ and on ∂D we know that $u = \varphi$ and $v \geq \varphi$ so $v - u \geq 0$. Hence $v - u \geq 0$ in C which implies that $v \geq u$ everywhere.

Question 29: [NOT DONE] **(a)** We know that there exists $r > 0$ such that $u \in C^{1,1}(B_1 \setminus B_{1-r})$. Fix $\delta \in (0, \frac{r}{2})$. We will let N denote $\max(\|\varphi\|_{C^{1,1}(B_1)}, \|u\|_{C^{1,1}(B_1 \setminus B_{1-r})})$. Let $h \in B(0,\delta) \subset \mathbb{R}^d$.

Claim: $v_h \geq u$ on $\partial B_{1-\delta}$.

If we fix $x \in \partial B_{1-\delta}$, then we know from the mean value theorem that there exists $x^*, y^* \in \{x + th : t \in (-1,1)\}$ such that

$$\begin{aligned}\left|\frac{u(x+h)+u(x-h)}{2} - u(x)\right| &= \left|\frac{u(x+h)-u(x)}{2} - \frac{u(x)-u(x-h)}{2}\right| \\ &= \left|\frac{\nabla u(x^*)\cdot h}{2} - \frac{\nabla u(y^*)\cdot h}{2}\right| \leq \frac{|\nabla u(x^*) - \nabla u(y^*)||h|}{2} \\ &\leq N|h|^2.\end{aligned}$$

where we use the fact that $|x^* - y^*| \leq 2|h|$. It follows that for any $x \in \partial B_{1-\delta}$ we have

$$v_h(x) - u(x) = N|h|^2 + \frac{u(x+h)+u(x-h)}{2} - u(x) \geq 0.$$

Claim: $v_h \geq \varphi$ in B_1.

$$\begin{aligned}v_h(x) - \varphi(x) &= N|h|^2 + \frac{u(x+h)-\varphi(x+h)}{2} + \frac{u(x-h)-\varphi(x-h)}{2} + \frac{\varphi(x+h)+\varphi(x-h)}{2} - \varphi(x) \\ &\geq N|h|^2 - \left|\frac{\varphi(x+h)+\varphi(x-h)}{2} - \varphi(x)\right|.\end{aligned}$$

The same computation as above (i.e. the mean value theorem) gives us our result again.

(b) v_h is the sum of three superharmonic functions, and is therefore itself superharmonic. It follows from Question 28 and part (a) of this question that $v_h \geq u$ in $B_{1-\delta}$.

(c) In $B_{1-\delta}$, $h \in B(0,\delta)$, we get

$$\begin{aligned}0 &\leq 2(v_h(x) - u(x)) \\ \Rightarrow -2N &\leq \frac{u(x+h)+u(x-h)-2u(x)}{|h|^2}.\end{aligned}$$

Now if we let h be a small real scalar, e be a unit vector, and x be a point such that $\partial_{ee}u(x)$ exists, then we see that

$$\liminf_{h\to 0} \frac{\partial_e u(x+he) - \frac{u(x+he)-u(x)}{h}}{h} = \liminf_{h\to 0} \frac{\partial_e u(x+he) - \partial_e u(x)}{h} + \frac{\partial_e u(x) - \frac{u(x+he)-u(x)}{h}}{h}$$

$$\geq \liminf_{h\to 0} \frac{\partial_e u(x+he) - \partial_e u(x)}{h},$$

and

$$\liminf_{h\to 0} \frac{-\partial_e u(x-he) - \frac{u(x-he)-u(x)}{h}}{h} = \liminf_{h\to 0} \frac{-\partial_e u(x-he) + \partial_e u(x)}{h} + \frac{-\partial_e u(x) - \frac{u(x-he)-u(x)}{h}}{h}$$
$$\geq \liminf_{h\to 0} \frac{\partial_e u(x) - \partial_e u(x-he)}{h}.$$

It follows that

$$\liminf_{h\to 0} \frac{\partial_e u(x+he) - \partial_e u(x-he)}{2h}$$
$$= \liminf_{h\to 0} \frac{1}{2}\frac{\partial_e u(x+he) - \frac{u(x+he)-u(x)}{h}}{h} + \frac{1}{2}\frac{-\partial_e u(x-he) - \frac{u(x-he)-u(x)}{h}}{h} + \frac{u(x+h)+u(x-h)-2u(x)}{2|h|^2}$$
$$\geq \liminf_{h\to 0} \frac{\partial_e u(x+he) - \partial_e u(x)}{h} + \liminf_{h\to 0} \frac{\partial_e u(x) - \partial_e u(x-he)}{h}$$
$$+ \liminf_{h\to 0} \frac{u(x+h)+u(x-h)-2u(x)}{2|h|^2}.$$

2 Hausdorff Measure

2.1 Problem 30

First, note that since the expression:

$$\inf\{\sum_{i=1}^{\infty}(\mathrm{Diam}\, U_i)^m : \bigcup_i U_i \supset A, \mathrm{Diam}\, U_i < \delta\} \tag{2.1.1}$$

is decreasing is δ the sup in the definition of Hausdorff measure can be replaced with a lim sup. Now fix r and write:

$$A + B_r = \bigcup_{a\in A} B_r(a) \tag{2.1.2}$$

By the Vitali covering lemma we can find a (possibly finite but at most countable) subset of these balls $\{B_i\}_{i\in I}$ such that all the B_i are disjoint and $A + B_r \subseteq \cup_{i\in I} 5B_i$. Now we clearly have:

$$|A + B_r| \geq \sum_{i\in I} |B_i| = C_1|I|r^d \tag{2.1.3}$$

where C_1 is the volume of the unit ball in $\mathbb{R}^d$. Now set $\delta > 10r$. Clearly the set of $\{5B_i\}$ form an admissible set for expression (1). Hence, we have:

$$\inf\{\sum_{i=1}^{\infty}(\mathrm{Diam}\, U_i)^m : \bigcup_i U_i \supset A, \mathrm{Diam}\, U_i < \delta\} \leq \sum_{i\in I}(10r)^m \leq \frac{10^d}{C_1}\frac{|A + B_r|}{r^{d-m}} \tag{2.1.4}$$

Hence, taking lim sup of both sides and noting that as $r \to 0$ our choice of $\delta \to 0$ we get the desired result:

$$H^m(A) \leq CM^m(A) \tag{2.1.5}$$

The converse does not hold. Set $A = \mathbb{Q}^d$. For all r we have $A + B_r = \mathbb{R}^d$ by density of the rationals. Hence, the Minkowski content of A is infinite for all m while the Hausdorff measure of it is 0 for all $m \neq 0$.

2.2 Problem 31

Note: for this question we will be using the alternate definition found online of perimeter:

$$\operatorname{Per}(A) = \sup\{|\int_A \operatorname{div}\varphi| : \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n), ||\varphi||_{L^\infty} \leq 1\} \tag{2.2.1}$$

It is possible that this is equivalent to the definition given.
Now we have by the divergence theorem that for any set A with C^1 boundary:

$$\int_A \operatorname{div}\varphi = \int_{\partial A} \varphi \cdot v dS \tag{2.2.2}$$

Clearly when $||\varphi||_{L^\infty} \leq 1$ we have that

$$\int_{\partial A} \varphi \cdot v dS \leq \int_{\partial A} |\varphi| \times |v| dS = \int_{\partial A} dS \tag{2.2.3}$$

All we have to prove that this equality is attained is find a function $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ such that φ is equal to the unit normal on ∂A. Doing this is a little complicated. What we want to do is find a *tubular neighborhood* of ∂A, that is, a neighborhood U of ∂A that is diffeomorphic to $\partial A \times (-\epsilon, \epsilon)$. We construct one of these as follows:
Using the fact that ∂A is C^1 find a collection of open sets (in ∂A) U_i such that there exists $0 \leq n_i \leq d$ so that the projection of U_i onto $d-1$ coordinates is a C^1 map with C^1 inverse:

$$p_i : U_i \to V_i \subset \mathbb{R}^{d-1} :: (x_1, ..., x_d) \mapsto (x_1, ..., x_{n_i-1}, x_{n_i+1}, ..., x_d) \tag{2.2.4}$$

Now using problem 13 that has been assigned we know we can extend this to f_i, a C^1 map with C^1 inverse, from an open neighborhood of U_i to $\mathbb{R}^d$ such that the image of U_i has d-th coordinate equal to 0. Now consider the set $D_i = f_i(U_i) \times (-\epsilon, \epsilon) \subseteq \mathbb{R}^d$ where ϵ is chosen sufficiently small. We have that $U_i \subset f_i^{-1}(D_i) = W_i$. Now let ψ be a bump function on $(-1, 1)$ that attains its maximum of 1 at 0. Now consider the functions

$$\varphi_i : W_i \to \mathbb{R}^d :: x \mapsto v(x)\psi(\frac{f(x)_d}{\epsilon}) \tag{2.2.5}$$

where $v(x)$ gives the unit normal on ∂A corresponding to the first $d-1$ coordinates of $f_i(x)$ and $f_i(x)_d$ represents the d-th coordinate of $f_i(x)$. For each i this is clearly a C^1 function that is equal to the unit normal on ∂A. Hence, take a partition of unity u_i for the W_i and define

$$\varphi(x) = \sum u_i(x)\varphi_i(x) \tag{2.2.6}$$

and we clearly get the required function.
Now we have to prove that

$$\int_{\partial A} dS = H^{d-1}(\partial A) \tag{2.2.7}$$

but looking at the definition of the surface integral makes this obvious.

2.3 Problem 32

Take $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ with $||\varphi||_{L^\infty} \le 1$ and let $\{U_i\}_{i\in I}$ be a cover of $\partial A \cap \operatorname{supp}(\varphi) = \Omega$ with $\operatorname{Diam}(U_i) < \delta$ for all i. Note that Ω is compact so we can assume I is finite. Now for all $i \in I$ take $x_i \in U_i$ and set $B_i = B_{\operatorname{Diam}(U_i)}(x_i) \supset U_i$. This clearly still covers Ω and we have:

$$C_1 \sum \operatorname{Diam}(U_i)^{d-1} \ge \sum \operatorname{Diam}(B_i)^{d-1} \tag{2.3.1}$$

for some constant C_1 depending only on the dimension. Now we seek prove some lemmas:

Lemma 2.1. *For all $x \in \operatorname{Int} A$ with $B_r(x) \subset \operatorname{Int} A$ there exists a function $\varphi_1 \in C_c^1$ such that $\varphi_1 = 0$ on $B_{r/2}(x)$, $\varphi_1 = \varphi$ outside of $B_r(x)$, $||\varphi_1||_{L^\infty} \le ||\varphi||_{L^\infty}$ and:*

$$\int_A \operatorname{div}\varphi = \int_A \operatorname{div}\varphi_1 \tag{2.3.2}$$

Proof. Let h be a smooth bump function that is equal to 1 on $B_{r/2}(x)$ and 0 outside $B_r(x)$. Now write:

$$\int_A \operatorname{div}\varphi = \int_A \operatorname{div}(1-h)\varphi + \int_A \operatorname{div} h\varphi \tag{2.3.3}$$

Note that the second integral on the left hand side is just:

$$\int_A \operatorname{div} h\varphi = \int_{B_r(x)} \operatorname{div} h\varphi = \int_{\partial B_r(x)} h\varphi \cdot v = 0 \tag{2.3.4}$$

Hence $\phi_1 = (1-h)\varphi$ satisfies the conditions of the theorem. □

Lemma 2.2. *For all $\epsilon > 0$ there exists φ_1 such that $\varphi_1 = 0$ when $x \in \operatorname{Int} A$ and $d(x,\Omega) \ge \epsilon$, $||\varphi_1||_{L^\infty} \le ||\varphi||_{L^\infty}$, and:*

$$\int_A \operatorname{div}\varphi = \int_A \operatorname{div}\varphi_1 \tag{2.3.5}$$

Proof. Note that $\operatorname{supp}(\varphi) \cap \{x \in \operatorname{Int} A : d(x,\Omega) \ge \epsilon\}$ is a compact set. Hence, for each point x in this set put a ball of radius $\frac{r_x}{2}$ around it such that $B_{r_x}(x) \subseteq \operatorname{Int} A$. Now take a finite subcover of this and apply the above lemma finitely many times for each ball in this subcover to get the result. □

Lemma 2.3. *If $B = C \cup D$ then we have:*

$$|\int_B \operatorname{div} \varphi| \leq \operatorname{Per}(C) + \operatorname{Per}(D) \tag{2.3.6}$$

and specifically:

$$\operatorname{Per}(B) \leq \operatorname{Per}(C) + \operatorname{Per}(D) \tag{2.3.7}$$

Proof. This is true in general but we only need it for finite unions of balls so we only prove that case. Let C and D be some finite union of balls. Note that $\partial B \subseteq \partial C + \partial D$. Now we use Stokes' Theorem and the fact that ∂B, ∂C, and ∂D are piecwise C^1 to get:

$$|\int_B \operatorname{div} \varphi| \leq \operatorname{Per}(B) = \int_{\partial B} dS \leq \int_{\partial C} dS + \int_{\partial D} dS = \operatorname{Per}(C) + \operatorname{Per}(D) \tag{2.3.8}$$

□

Now we are ready to complete the proof of the result. Note that since $\mathbb{R}^n \setminus \bigcup_i B_i$ is closed we have:

$$\inf_{x \in A \setminus \cup(B_i)} d(x, \Omega) = \epsilon > 0 \tag{2.3.9}$$

Hence, apply Lemma 2 to find a φ_1 such that $\varphi_1 = 0$ when $d(x, \Omega) \geq \frac{\epsilon}{2}$ and:

$$\int_A \operatorname{div} \varphi = \int_A \operatorname{div} \varphi_1 \tag{2.3.10}$$

Now cover A by the B_i and a set $A' \in \operatorname{Int} A$ defined by $A' = \{x \in A : d(x, \Omega) > \frac{3}{4}\epsilon$. Let $B = A' \cup \bigcup_i U_i \supset A$. We have:

$$\int_B \operatorname{div} \varphi = \int_B \operatorname{div} \varphi_1 = \int_{\bigcup_{i \in I} B_i} \operatorname{div} \varphi_1 \tag{2.3.11}$$

since by construction of φ_1 we have that $\varphi_1 = 0$ on A'. Now apply Lemma 3 finitely many times to get:

$$|\int_B \operatorname{div} \varphi| = |\int_{\bigcup_{i \in I} B_i} \operatorname{div} \varphi_1| \leq \sum_i \operatorname{Per}(B_i) \leq C \sum_i \operatorname{Diam}(U_i)^{d-1} \tag{2.3.12}$$

where the last equality is from the above problem applied to balls.
Now as $\delta \to 0$ choose our selection of open sets U_i for each δ such that: (i) we have

$$\lim_{\delta \to 0} \sum_i \operatorname{Diam}(U_i)^{d-1} = H^{d-1}(\Omega) \tag{2.3.13}$$

(we can do this just by the definition of the Hausdorff measure), and (ii) we have:

$$\lim_{\delta \to 0} \mu((A' \cup \bigcup_i U_i) \setminus A) = 0 \tag{2.3.14}$$

(this is also possible by definition of Hausdorff measure). Note that this last equality is the same as the indicator functions of the sets B as defined above converging to the indicator function on A in the L^1 norm. But this then gives:

$$\int_B \operatorname{div}\varphi = \int \chi_B \operatorname{div}\varphi \to \int \chi_A \operatorname{div}\varphi = \int_A \operatorname{div}\varphi \tag{2.3.15}$$

(because $\operatorname{div}\varphi \in L^\infty$. Hence, take limits as $\delta \to 0$ in equation (23) above to get the final result:

$$|\int_A \operatorname{div}\varphi| \leq CH^{d-1}(\Omega) \leq CH^{d-1}(\partial A) \tag{2.3.16}$$

to get the required result.

2.4 Problem 33

If

$$\int_A \nabla \cdot \varphi$$

is supposed to converge, which it does for any $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, because of the compact support, then we have that

$$A_n = \bigcup_{i=0}^{n} B_{2^{-i}}(q_i), \qquad R_n = \bigcup_{i=n+1}^{\infty} B_{2^{-i}}(q_i)$$

$$\exists N \text{ s.t. } \forall n > N, \qquad \Big|\int_{R_n} \nabla \cdot \varphi\Big| < \epsilon$$

so that

$$\int_A \nabla \cdot \varphi = \int_{A_n} \nabla \cdot \varphi + c\epsilon$$

for $c \in (-1, 1)$. But note that A_n has a local C^1 boundary except at a set of measure zero (i.e. the "corners" created by the intersecting balls). And thus we can apply the divergence theorem so that

$$\Big|\int_{A_n} \nabla \cdot \varphi\Big| = \Big|\int_{A_n} \varphi \cdot d\vec{S}\Big| \leq ||\varphi||_{L^\infty} \int_{A_n} d\vec{S}$$

$$\leq 1 \cdot C \sum_{i=1}^{n} H^{d-1}(B_{2^{-i}}(q_i)) \leq C \sum_{i=1}^{\infty} H^{d-1}(B_{2^{-i}}(q_i)) \leq C \sum_{i=1}^{\infty} (2^{-i})^{d-1} = K < \infty$$

which is a bound uniform in n. Repeat this for every such φ and corresponding n, to get finite perimeter.

Now using the definition that

$$\partial A = \overline{A} \backslash A^o = \mathbb{R}^d \backslash A$$

$$\implies \mu(\partial A) = \mu(\mathbb{R}^d) - \mu(A) = +\infty$$

Note that $H^d(\partial A) = C\mu(\partial A) = \infty$, because the d-dimensional hausdorff measure in $\mathbb{R}^d$ is a scaled multiple of the lebesgue measure. So because

$$\dim_H(A) = \inf\{d \geq 0 \ : \ H^d(A) = 0\} = \sup(\{d \geq 0 \ : \ H^d(A) = \infty\} \cup \{0\})$$

where $\inf(\emptyset) = \infty$, then we see that $\dim_H(A) \geq d$. Note that because

$$\partial A \subseteq \mathbb{R}^d$$

if we show that $\dim_H(\mathbb{R}^d) = d$, then we're done because the Hausdorff measure is an outer measure. Using the countable subadditivity property of outer measures, we note that

$$\mathbb{R}^d = \bigcup_{i=1}^{\infty} (-i, i)^d$$

$$\implies H^m(\mathbb{R}^d) \leq \sum_{i=1}^{\infty} H^m((i, i)^d)$$

we show that $(0, 1)^d$ has zero Hausdorff measure for any $m = d + \epsilon$ with $\epsilon > 0$, and by an analagous argument, this will show $H^{d+\epsilon}((-i, i)^d) = 0$ for any $i > 0$.

For fixed $\epsilon > 0$, take any $\delta > 0$, and canonically partition $(0, 1)^d$ into 2^{nd} boxes of side length 2^{-n}. Note that from a volumetric perspective, this works out because

$$V((0, 1)^d) = 1 = 2^{nd}(2^{-n})^d$$

now cover each box (some of which may have parts of their boundaries, which is ok) with boxes of side length 2^{-n+1} but centered at the same centers of the original boxes in our partition. Note that

$$diam(C_{2^{-n+1}}) = \sqrt{\sum_{i=1}^{d} (2^{-n+1})^2} = \sqrt{d} 2^{-n+1}$$

which happens to be the length of the diagonal. Then, choosing N large so that

$$n > N \implies 2^{-N+1}\sqrt{d} < \delta$$

we take our cover of $(0, 1)^d$ with these boxes (which overlap and cover all of $(0, 1)^d$ because they are twice the size of the boxes of side length 2^{-n} in our partition) and note that

$$\sum_{i=1}^{2^{nd}} diam(C_{2^{-n+1},i})^{d+\epsilon} = 2^{nd} \left(\sqrt{d} 2^{-n+1}\right)^{d+\epsilon} = (2\sqrt{d})^{d+\epsilon} 2^{-n\epsilon}$$

$$\implies \forall n > N \quad \inf\{\sum_{i=0}^{\infty} (diam(U_i))^m \ : \ \bigcup_{i=1}^{\infty} U_i \supseteq A, \ diam(U_i) < \delta\} \leq (2\sqrt{d})^{d+\epsilon} 2^{-n\epsilon}$$

given our construction of the valid cover, we can repeat this process for any δ by choosing a larger and larger n, and for fixed δ any larger n works so that clearly, the inf is 0. From this, it is clear that

$$\limsup_{\delta \to 0^+} \inf\{\sum_{i=0}^{\infty}(diam(U_i))^m \ : \ \bigcup_{i=1}^{\infty} U_i \supseteq (0,1)^d, \ diam(U_i) < \delta\} = 0$$

Now noting that any $(-i,i)^d$ can be covered by a finite union of cubes of the form $\{x+(0,1)^d\}$ and noting that the hausdorff measure is translation invariant, we use subadditivity and monotoncity to conclude

$$H^{d+\epsilon}((-i,i)^d) = 0 \qquad \forall i > 0$$
$$\implies H^{d+\epsilon}(\mathbb{R}^d) = 0$$

thus $\dim_H(\mathbb{R}^d) = d$ and so $\dim_H(\partial A) = d$.

2.5 Problem 34

Take $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ with $||\varphi||_{L^\infty} \leq 1$. Write $D_r = A + B_r$. Now use Lemma 2 from above to find a function φ_1 with norm less than or equal to that of φ, $\varphi_1 = 0$ on $D_{r/2}$ and:

$$\int_{D_r} \operatorname{div} \varphi_1 = \int_{D_r} \operatorname{div} \varphi \tag{2.5.1}$$

Now cover $D_r \setminus D_{r/2}$ by balls of radius $r/2$. Use Vitali covering lemma t find a disjoint subset of these balls, call them B_i for $i \in I$, such that $\bigcup_i 5B_i \supset (D_r \setminus D_{r/2})$. We want to bound the cardinality of our index set I. Note that:

$$\bigcup_i B_i \subseteq (D_{2r} \setminus A) \tag{2.5.2}$$

Hence, we have by disjointness and our bounds from above:

$$\sum |B_i| = C_1|I|r^d \leq 2Cr \tag{2.5.3}$$

and hence for some universal constant C_2 we have:

$$|I| \leq \frac{C_2}{r^{d-1}} \tag{2.5.4}$$

Let $E = \bigcup_i 5B_i \cup D_r \subseteq D_{5r}$Now note that by our construction of φ_1 we have:

$$\int_E \operatorname{div} \varphi = \int_{\bigcup_i 5B_i} \operatorname{div} \varphi_1 \tag{2.5.5}$$

Hence, since this is a finite union we apply Lemma 3 to get:

$$|\int_{\bigcup_i 5B_i} \operatorname{div} \varphi_1| \leq \sum_i \operatorname{Per}(5B_i) = |I|C_3 r^{n-1} \leq C_4 \tag{2.5.6}$$

For some universal constant C_4. Hence, we have proven that the perimeter of E is bounded. Now we just let r go to 0 and note that $E \subseteq D_{5r}$ while $\chi_{D_{5r}} \to \chi_A$ in L^1 and hence we get:

$$|\int_E \operatorname{div} \varphi| \to |\int_A \operatorname{div} \varphi| \leq C_4 \tag{2.5.7}$$

It is not true that $H^{d-1}(\partial A)$ is comparable with C. To see this consider $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$.

Question 36: *Prove that the symmetric bilinear form*

$$\int_\Omega \langle A\nabla \cdot, \nabla \cdot \rangle \, dx$$

is an inner product on $H_0^1(\Omega)$ *that induces a norm equivalent to* $\| \cdot \|_{H^1}$.

We know from Poincaré's inequality that since Ω is bounded we have $\| \cdot \|_{H^1} \sim \|\nabla \cdot \|_2$ on $H_0^1(\Omega)$. Let $A(x) = \{a_{ij}(x)\}$. Then Cauchy-Schwarz gives us

$$\int \langle A(x)\nabla u(x), \nabla u(x)\rangle \, dx \leq \int \langle \Lambda I \nabla u(x), \nabla u(x)\rangle \, dx = \Lambda \|\nabla u\|_2^2.$$

Similarly, we also know that

$$\int \langle A(x)\nabla u(x), \nabla u(x)\rangle \, dx \geq \lambda \|\nabla u\|_2^2.$$

Therefore, $u_n \xrightarrow{H_0^1} u$ if and only if $\int \langle A\nabla[u_n - u], \nabla[u_n - u]\rangle \, dx \to 0$. If follows that $\in \langle A\nabla \cdot, \nabla \cdot \rangle \, dx$ is a positive definite form on $H_0^1(\Omega)$, and the norm it induces must be equivalent to $\| \cdot \|_2$. $\square$

Question 37: *Prove that the minimizer of*

$$\min \left\{ \int_\Omega \langle A(x)\nabla u(x), \nabla u(x)\rangle \, dx : u \in H^1(\Omega), u = f \text{ on } \partial\Omega \right\},$$

is attained by a function that solves the equation

$$u = f \text{ on } \partial\Omega,$$

$$\nabla \cdot A(x)\nabla u(x) = 0 \text{ in } \Omega.$$

This is a special case of Question 7 with $g = 0$. $\square$

Question 38: *Prove that if* $u \in H^1(\Omega)$ *is a subsolution, then*

$$\int_\Omega \langle A(x)\nabla u(x), \nabla \varphi(x)\rangle \, dx \leq 0 \quad \forall \, \varphi \in H_0^1(\Omega).$$

Notice that for a fixed $u \in H^1(\Omega)$ the map

$$f \to \int_\Omega \langle A(x)\nabla u(x), \nabla f(x)\rangle \, dx$$

is a linear functional on $H^1(\Omega)$. We get from Cauchy-Schwarz that

$$\left| \int \langle A(x)\nabla u(x), \nabla f(x)\rangle dx \right| \leq \int \|A(x)\| |\nabla u(x)| |\nabla f(x)| dx \leq \Lambda \|\nabla u\|_2 \|\nabla f\|_2 \leq \Lambda \|\nabla u\|_2 \|f\|_{H^1}.$$

It follows that the linear functional $f \to \int \langle A(x)\nabla u(x), \nabla f(x)\rangle \, dx$ is bounded. Since convergence implies weak convergence, our result follows immediately from the fact that we can approach φ in $H^1(\Omega)$ with nonnegative functions $\varphi_n \in C_c^\infty(\Omega)$. □

Question 39: *Let $f \in C^1(\mathbb{R})$ and let $u \in W^{1,p}(\Omega)$ with $p \in [1, \infty)$. If one of the two following holds*

1. *$u \in L^\infty(\Omega)$ and Ω is bounded.*
2. *$f(0) = 0$ and $\|f'\|_\infty = L < \infty$*

then $f \circ u \in W^{1,p}(\Omega)$, and $\nabla(f \circ u) = f'(u)\nabla u$.

In case (1), we know that $f \circ u$ is bounded (by the continuity of f on $\mathbb{R}$), and since Ω is bounded we conclude that $f \circ u \in L^p(\Omega)$. Notice that this does not follow if Ω is unbounded, and that Ω being bounded is a necessary assumption in this case:

Counterexample: Let $\Omega = \mathbb{R}$ and let $f = u = e^{-x^2}$.

In case (2), since f is Lipschitz with $f(0) = 0$, we have $|f \circ u| \leq L|u|$ everywhere, and $f \circ u \in L^p(\Omega)$. Since $\|f'\|_\infty = L$, we know that $(f \circ u)\frac{\partial u_n}{\partial x_i} \in L^p(\Omega)$ as well. Notice that we really needed $f(0) = 0$ in order to get this to work:

Counterexample: Let $\Omega = \mathbb{R}$ and let $f = u = e^{-x^2}$.

Using Fubini's theorem and basic calculus, one can check that if f is differentiable we get

$$\int_\Omega (f \circ u_n) \frac{\partial \phi}{\partial x_i} \, dx = \int_{\mathbb{R}^{n-1}} \int_{\{(x',t) \in \Omega\}} (f \circ u_n)(x', t) \frac{\partial \phi}{\partial x_i}(x', t) \, dt dx'$$

$$= - \int_{\mathbb{R}^{n-1}} \int_{\{(x',t) \in \Omega\}} (f' \circ u_n)(x', t) \frac{\partial u_n}{\partial x_i}(x', t) \phi(x', t) \, dt dx' = - \int_\Omega (f' \circ u) \frac{\partial u_n}{\partial x_i} \phi \, dx.$$

Since f is continuous, we know that (passing to a subsequence) $f \circ u_n \to f \circ u$ pointwise almost everywhere. In case (1), since we know (see appendix) that $\|u_n\|_\infty \leq \|u\|_\infty$ everywhere, it follows that $f \circ u_n \leq M = \max_{x \in [-\|u\|_\infty, \|u\|_\infty]} f(x)$ for all n. Since supp ϕ

is compact, the constant function M is in $L^p(\text{supp } \phi)$ and dominated convergence theorem gives us

$$(\star) \qquad \int_\Omega (f \circ u_n) \frac{\partial \phi}{\partial x_i}\, dx \to \int_\Omega (f \circ u) \frac{\partial \phi}{\partial x_i}\, dx.$$

In case (2), since f is Lipschitz, we know that $|(f \circ u)(x) - (f \circ u_n)(x)| \leq L|u_n(x) - u(x)|$ everywhere, and therefore $f \circ u_n \xrightarrow{L^p(\Omega)} f \circ u$, and we get $(\star)$ again.

Since f' is continuous, we know that (passing to a subsequence) $(f' \circ u_n) \to (f' \circ u)$ pointwise almost everywhere. It follows that (passing to yet another subsequence) $(f' \circ u_n)\frac{\partial u_n}{\partial x_i} \to (f' \circ u)\frac{\partial u}{\partial x_i}$ pointwise almost everywhere. In case (1), we can use the continuity of f' to replicate the argument in the above paragraph and conclude that $f' \circ u_n \xrightarrow{L^q(\omega)} f' \circ u$ for any $\omega \subset\subset \Omega$, including some $\omega \supset \text{ supp } \phi$ (here q is the Hölder conjugate of p). It follows that since $\frac{\partial u_n}{\partial x_i}\phi \xrightarrow{L^p(\Omega)} \frac{\partial u}{\partial x_i}\phi$, we get

$$(\star\star) \qquad \int_\Omega (f' \circ u_n) \frac{\partial u_n}{\partial x_i} \phi\, dx \to \int_\Omega (f' \circ u) \frac{\partial u}{\partial x_i} \phi\, dx.$$

In case (2), since $\|f'\|_\infty = L$, we know that $(f \circ u_n)$ is uniformly bounded, and therefore $(f' \circ u_n) \xrightarrow{L^p(\omega)} f' \circ u$ for any $\omega \subset\subset \Omega$, including some $\omega \supset \text{ supp } \phi$. This gives us $(\star\star)$ once again. $\square$

Let $u \in H^1(\Omega)$. Prove that $u^+ \in H^1(\Omega)$ and

$$\nabla u_+ = \nabla u \chi_{\{u>0\}}.$$

Stephen already proved this for us in his handout:

We know that if $G : \mathbb{R} \to \mathbb{R}$ is C^1 with bounded derivative, then $G(u) \in H^1(\Omega)$ for all $u \in H^1(\Omega)$ with $\nabla(G(u)) = G'(u)\nabla u$ (You should have seen this proof in graduate functional analysis. You can find it in Brezis at least).

With that in mind, for each $\epsilon > 0$ we define G_ϵ by

$$G_\epsilon(t) = \begin{cases} t - \epsilon/2, & t \geq \epsilon \\ t^2/2\epsilon, & 0 \leq t \leq \epsilon \\ 0, & t \leq 0 \end{cases}. \qquad (1.0.35)$$

Thus $G_\epsilon(u) \in H^1(\Omega)$ for all $\epsilon > 0$ with $||G_\epsilon(u)||_{H^1} \leq ||u||_{H^1}$.

It's clear that $G_\epsilon(u)(x) \to u_+(x)$ and $\nabla(G_\epsilon(u))(x) = G'_\epsilon(u)\nabla u(x) \to \chi_{\{u>0\}}(x)\nabla u(x)$ pointwise. Since both sequences $\{G_\epsilon(u)\}_{\epsilon>0}$ and $\{G'_\epsilon(u)\nabla u(x)\}_{\epsilon>0}$ are bounded in $L^2(\Omega)$, we know that along some subsequence $\epsilon_i \to 0$, we must have that $G_{\epsilon_i}(u) \rightharpoonup \varphi \in L^2(\Omega)$, $G'_{\epsilon_i}(u)\nabla u \rightharpoonup \Phi \in L^2(\Omega; \mathbb{R}^d)$ for some functions φ, Φ. But since we knew what these sequences already converged pointwise, we thus have that $G_{\epsilon_i}(u) \rightharpoonup u_+$ and $G'_{\epsilon_i}(u)\nabla u \rightharpoonup \chi_{\{u>0\}}\nabla u$.

So, all that remains to do is to prove that $u_+ \in H^1(\Omega)$ is to prove that $\nabla(u_+) = \chi_{\{u>0\}}\nabla u$ in the sense of distributions. So, let $\varphi \in C_c^\infty(\Omega; \mathbb{R}^d)$ be arbitrary. Then by definition of weak convergence, we have that

$$\int_\Omega u_+ \operatorname{div}\varphi dx = \lim_{\epsilon_i \to 0} \int_\Omega G_{\epsilon_i}(u) \operatorname{div}\varphi dx = \lim_{\epsilon_i \to 0} -\int_\Omega G'_{\epsilon_i}(u)\nabla u \cdot \varphi dx = -\int_\Omega \chi_{\{u>0\}}\nabla u \cdot \varphi dx. \quad (1.0.36)$$

Thus $\nabla(u_+) = \chi_{\{u>0\}}\nabla u$, so we're done.

Prove that $\nabla u = 0$ almost everywhere on $\{u = 0\}$.

$u = u^+ - u^-$. We know that $\nabla u^+ = \nabla u^- = 0$ on $\{u = 0\}$. $\square$

Prove that if Ω is connected and there is a measurable set $A \subset \Omega$ and the function

$$u(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $H^1(\Omega)$, then either $|A| = 0$ or $|\Omega \setminus A| = 0$.

We can reduce the case where Ω is bounded and smooth, because if Ω is unbounded and $\chi_A \in H^1(\Omega)$, then $\chi_A\big|_{\Omega'} \in H^1(\Omega')$ for all bounded, smooth open subsets Ω' of Ω. Assume that A is such that $u \in H^1(\Omega)$. We know that $\nabla u \chi_{\Omega \setminus A} = 0$ because $u = u^+$. Since Ω is bounded, $1 - u \in H^1(\Omega)$. Since $1 - u = (1 - u)^+$, we conclude that $\nabla(1 - u)\chi_A = 0$. Since $\nabla 1 = 0$, we conclude that $\nabla u \equiv 0$.Since Ω is connected, it follows that u is constant, and is therefore 1 or 0. $\square$

3 Problem 40

3.1 Part 1

Set $v = F \circ u$. Note that $\partial_i v = F'(u)\partial_i u$ and thus for any test function φ we have:

$$\int_\Omega a_{ij}\partial_i v \partial_j \varphi = \int_\Omega F'(u) a_{ij} \partial_i u \partial_j \varphi \tag{3.1.1}$$

Now let $h = F'(u)\varphi$. This is an admissible test function because F is monotone. Note that $\partial_i h = F'(u)\partial_i \varphi + \varphi F''(u)\partial_i u$. But we have:

$$\int_\Omega a_{ij}\partial_i u \partial_j h = \int_\Omega a_{ij}\partial_i u (F'(u)\partial_j \varphi + \varphi F''(u)\partial_j u) \leq 0 \tag{3.1.2}$$

Now note that:

$$\int_\Omega F'(u) a_{ij}\partial_i u \partial_j \varphi = \int_\Omega a_{ij}\partial_i u \partial_j h - \int_\Omega F''(u) a_{ij} \partial_i u \partial_j u \tag{3.1.3}$$

But the first term here is negative because u is a subsolution, and the second is negative because a_{ij} is a positive matrix and F is convex so F'' is negative. And thus the whole expression is negative so we get:

$$\int_\Omega a_{ij}\partial_i v \partial_j \varphi \leq 0 \tag{3.1.4}$$

and thus $v = F \circ u$ is a subsolution.

3.2 Part 2

This is just an application of problem 11a) to $-u$ and $-v$.

4 Problem 41

First, note that since we have by ellipticity that $A = \{a_{ij}\} \leq \Lambda I$ then for any u, v:

$$|u^t Av| = |\langle u, Av\rangle| \leq |u| \ |Av| \leq C|u| \ |v| \tag{4.0.1}$$

for some constant C. Now use ellipticity to write:

$$\int_{B_{1+\delta}} \varphi^2 |\nabla u|^2 dx \leq C \int_{B_{1+\delta}} \varphi^2 a_{ij} \partial_i u \partial_j u \ dx \tag{4.0.2}$$

Now let $h = \varphi^2 u$. Note that h is positive and has $\partial_i h = 2\varphi u \partial_i \varphi + \varphi^2 \partial_i u$. Now we have because u is a subsolution that:

$$\int_{B_{1+\delta}} a_{ij} \partial_i u \partial_j h \ dx = \int_{B_{1+\delta}} a_{ij} \partial_i u (2\varphi u \partial_j \varphi + \varphi^2 \partial_j u) \ dx \leq 0 \tag{4.0.3}$$

From this we see that:

$$C \int_{B_{1+\delta}} \varphi^2 a_{ij} \partial_i u \partial_j u \ dx \leq C \int_{B_{1+\delta}} u\varphi |a_{ij} \partial_i u \partial_j \varphi| \ dx \leq C \int_{B_{1+\delta}} u |\nabla \varphi| \cdot \varphi |\nabla u| \ dx \tag{4.0.4}$$

However, by Young's inequality with exponent 2 we have for any a, b that $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$. Hence, we have:

$$\int_{B_{1+\delta}} \varphi^2 |\nabla u|^2 dx \leq C \int_{B_{1+\delta}} u |\nabla \varphi| \cdot \varphi |\nabla u| \ dx \leq \epsilon \int_{B_{1+\delta}} \varphi^2 |\nabla u|^2 \ dx + \frac{C}{\epsilon} \int_{B_{1+\delta}} u^2 |\nabla \varphi|^2 \ dx \tag{4.0.5}$$

and taking $\epsilon = \frac{1}{2}$ proves the result.
To get the concluding inequality let φ be a function that is equal to 1 on B_1, is equal to 0 on $\partial B_{1+\delta}$ and has gradient bounded by $\frac{C}{\delta}$ for some constant C. Then we get:

$$\int_{B_1} |\nabla u|^2 \leq \int_{B_{1+\delta}} \varphi^2 |\nabla u|^2 \leq C \int_{B_{1+\delta}} u^2 |\nabla \varphi|^2 \leq \frac{C}{\delta^2} \int_{B_{1+\delta}} u^2 \tag{4.0.6}$$

and thus taking squareroots gives the desired result.

Question 42

This is easy, just set

$$w = u \frac{\delta_0}{||u||_{L^2(B_2)}} \implies ||w||_{L^2(B_2)} \leq \delta$$

$$\implies ||w||_{L^\infty(B_1)} \leq 1 \implies ||u||_{L^\infty(B_1)} \leq \frac{1}{\delta_0} ||u||_{L^2(B_2)}$$

Now note that the essential supremum coincides with the $||\cdot||_{L^\infty(B_1)}$ norm because $u \geq 0$ everywhere. □

Question 43 Assume the statement. Note that $A_0 \leq \delta_0$ means exactly that $||u_0||_{L^2(B_2)} \leq \delta_0$. Yet $u_0 = (u - l_0)_+ = u_+ = u$ because u is non-negative. Thus in fact $||u||_{L^2(B_2)} \leq \delta_0$. Now assume that $||u||_{L^\infty(B_1)} > 1$, then for

$$S_n = \{x \in B_1 \mid |u(x)| \geq 1 + \frac{1}{n}\}$$

$$\exists N \text{ s.t. } \mu(S_N) > 0$$

which is contradictory because then

$$\forall n > N, \quad A_n^2 = \int_{B_{r_k}} |u_k|^2 \geq \int_{S_n} |u_k|^2 = \int_{S_n} (u - 1 + 2^n)^2 \geq \int_{S_n} 1/N^2 = \mu(S_n)/N^2 > 0$$

$$\implies \lim_{n\to\infty} A_n > 0$$

a contradiction, so we must have that $||u||_{L^\infty(B_1)} \leq 1$, from which we apply the previous problem to get the conclusion.

For the other direction, note that in the statement of 5.1, if we set $\delta_0 = 1/C$, then we get exactly the statement of question 42. Thus we can assume such a value of δ_0, which makes question 42 true and thus makes question 43 sensical so that

$$esssup_{B_1} u = ||u||_{L^\infty(B_1)} \leq C||u||_{L^2(B_2)} = C\delta_0 = 1$$

From this, we can apply the dominated convergence theorem because we know that $u_k^2 \leq u^2$, and u^2 is an integrable function over B_2. From the form of u_k, it is clear that pointwise $u_k(x) \to 0$ as $k \to \infty$ on B_1 and the measure of the integral outside of B_1 is negligible in the sense that, thus we have that

$$A_k^2 = ||u_k||^2_{L^2(B_{r_k})} = \int_{B_1} |u_k|^2 + \int_{B_{r_k}\setminus B_1} |u_k|^2 \leq \int_{B_1} |u_k|^2 + \int_{B_{r_k}\setminus B_1} |u|^2$$

for any $\epsilon > 0$, the latter term is less than ϵ for k sufficiently large, and the former term goes to 0 by dominated convergence on B_1. Thus

$$\lim_{k\to\infty} A_k = 0$$

□

Question 44

First extend the functions $\{u_{k+1}\}$ from B_{r_k} to $\mathbb{R}^n$ to use the sobolev inequalities. Note that after extending, that

$$||u_{k+1}||_{L^p(\mathbb{R}^n)} \leq C||u_{k+1}||_{L^p(B_{r_{k+1}})}, \qquad ||u_{k+1}||_{H^1(\mathbb{R}^n)} \leq C||u_{k+1}||_{H^1(B_{r_{k+1}})}$$

$$||u_{k+1}||_{L^p(B_{r_{k+1}})} \leq ||u_{k+1}||_{L^p(\mathbb{R}^n)}, \qquad ||u_{k+1}||_{H^1(B_{r_{k+1}})} \leq ||u_{k+1}||_{H^1(\mathbb{R}^n)}$$

because B_{r_k} has a C^1 boundary (see Brezis p.273). Then applying the sobolev inequality, we get

$$||u_{k+1}||_{L^p(B_{r_{k+1}})} \leq ||u_{k+1}||_{L^p(\mathbb{R}^n)} \leq C||\nabla u_{k+1}||_{L^2(\mathbb{R}^n)} \leq C'||\nabla u_{k+1}||_{L^2(B_{r_{k+1}})}$$

Now from cacciopoli's inequality applied with a bump function φ which satisfies

$$\varphi = \begin{cases} 1 & |x| \leq r_{k+1} \\ 0 & |x| \geq r_k \end{cases}$$

which decays almost linearly so that $|\nabla\varphi| \leq 2^{-k}$ (note that $r_k - r_{k+1} = 2^{-k-1}$) everywhere (such a bump function is always possible to construct, ask David), then we have that

$$C'||\nabla u_{k+1}||_{L^2(B_{r_{k+1}})} \leq 2C'2^k||\nabla u_{k+1}||_{L^2(B_{r_k})}$$

from which we finish the proof. □

Question 45

This is pretty trivial, but I assume that the first norm should be $||\cdot||_{L^p}$ for $1/p = 1/2 - 1/d$ as before. Note that

$$1 = \frac{2}{p} + \frac{2}{d}$$

$$||u_{k+1}||^2_{L^2(B_{r_{k+1}})} = ||u^2_{k+1}||_{L^1(B_{r_{k+1}})} = ||u^2_{k+1}g||_{L^1(B_{r_{k+1}})}$$

$$g(x) = \chi_{\{u_{k+1}>0\}\cap B_{r_{k+1}}}$$

$$||u_{k+1}||^2_{L^2(B_{r_{k+1}})} \leq ||u^2_{k+1}||_{L^{p/2}}||g||_{L^{d/2}}$$

Note that

$$||u^2_{k+1}||_{L^{p/2}(B_{r_{k+1}})} = \left(\int_{B_{r_{k+1}}} |u_{k+1}|^p\right)^{2/p} = ||u_{k+1}||^2_{L^p(B_{r_{k+1}})}$$

$$||g||_{L^{d/2}} = |\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{2/d}$$

$$\implies ||u_{k+1}||_{L^2(B_{r_{k+1}})} \leq ||u_{k+1}||_{L^p(B_{r_{k+1}})}|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{/d}$$

$$\leq C2^k||u_{k+1}||_{L^2(B_{r_k})}|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{1/d}$$

□

Question 46

Consider

$$|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|$$

Note that

$$u_{k+1} = (u - 1 + 2^{-k-1})_+ = (u - 1 + 2^{-k} - 2^{-k-1})_+ = (u - l_k - 2^{-k-1})_+$$

$$\implies |\{u_{k+1} > 0\}| = |\{u - l_k > 2^{-k-1}\}| = |\{u_k > 2^{-k-1}\}| \leq 2^{2k+2}||u_k||^2_{L^2(B_{r_{k+1}})}$$

$$\leq 2^{2k+2}||u_k||^2_{L^2(B_{r_k})} = (2^{k+1}A_k)^2$$

where we get this bound by consider u_k as a function defined on $B_{r_{k+1}}$ initially in order to apply chebyshev. From the previous problem, we get

$$||u_{k+1}||_{L^2(B_{r_{k+1}})} \leq C2^k||u_{k+1}||_{L^2(B_{r_k})}|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{1/d}$$

now noting that $0 \leq u_{k+1} \leq u_k$, we can replace

$$||u_{k+1}||_{L^2(B_{r_k})} \leq ||u_k||_{L^2(B_{r_k})} = A_k$$

$$|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{1/d} \leq 4^{1/d}2^{2k/d}A_k^{2/d}$$

$$\implies A_{k+1}||u_{k+1}||_{L^2(B_{r_{k+1}})} \leq C2^{k+2k/d}A_k^{1+2/d}$$

where we absorb $4^{1/d}$ into the constant (its bounded above and below for all values of d). Note that this bound is different from the problem statement, but makes sense given the adjustment to the previous problem.

Now choose

$$\delta = \min(0.5, (2C^2)^{-d/2})$$

and we'll show that

$$A_n \leq \frac{\delta}{(2C)^n}$$

by induction (we assume $2C > 1$, else we can always increase C).

Set

$$\delta = \alpha^{-d^2}C^{-d} \qquad \alpha = 2^{1+2/d}$$

Again, taking C large, this is always less than 1. Then we want to show that $A_k \leq \delta^k$.

For the base case, we have

$$A_1 \leq C2^0\delta^{1+2/d} \leq C\delta^1\delta^{2/d} \leq \delta\alpha^{-d^2}C^{1-d} \leq \delta^1$$

Now assume that the inductive hypothesis holds for $k = n$, then

$$A_{n+1} \leq C2^{n+2n/d}A_nA_n^{2/d}$$

$$A_n \leq \delta^n$$

$$A_n^{2/d} = \delta^{2n/d} \implies C2^{n+2n/d}A_n^{2/d} = C[\alpha\delta^{2/d}]^n$$

$$\delta^{2/d} = \alpha^{-2d}C^{-2} \implies C2^{n+2n/d}A_n^{2/d} \leq C^{1-2n}\alpha^{(1-2d)n} = C^{1-2n+d}\alpha^{(1-2d)n+d^2}\delta$$

$$\implies A_{n+1} \leq \delta^{n+1}[C^{1-2n+d}\alpha^{(1-2d)n+d^2}] = \delta^{n+1}K(n,d)$$

Set $N_0 = \max(d, 2)$. This is a really ugly bound, but the point is, we can choose $A_0 < \delta_0$, which is even smaller, but small enough so that

$$A_{N_0} \leq C2^{N_0+2N_0/d}C2^{N_0-1+2(N_0-1)/d}\dots C2^0\delta_0 \leq \delta^{N_0}$$

(i.e. apply the bound naively starting with $A_0 < \delta_0$), from which we can use the fact that $n > N_0$ yields $K(n,d) < 1$, so that the inductive hypothesis holds true. Thus we produce a very small δ_0, so that the bound holds for finitely many n, but in fact a much sharper bound holds. Once the finitely many n are handled, we can proceed with the induction. Thus

$$A_n \leq \delta^n \quad \forall n > N_0$$

$$\implies \lim_{n\to\infty} A_n = 0$$

now we apply questions 43 and 42 to get theorem 5.1, having found such a δ_0. □

Question 47

Let $u : B_2 \to \mathbb{R}$ be a non-negative supersolution. Prove that there is a constant $\epsilon_0 > 0$ so that if

$$|\{x \in B_2 : u(x) \geq 1\}| \geq (1-\epsilon_0)|B_2|$$

then $u(x) \geq 1/2$ a.e. in B_1.

Proof. We define $v = (1-u)_+$. Note that this is a non-negative subsolution and $0 \leq v \leq 1$. If

$$|\{x \in B_2 : u(x) \geq 1\}| \geq (1-\epsilon_0)|B_2|$$

then we have

$$|\{x \in B_2 : v(x) > 0\}| \leq \epsilon_0 |B_2|.$$

We use this to estimate the L^2 norm of v:

$$\|v\|_{L^2(B_2)} = \left(\int_{B_2} v^2\right)^{1/2} = \left(\int_{\{v>0\}} v^2\right)^{1/2} \leq \left(\int_{\{v>0\}} 1\right)^{1/2} = |\{v>0\}|^{1/2} \leq (\epsilon_0|B_2|)^{1/2}$$

Applying Theorem 5.1, we obtain

$$\text{esssup}_{B_1} v \leq C\|v\|_{L^2(B_2)} \leq C(\epsilon_0|B_2|)^{1/2}.$$

Notice that if $\text{esssup}_{B_1} v \leq 1/2$, then a.e. in B_1, we have $\max(1-u,0) \leq 1/2$, so $1-u \leq 1/2$ and thus $u \geq 1/2$. Setting $C(\epsilon_0|B_2|)^{1/2} = 1/2$, or taking $\epsilon_0 = \dfrac{1}{4C^2|B_2|}$, we guarantee $\text{essinf}_{B_1} u \geq 1/2$. □

Question 48

Suppose for contradiction that no such ϵ existed for fixed C, δ_0, δ_1. Then we have that

$$\forall n \in \mathbb{N},\ \exists u_n \text{ s.t. } u_n : B_1 \to [0,1], \quad ||u_n||_{H^1(B_1)} \leq C, \quad |\{u_n = 0\}| \geq \delta_0, \quad |\{u_n = 1\}| \geq \delta_1$$

$$\& \quad |\{0 < u_n(x) < 1\}| < \frac{1}{n}$$

then via Rellich-Kondravich, we know that $H^1(B_1)$ has compact injection into L^2 so that given our bounded sequence of $\{u_n\}$ w.r.t. $||\cdot||_{H^1(B_1)}$, we can extract a subsequence of the $\{u_n\}$ which are cauchy in the $L^2(B_1)$ norm and converge to some $u \in L^2(B_1)$.

We want to take d more subsequences so that $\{u_{n_j}\}$ and $\{\nabla u_{n_j}\}$ are all cauchy in the L^2 norm, but Rellich-Kondravich fails us here because the partials are not in $H^1(B_1)$. However, note that by theorem 9.3 in Brezis, we have that

$$\Big|\int_{B_1} u\frac{\partial \varphi}{\partial x_i}\Big| \leq \Big|\lim_{j\to\infty}\int_{B_1}\frac{\partial u_{n_j}}{\partial x_i}\varphi\Big| \leq \limsup_{j\to\infty}||\frac{\partial u_{n_j}}{\partial x_i}||_{L^2}||\varphi||_{L^2} \leq C||\varphi||_{L^2}$$

which by our theorem implies that $u \in W^{1,2} = H^1$. Here we use the fact that

$$\int_{B_1} u\frac{\partial \varphi}{\partial x_i} = \lim_{j\to\infty}\int_{B_1} u_{n_j}\frac{\partial \varphi}{\partial x_i} = \lim_{j\to\infty}\int_{B_1}\frac{\partial u_{n_j}}{\partial x_i}\varphi$$

Thus the limiting function u must satisfy

$$|\{f=0\}| \geq \delta_0 > 0, \quad |\{f=1\}| \geq \delta_1 > 0, \quad |\{0<f<1\}| = 0$$

which means that f is equivalent to an indicator function a.e. Note that we showed in problem 39 that indicator functions for sets of non-zero and non-full measure are not in H^1. Yet the subsequence must converge to such an f, which would lie in H^1. This is a contradiction, so such an ϵ must exist. □

Question 49[Not done?]
Consider the functions $v_k = (1-2^k u)_+$ restricted to $B_{3/2}$. Note that

$$|\{v_k = 0\}| = |B_{3/2}| - \delta_k, \quad |\{v_k > 0\}| = \delta_k, \quad |\{v_k > 1/2\}| = \delta_{k+1}$$

we want to apply question 48. Note the v_k are subsolutions, so by Cacciopoli from question 41, we know that

$$||\nabla v_k||_{L^2(B_{3/2})} \leq C2||v_k||_{L^2(B_2)}$$

but of course $|v_k| \leq 1$ everywhere, so

$$\forall k, \quad ||\nabla v_k||_{L^2(B_{3/2})} \leq 4C$$

where C depends ellipticity constants, dimension, etc. Now assume that $\delta_k \to \epsilon > 0$. Then we apply problem 48 to the functions

$$g_k = 2\min(v_k, 1/2)$$

from problem 10, we know that $g_k \in H^1(B_{3/2})$ is still uniformily bounded in the $||\cdot||_{H^1(B_{3/2})}$ norm because

$$\min(u,v) = -\max(u-v,0) + u$$

$$\implies \nabla(\min(u,v)) = \begin{cases} \nabla v & u \geq v \\ \nabla u & u \leq v \end{cases}$$

by problem 39 and using the fact that $\max(u,0) = f(u)$ where f is the monotone convex function

$$f = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

so that the g_k still satisfy $||g_k||_{H^1(B_{3/2})} \leq 4C$. Moreover it is clear that

$$|\{g_k = 0\}| \geq |B_{3/2}| - 2\epsilon, \quad |\{g_k = 1\}| > \epsilon$$

$$\lim_{k\to\infty} |\{g_k = 0\}| = |B_{3/2}| - \epsilon, \quad \lim_{k\to\infty} |\{g_k = 1\}| = \epsilon$$

for all k sufficiently large, applying problem 48 yields that all g_k for k sufficiently large must satisfy

$$|\{0 < g_k < 1\}| > \alpha > 0$$

but this is a contradiction as for k increases implies that the measure of this set goes to 0. Thus we must have $\delta_k \to 0$ as $k \to \infty$.

We now want to apply problem 48 to the $g_k : B_{2-\sigma} \to [0,1]$ with σ to be determined. Note that because

$$|\{x \in B_2 \ : \ u(x) \geq 1\}| \geq \delta$$
$$\implies |\{x \in B_{2-\sigma} \ : \ u(x) \geq 1\}| \geq \delta/2$$
$$\sigma < 2 \ \& \ |B_2| - |B_\sigma| = \delta/2$$

basically, we can't have more than δ of measure in an aribtrarily small region near ∂B_2. Now apply cacciopoli, so that

$$||g_k||_{L^2(B_{2-\sigma})} \leq C\sigma^{-1}||g_k||_{L^2(B_2)} \leq C\sigma^{-1}$$

for C dependent only on dimension and elliptical coefficients. We can now apply 48 in the following manner.

Choose an $\alpha > 0$ so that $\delta_1 > \alpha$. Then

$$\exists N \text{ s.t. } \ \delta_k \geq \alpha \quad \forall k+1 \leq N$$

then we have that

$$B_{2-\sigma} \supseteq \{g_k = 0\} = \{(1-2^k u)_+ = 0\} \cap B_{2-\sigma} = \{u \geq 2^{-k}\} \cap B_{2-\sigma} \supseteq \{u \geq 1\} \cap B_{2-\sigma}$$

$$\implies |\{g_k = 0\}| \geq |\{x \in B_{2-\sigma} \ : \ u \geq 1\}| \geq \delta/2$$

and

$$\{g_k = 1\} = \{1 - 2^k u \geq 1/2\} = \{u \leq 2^{-k-1}\} \supseteq \{u < 2^{-k-1}\}$$
$$\implies |\{g_k = 1\}| \geq \delta_{k+1} > \alpha$$

Thus we get that

$$|\{0 < g_k < 1\}| > \epsilon(\delta, \alpha, C\sigma^{-1})$$

$$S_k = \{0 < g_k < 1\} = \{0 < 1 - 2^k u < 1/2\} = \{2^{-k-1} < u < 2^{-k}\}$$
$$\implies \delta_{k+1} = |\{0 < g_k < 1\}| > \epsilon(\delta, \alpha, C\sigma^{-1})$$

but it is clear that the $\{S_k\}$ are disjoint for different values of k. And thus

$$\mu(B_2) = 2 > \mu\left(\bigcup_{k=1}^{N-1} S_k\right) = \sum_{k=1}^{N-1} \mu(S_k) > \epsilon(N-1)$$
$$\implies N - 1 < \frac{2}{\epsilon}$$

from this bound, we see that the maximal value of N is $N_0 =]2/\epsilon[+ 1$, and so we must have $\delta_{N_0+1} \leq \alpha$, and in general

$$\delta_n \leq \min\{\alpha \text{ s.t. } \epsilon(\alpha, \delta/2, C\sigma^{-1})(n-1) < 2\}$$

we can write the α from 48 as a function of the $\{a_{ij}\}$, dimension d, and the constants δ_0 and δ_1. But for the g_k, we have

$$notdone$$

still not.

Question 50
Consider the ϵ_0 in question 47. Apply question 49, but repeated in B_{r_0} with r_0 to be determined, so that

$$\forall k > K, \quad \delta_k < \epsilon$$

then we have that

$$|\{x \in B_{r_0} \ : \ u(x) \geq 2^{-K}| = |B_{r_0}| - \delta_K = \gamma |B_2|$$

we want $\gamma > (1 - \epsilon_0)$, which is possible if we first choose r_0 so that

$$|B_{r_0}| > (1 - \epsilon_0/2)|B_2|$$

and then make K large so that ϵ can be chosen small enough so

$$|B_{r_0}| - \delta_K = \gamma |B_2| > (1 - \epsilon_0)|B_2|$$

Now consider $2^K u(x)$, which satisfies the conditions of problem 47, and we get that

$$\text{ess-inf}_{B_1} 2^K u \geq 1/2 \implies \text{ess-inf}_{B_1} u \geq 2^{-k-1} > 0$$

finishing the proof. □

Question 51
Let $u : B_2 \to [0, 1]$ be a solution. Prove that

$$\text{osc}_{B_1} u := \big(\text{ess-sup}_{B_1} u - \text{ess-inf}_{B_1} u\big) \leq (1 - \theta)$$

for some $\theta > 0$ depending only on dimension and ellipticity constants.

Proof. If $|\{x \in B_2 : u(x) \geq 1/2\}| \geq |B_2|/2$, we apply the result of question 50 to $2u$ and obtain $\text{ess-inf}_{B_1} 2u \geq \theta_0$ for some $\theta_0 > 0$. Thus, $\text{ess-inf}_{B_1} u \geq \theta$ for some $\theta > 0$ depending only on dimension and ellipticity constants. It follows that since $\text{ess-sup}_{B_1} u \leq 1$, we have $\text{osc}_{B_1} u \leq 1 - \theta$.

Otherwise, $|\{x \in B_2 : u(x) \leq 1/2\}| \geq |B_2|/2$. Notice that $v(x) := 1 - u(x)$ is then also a solution and v satisfies the case above. It follows that $\text{osc}_{B_1} v \leq 1 - \theta$. But $\text{ess-sup}\, v = 1 - \text{ess-inf}\, u$ and $\text{ess-inf}\, v = 1 - \text{ess-sup}\, u$, so $\text{osc}_{B_1} v = \text{osc}_{B_1} u$, and we have the desired result. □

Question 52

Via the hint, I'll prove

$$||u||_{C^\alpha(B_{1/2})} \leq C||u||_{L^2(B_2)}$$

because if it was meant to be proved with B_1, then simply adjust all of the previous questions which make reference to B_1 to $B_{3/2}$ and the constants will be adjusted, but we should be able to do everything to get a bound on B_1. The key idea is that we need to take $B_r(x_0) \subseteq B_{3/2}$ with $x_0 \in B_1$ and r bounded below uniformly in x_0. This will come up in the proof later.

First note that WLOG we can assume $u \geq 0$ everywhere. First note that u has a continuous representative by nature of it being in $H^1(B_2)$. From problem 40 part 2, we have that $\max(0, u)$ and $\max(0, -u)$ are both non-negative subsolutions so that theorem 5.1 applies to both and we get that

$$\text{ess-sup}_{B_1} \max(0, u) \leq C||u||_{L^2(B_2)}, \quad \text{ess-sup}_{B_1} \max(0, -u) \leq C||u||_{L^2(B_2)}$$

$$\implies 0 \leq u(x) + C||u||_{L^2(B_2)} \leq 2C||u||_{L^2(B_2)}$$

everywhere in B_2 having chosen the continuous representative of u. Now we might as well absorb the 2 into the C and assume that $u \geq 0$ everywhere because subsolutions and holder norms are not affected by constant shifts.

We'll need the following fact:

Lemma 4.1. *If $u : \Omega \to \mathbb{R}$ is a solution of the elliptic PDE, then $u\big|_{\Omega'}$ for any $\Omega' \subseteq \Omega$.*

Proof Really we want that the solution condition as described in problem 37 holds. Both

$$C_c^1(\Omega') \subseteq H_0^1(\Omega')$$

$$C^1(\Omega') \cap \{\varphi \mid \varphi\big|_{\partial\Omega'} = 0\} \subseteq H_0^1(\Omega')$$

hold, and because their closures in $H_0^1(\Omega')$ are the same (i.e. all of $H_0^1(\Omega')$), we can show that u is a solution in the sense of problem 37 (i.e. for the second collection of C^1 functions) by showing it for $C_c^1(\Omega')$ and then noting that it will hold for all of $H_0^1(\Omega')$ by nature of $\overline{C_c^1(\Omega')} = H_0^1(\Omega')$ (see the proof of 38, basically this is because the operator given by the elliptic PDE is a bounded linear functional on $H^1(\Omega')$). But clearly any $\varphi \in C_c^1(\Omega')$ is also in $C_c^1(\Omega)$ because we can just extend φ to be 0 outside of Ω' which will maintain the C^1 property. Thus

$$\partial_i[a_{ij}(x)\partial_j u] = 0 \text{ in } \Omega'$$

will hold. $\square$

With the lemma and the positivity assumption, define

$$w_0 = \frac{u}{C||u||_{L^2 B_2}}$$

so that $w : B_1 \to [0,1]$. Now for any $x_0 \in B_{1/2}$, choose r_0 with

$$1/2 \geq r_0 = (1 - |x_0|)/2 \geq 1/4$$

so that $B_{r_0}(x_0) \subseteq B_1$ and so by the lemma, w_0 is a solution on both $B_{r_0}(x_0)$ and $B_{2r_0}(x_0) \subseteq B_1$ with the same range. Thus

$$\operatorname{osc}_{B_{r_0}(x_0)} w_0 \leq (1-\theta)$$
$$\implies \operatorname{osc}_{B_{r_0}(x_0)} u \leq (1-\theta)C||u||_{L^2(B_2)}$$

now define

$$w_n = \frac{1}{C||u||_{L^2(B_2)}} \frac{u}{(1-\theta)^n}$$
$$r_n = \frac{r_0}{2^n}$$

recall here that θ only depends on the dimension and ellipticity constants, and not the radius or location of the ball in question (it should only depend on the ratio of the balls, **by a scaling argument**). Thus, we note that on $B_{r_1}(x_0)$ we have that $w_1 : B_{r_1}(x_0) \to [0,1]$, so applying 51 again we get

$$\operatorname{osc}_{B_{r_2}(x_0)} w_1 \leq (1-\theta) \implies \operatorname{osc}_{B_{r_2}} w_2 \leq 1$$

in general, we'll have

$$\operatorname{osc}_{B_{r_n}(x_0)} w_n \leq 1 \iff \operatorname{osc}_{B_{r_n}(x_0)} u \leq (1-\theta)^n C||u||_{L^2(B_2)} = r_0^\alpha (2^\alpha)^{-n} \frac{C||u||_{L^2(B_2)}}{r_0^\alpha} \leq [r_0 2^{-n}]^\alpha 4^\alpha C||u||_{L^2(B_2)}$$

for

$$\alpha = \frac{-\log(1-\theta)}{\log(2)} > 0$$

note that the θ in problem 50 is always at most 1/2, so that $0 < \alpha \leq 1$. For $|x-y| < 1/4$, we have that

$$\exists n \text{ s.t. } r_{n+1} \leq |x-y| < r_n$$
$$\implies |u(x)-u(y)| \leq \operatorname{osc}_{B_{r_n}} u \leq [r_0 2^{-n}]^\alpha 4^\alpha C||u||_{L^2(B_2)} \leq r_{n+1} 8^\alpha C||u||_{L^2(B_2)} \leq |x-y|^\alpha K||u||_{L^2(B_2)}$$

in this case $K = 8^\alpha C$ which is only dependent on ellipticity constants, and so

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \le K||u||_{L^2(B_2)}$$

as desired. For $|x - y| \ge 1/4$, you can create a sequence of balls to get from x to y with radius at least $1/4$ as prescribed above in at most 4 balls, so

$$|u(x)-u(y)| \le |u(x)-u(x_1)|+|u(x_1)-u(x_2)|+|u(x_2)-u(y)| \le K||u||_{L^2(B_2)}\left[|x_1 - x|^\alpha + |x_2 - x_1|^\alpha + |y - x_2|^\alpha\right.$$

$$\le 3K||u||_{L^2(B_2)} \le 3|x-y|^\alpha 4^\alpha K||u||_{L^2(B_2)}$$

and so in fact $3K$ works for all $x, y \in B_{1/2}$. □

Question 53': *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $u_n : \Omega \to \mathbb{R}$, $f_n : \Omega \to \mathbb{R}$ and $a^n : \Omega \to \mathbb{R}^{d\times d}$ be sequences so that*

- *For each $n = 1, 2, 3, \ldots,$*

$$\partial_i[a^n_{ij}(x)\partial_j u_n] = f_n \qquad \text{in } \Omega.$$

- *The coefficients a^n_{ij} are uniformly elliptic, with constants uniform in n. Moreover $a^n_{ij} \to a_{ij}$ almost everywhere in Ω.*
- *$f_n \to f$ in $H^{-1}(\Omega)$.*
- *$u_n \to u$ in $H^1(\Omega)$.*

Then,

$$\partial_i[a_{ij}(x)\partial_j u] = f \qquad \text{in } \Omega. \tag{4.0.7}$$

Conversely, if we have a solution to (4.0.7), there are sequences u_n, f_n and a^n of C^∞ functions as above.

Answer 1[Need to check other direction]
We only need a solution in the weak sense so because

$$\int_\Omega [a^n_{ij}\partial_j u_n]\frac{\partial\varphi}{\partial x_i} = -\int_\Omega f_n\varphi \qquad \forall\varphi \in C^\infty_c(\Omega) \subseteq H^1(\Omega)$$

and that integration is a linear functional, i.e.

$$T_{f_n}(\varphi) = -\int f_n\varphi$$

we know that $T_{f_n} \to T_f$ by the problem statement so that

$$\int f_n\varphi \to \int f\varphi \qquad \forall\varphi \in C^\infty_c(\Omega)$$

and also from the H^1 convergence of the $\{u_n\}$ we have that

$$\lim_{n\to\infty}\int_\Omega [a_{ij}^n \partial_j u_n]\frac{\partial\varphi}{\partial x_i} = \lim_{n\to\infty}\int_\Omega [\nabla u_n]^T A\nabla\varphi$$

$$\int_\Omega \big([\nabla u_n]^T A_n - [\nabla u]^T A\big) = \langle A_n\nabla u_n - A\nabla u, \nabla\varphi\rangle = \langle A_n u_n - A_n u, \varphi\rangle + \langle A_n u - Au, \varphi\rangle$$

note that

$$|\langle A_n u_n - A_n u, \varphi\rangle| \le ||A_n||\ ||u_n - u||_{H^1}||\varphi||_{H^1} \le \Lambda||u_n - u||_{H^1}||\varphi||_{H^1}$$

which goes to 0 for fixed φ as $n\to\infty$ by uniform ellipticity and $\{u_n\}$ convergence. The second term is as follows

$$|\langle A_n u - Au, \varphi\rangle| = \Big|\int_\Omega [a_{ij}^n - a_{ij}]\partial_j u\frac{\partial\varphi}{\partial x_i}\Big| \le \Big|\Big|(a_{ij}^n - a_{ij})\frac{\partial\varphi}{\partial x_i}\Big|\Big|_2\ ||\partial_j u||_2$$

to show that

$$\Big|\Big|(a_{ij}^n - a_{ij})\frac{\partial\varphi}{\partial x_i}\Big|\Big|_2 \to 0$$

we can use Egorov's theorem (because $\mu(\Omega) < \infty$ by nature of being bounded), to bound the above on

$$A\subseteq\Omega \quad s.t. \quad \mu(A^c) < \epsilon$$

and then on A^c, we use the fact that the $\{a_{ij}^n\}$ are uniformly elliptic, which gives the following bound on their L^∞ norms

$$|e_i A_n(x)e_j| = |a_{ij}(x)| \le ||A_n||\ |e_i|\ |e_j| \le \Lambda$$

where $||\cdot||$ is the standard operator norm of a matrix on vectors in $\mathbb{R}^d$. Thus each a_{ij}^n is bounded uniformily in n and i,j in their L^∞ norm on Ω. So

$$\Big|\Big|(a_{ij}^n - a_{ij})\frac{\partial\varphi}{\partial x_i}\Big|\Big|_2 \le 2\Lambda\Big|\Big|\frac{\partial\varphi}{\partial x_i}\Big|\Big|_2$$

but we can make

$$\Big|\Big|\frac{\partial\varphi}{\partial x_i}\Big|\Big|_{L^2(A^c)} < \epsilon$$

by choosing A^c sufficiently small because $\partial_i\varphi$ is L^1 and thus uniformily integrable. Thus as $n\to\infty$, we get

$$-\int_\Omega f\varphi = \lim_{n\to\infty}\int_\Omega [a_{ij}^n\partial_j u_n]\frac{\partial\varphi}{\partial x_i} = \int_\Omega [a_{ij}\partial_j u]\frac{\partial\varphi}{\partial x_i}$$

which implies that $\partial_i[a_{ij}\partial_j u] = f$.

For the other direction, take $u_n = \varphi_{1/n}\star u$, $a_{ij}^n(x) = \varphi_{1/n}\star a_{ij}(x)$, where we first extend u and $\{a_{ij}\}$, the former as a member of $H^1(\Omega)$ to $H^1(\mathbb{R}^d)$, and the latter as a member of

$L^1(\Omega)$ and then use Lemma 9.1 of Brezis to get convergence of u and its weak derivatives, i.e.

$$||u_n - u||_{L^2(\Omega)} \leq ||u_n - u||_{L^2(\Omega)} \to 0, \quad ||\partial_i u_n - \partial_i u||_{L^2(\Omega)} \leq ||\partial_i u_n - \partial_i u||_{L^2(\Omega)} \to 0$$

The pointwise convergence of $\{a_{ij}^n\}$ is self-evident and we also have that

$$\partial_i[a_{ij}^n \partial_j u_n] \in C^\infty(\Omega)$$

Now we can define

$$f_n = \partial_i[a_{ij}^n \partial_j u_n]\Big|_\Omega \in C^\infty(\Omega)$$

which will converge to f. By nature of the pointwise convergence of the $\{a_{ij}^n\}$ and the convergence of $\{u_n\} \to u$ in H^1, we automatically get that $f_n \to f$ as an element of $H^{-1}(\Omega)$ from our above work of equating the two limits.

Question 54': *Let $u \in H^1(B_1^+)$ be a solution of the equation*

$$\partial_i[a_{ij}(x)\partial_j u] = 0 \qquad \text{in } B_1^+,$$

where $\{a_{ij}\} : B_1^+ \to \mathbb{R}^{d\times d}$ are uniformly elliptic measurable coefficients. Assume that the trace of u on $B_1 \cap \{x_n = 0\}$ is zero. Consider the reflection:

$$\begin{aligned} u(x', -x_n) &= -u(x', x_n) \qquad \text{for } (x', x_n) \in B_1^+, \\ a_{ij}(x', -x_n) &= a_{ij}(x', x_n). \end{aligned}$$

Prove that this extended function $u : B_1 \to \mathbb{R}$ (yes, I still call it u) satisfies the equation

$$\partial_i[a_{ij}(x)\partial_j u] = 0 \qquad \text{in } B_1.$$

We will start with the following general fact.

Fact: If $u \in H^1(B_1^+)$, then the odd extension of u to B_1 is in $H^1(B_1)$ if and only if the trace of u on $\{x_n = 0\} \cap B_1$ is 0.

Let Γ denote $\{x_n = 0\} \cap B_1$. It is obvious that the odd extension $\tilde{u}$ of u is in $H^1(B_1 \setminus \Gamma)$. Let $\varphi \in C_c^1(B_1)$. If we let $T_+ : H^1(B_1^+) \to L^2(\Gamma), T_- : H^1(B_1^-) \to L^2(\Gamma)$ denote trace operators and we let u denote $\tilde{u}$ and the restriction of $\tilde{u}$ to appropriate portions of B_1, then it follows from Green's formula that

$$\int_{B_1} u \frac{\partial \varphi}{\partial x_i}\, dx = \int_{B_1^+} u \frac{\partial \varphi}{\partial x_i}\, dx + \int_{B_1^-} u \frac{\partial \varphi}{\partial x_i}\, dx$$

$$= -\int_{B_1^+} \varphi \frac{\partial u}{\partial x_i}\, dx + \int_\Gamma T_+ u \varphi(-e_n \cdot e_i)\, d\sigma - \int_{B_1^-} \varphi \frac{\partial u}{\partial x_i}\, dx + \int_\Gamma T_- u \varphi(e_n \cdot e_i)$$

$$= -\int_{B_1} \varphi \frac{\partial u}{\partial x_i}\, dx + \int_\Gamma (T_- - T_+) u\, \varphi (e_n \cdot e_i)\, d\sigma.$$

Therefore, $\tilde{u}$ is in $H^1(B_1)$ if and only if $T_- \tilde{u}\big|_{B_1^-} = T_+ u$. Since $\tilde{u}(x', x_n) = -\tilde{u}(x', -x_n)$, it follows that $\tilde{u} \in H^1(B_1)$ if and only if $T_+ u = -T_+ u$ if and only if $T_+ u = 0$. ☺

We now prove our result. Let $\varphi \in C_c^1(B_1)$, let u be a solution of $\nabla \cdot A(x) \nabla u(x) = 0$ in B_1^+, and let u denote the odd extension of u to $H^1(B_1)$. Let $\eta \in C^\infty(\mathbb{R})$ be an even function such that

$$(-\infty, -1/2] \cup [1/2, \infty) \prec \eta \prec (-\infty, -1/4) \cup (1/4, \infty).$$

Furthermore, define $\eta_\epsilon(t) = \eta(t/\epsilon)$ and $\varphi_\epsilon(x) = \eta_\epsilon(x_n) \nabla \varphi(x', x_n)$. Since

$$\nabla \varphi_\epsilon(x) = \eta_\epsilon(x_n) \nabla \varphi(x) + \eta_\epsilon'(x_n) \varphi(x) e_n,$$

we know that

$$\int_{B_1} \langle A(x) \nabla u(x), \nabla \varphi_\epsilon(x) \rangle dx = \int_{B_1} \langle A(x) \nabla u(x), \nabla \eta_\epsilon(x_n) \varphi(x) \rangle dx + \int_{B_1} \langle A(x) \nabla u(x), \eta_\epsilon'(x_n) \varphi(x) e_n \rangle dx.$$

We know from dominated convergence theorem that

$$\int_{B_1} \langle A(x) \nabla u(x), \nabla \varphi(x) \rangle\, dx = \lim_{\epsilon \to 0^+} \int_{B_1} \langle A(x) \nabla u(x), \eta_\epsilon(x_n) \nabla \varphi(x) \rangle\, dx.$$

Since $\operatorname{supp} \varphi_\epsilon \cap \{x_n = 0\} = \emptyset$ for each $\epsilon > 0$, it follows that

$$\int_{B_1} \langle A(x) \nabla u(x), \nabla \varphi(x) \rangle\, dx = -\int_{B_1} \langle A(x) \nabla u(x), \eta_\epsilon'(x_n) \varphi(x) e_n \rangle\, dx.$$

Using the fact that $\eta'(-x_n) = -\eta'(x_n)$, if we let $\psi(x', x_n) = \varphi(x', -x_n)$, we compute that

$$\int_{B_1} \langle A(x) \nabla u(x), \eta_\epsilon'(x_n) \varphi(x) e_n \rangle dx = \int_{B_1^+} \langle A(x) \nabla u(x), \eta_\epsilon'(x_n) \varphi(x) e_n \rangle dx + \int_{B_1^-} \langle A(x) \nabla u(x), \eta_\epsilon'(x_n) \varphi(x) e_n \rangle dx$$

$$= \int_{B_1^+} \eta_\epsilon'(x_n) \langle A(x) \nabla u(x), [\varphi(x) - \psi(x)] e_n \rangle\, dx$$

$$= \int_{\{0 \le x_n < \epsilon\} \cap B_1} \frac{\eta'(x_n/\epsilon)}{\epsilon} \langle A(x) \nabla u(x), [\varphi(x) - \psi(x)] e_n \rangle\, dx.$$

Using the mean value theorem, we know that there exists $M = 2\|\partial_n \varphi\|_\infty$ such that

$$\left| \int_{\{0 \le x_n < \epsilon\} \cap B_1} \frac{\eta'(x_n/\epsilon)}{\epsilon} \langle A(x) \nabla u(x), [\varphi(x) - \psi(x)] e_n \rangle dx \right| \le \int_{\{0 \le x_n < \epsilon\} \cap B_1} \frac{|\eta'(x_n/\epsilon)|}{\epsilon} \Lambda |\nabla u(x)| |\varphi(x) - \psi(x)| dx$$

$$\le \int_{\{0 \le x_n < \epsilon\} \cap B_1} \Lambda \|\eta'\|_\infty M \frac{|x_n|}{\epsilon} |\nabla u(x)|\, dx$$

$$\leq \int_{\{0\leq x_n<\epsilon\}\cap B_1} \Lambda\|\eta'\|_\infty M|\nabla u(x)|\,dx \to 0. \quad \square$$

Question 55: *For any $\alpha > 0$, prove that there exists a solution to a uniformly elliptic equation in B_1 which is not C^α at the origin (of course, the uniform ellipticity constants will depend on α).*

Fix some $\alpha \in (0, \pi)$. We want to take the harmonic function

$$u(z) = \mathrm{Im}[z^{\pi/\alpha}]$$

defined in the sector $0 \leq \theta \leq \alpha$ of the unit disc (which we will call Ω) and precompose it with a change of variables $\phi : \Omega \to B_1^+$ to get the function

$$u \circ \phi^{-1}(r, \theta) = r^{\pi/\alpha} \sin\theta.$$

If we solve algebraically for ϕ, we see that ϕ is the map

$$\phi(r, \theta) = (r^{\pi^2/\alpha^2}, \frac{\pi}{\alpha}\theta).$$

ϕ is clearly a C^1 map with a C^1 inverse, so we know that $u \circ \phi^{-1}$ is a weak solution to the equation

$$\nabla \cdot B(y)\nabla(u \circ \phi^{-1})(y) = 0 \text{ in } B_1^+$$

$$\text{where } B(\phi(z)) = \frac{1}{\frac{d\phi^* z}{dz}(z)} D\phi(z) D\phi(z)^T.$$

Here $D\phi$ is the derivative of ϕ with respect to Cartesian coordinates, namely

$$D\phi = \begin{bmatrix} \frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_x}{\partial y} \\ \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} \end{bmatrix}.$$

After lots of computation, we get

$$D\phi(z) = \frac{\pi}{\alpha} r^{\frac{\pi^2}{\alpha^2}-2} \begin{bmatrix} \frac{\pi}{\alpha} x \cos(\frac{\pi}{\alpha}\theta) + y \sin(\frac{\pi}{\alpha}\theta) & \frac{\pi}{\alpha} y \cos(\frac{\pi}{\alpha}\theta) - x \sin(\frac{\pi}{\alpha}\theta) \\ \frac{\pi}{\alpha} x \sin(\frac{\pi}{\alpha}\theta) - y \cos(\frac{\pi}{\alpha}\theta) & \frac{\pi}{\alpha} y \sin(\frac{\pi}{\alpha}\theta) + x \cos(\frac{\pi}{\alpha}\theta) \end{bmatrix}$$

where $z = (x, y) = (r, \theta)$ in Cartesian and polar coordinates respectively.

It follows that

$$\frac{d\phi^* z}{dz}(z) = \det D\phi(z) = \big(\frac{\pi}{\alpha} r^{\frac{\pi^2}{\alpha^2}-2}\big) \frac{\pi}{\alpha} r^2.$$

Therefore

$$B(\phi(z)) = \begin{bmatrix} \frac{\pi}{\alpha}\cos^2(\frac{\pi}{\alpha}\theta) + \frac{\alpha}{\pi}\sin^2(\frac{\pi}{\alpha}\theta) & \frac{1}{2}(\frac{\pi}{\alpha} - \frac{\alpha}{\pi})\sin(2\frac{\pi}{\alpha}\theta) \\ \frac{1}{2}(\frac{\pi}{\alpha} - \frac{\alpha}{\pi})\sin(2\frac{\pi}{\alpha}\theta) & \frac{\pi}{\alpha}\sin^2(\frac{\pi}{\alpha}\theta) + \frac{\alpha}{\pi}\cos^2(\frac{\pi}{\alpha}\theta) \end{bmatrix}.$$

Now we B explicitly as

$$B(z) = \begin{bmatrix} \frac{\pi}{\alpha}\cos^2(\theta) + \frac{\alpha}{\pi}\sin^2(\theta) & \frac{1}{2}(\frac{\pi}{\alpha} - \frac{\alpha}{\pi})\sin(2\theta) \\ \frac{1}{2}(\frac{\pi}{\alpha} - \frac{\alpha}{\pi})\sin(2\theta) & \frac{\pi}{\alpha}\sin^2(\theta) + \frac{\alpha}{\pi}\cos^2(\theta) \end{bmatrix}$$

where $z = (x, y) \in B_1^+$, and $z = (r, \theta)$ in polar coordinates.

Note that this matrix B is meant to be applied to a gradient of partial derivatives with respect to Cartesian coordinates; the use of r and θ is just so that we have concise notation. Since ϕ is not bi-Lipschitz, we need to explicitly check to see that B is still uniformly elliptic. More computation yields that for all $z \in B_1^+$ the characteristic polynomial of B is

$$p(\lambda) = \lambda^2 - (\frac{\pi}{\alpha} - \frac{\alpha}{\pi})\lambda + 1.$$

Using the quadratic formula, we see that the eigenvalues are $\lambda = \frac{\alpha}{\pi}$ and $\Lambda = \frac{\pi}{\alpha}$. So B is uniformly elliptic. Therefore, the function $v : B_1^+ \to \mathbb{R}, v(z) = r^{\frac{\pi}{\alpha}}\sin\theta$ is the unique weak solution to the differential equation

$$\begin{aligned} \nabla \cdot B(z)\nabla v(z) = 0 \qquad & \text{in } B_1^+ \\ v = \sin\theta \qquad & \text{on } \partial B_1^+. \end{aligned}$$

We now conclude that since $v = 0$ on $\{y = 0\}$ we can create the odd reflection $u = r^{\frac{\pi}{\alpha}}\sin\theta$ which is the unique weak solution to the uniformly elliptic equation

$$\begin{aligned} \nabla \cdot B(z)\nabla u(z) = 0 \qquad & \text{in } B_1 \\ u = \sin\theta \qquad & \text{on } \partial B_1. \end{aligned}$$

Here B is reflected across $\{y = 0\}$ with an even reflection just as in Question 54'. u is $\frac{\alpha}{\pi}$-Hölder continuous, but not β-Hölder continuous for any $\beta > \frac{\alpha}{\pi}$ due to its behavior at the origin. $\square$

Question 56: *Let $f \in L^p(B_1)$ for some $p > d/2$. Let u be a solution of*

$$\begin{aligned} \partial_i[a_{ij}(x)\partial_j u] = f \ \text{ in } B_1, \\ u = 0 \ \text{ on } \partial B_1. \end{aligned}$$

Then

$$\|u\|_{L^\infty(B_1)} \le C\|f\|_{L^p(B_1)}.$$

Moreover, u is Hölder continuous in $\overline{B_1}$ with a norm depending on ellipticity, dimension and $\|f\|_{L^p}$ only.

First note that because $u \in H_0^1$ we have that

$$\int_\Omega |\nabla u|^2 \le \frac{1}{\lambda}\int_\Omega a_{ij}\partial_j u \partial_i u = -\frac{1}{\lambda}\int_\Omega fu$$

applying absolute value signs, we get

$$\frac{1}{\lambda}||\nabla u||_2^2 \leq ||f||_p||u||_q$$

for $1 = \frac{1}{p} + \frac{1}{q}$. Given that $p > d/2$, we have that

$$\frac{1}{q} = 1 - \frac{1}{p} > 1 - \frac{2}{d} = \frac{d-2}{d}$$

$$\implies q < \frac{d}{d-2} \leq 3$$

but $u \in H_0^1(B_1)$, so that we can extend $u \in H_0^1(\mathbb{R}^n)$ and then apply the sobolev inequality (really a corollary, see Corollary 9.10 in Brezis) to get that

$$||u||_{L^p(B_1)} \leq C||u||_{H_0^1(B_1)} \leq C'||\nabla u||_2 \qquad p \in [2, 2^*]$$

where I have passed back to the case of Ω at the expense of a constant, and also used Poincare's inequality in the last step. In this case,

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d} = \frac{d-2}{2d} \implies 2^* = \frac{2d}{d-2} > \frac{d}{d-2}$$

and so we have

$$\frac{1}{\lambda}||\nabla u||_2^2 \leq ||f||_p||u||_q \leq C||f||_p||\nabla u||_2$$

$$\implies ||\nabla u||_2 \leq \lambda C||f||_p$$

which gives $||u||_2 \leq K||f||_p$ by Poincare again. From this, we want to mimic the proof of theorem 5.1 from two weeks ago, because if we can prove that

$$\text{ess-sup}_{B_1} u \leq C||u||_{L^q(B_1)}$$

then from our bound $||\nabla u||_2 \leq K||f||_p$ and the fact that

$$||u||_q \leq C||u||_{2^*} \leq C||\nabla u||_2 \leq C||f||_p$$

we'll have a bound on $\text{ess-sup}\, u$ in terms of $||f||_p$, and we can then repeat the process for $\text{ess-sup}\, -u$. Formally, we want

Lemma 4.2. *For u the solution given in the statement, we have*

$$\text{ess-sup} \pm u \leq C||u||_q$$

for C dependent on only ellipticity constants, dimension, and potentially $||f||_p$.

Proof: The setup will be as follows

$$l_k = 1 - 2^{-k}, \quad u_k = (u - l_k)_+, \quad A_k = ||u_k||_{L^q(B_1)}$$

The proofs of question 42 and 43 from last time still hold (with all of the L^2 references replaced with L^q for our prescribed q), despite u being a solution with our given f. For our analogy of 44, , we have that

$$||u_{k+1}||_{L^{2^*}(B_1)} \leq C||\nabla u_{k+1}||_{L^2(B_1)} \leq \sqrt{||f||_p/\lambda}\sqrt{||u_{k+1}||_q} = K||u_{k+1}||_q^{1/2}$$

by sobolev inequality, and then pulling the same trick as in the beginning of this answer, but replacing the

$$\partial_i u \to \partial_i[u\chi_{u_{k+1}>0}]$$

which is valid because derivatives are not affected by constant shifts and we know that for $u \in H^1$, we have $u_+ = \max(0, u) \in H^1$ with

$$\nabla u_+ = \begin{cases} \nabla u & u > 0 \\ 0 & u \leq 0 \end{cases}$$

With this, our analogy of 45 is

$$||u_{k+1}||_q^q = \int |u_{k+1}|^q \leq ||u_{k+1}||_{2^*}^q \Big|\{u_{k+1} > 0\}\Big|^{1-\frac{q}{2^*}}$$

$$\implies ||u_{k+1}||_q \leq ||u_{k+1}||_{2^*}\Big|\{u_k > 0\}\Big|^{\frac{1}{q}-\frac{1}{2^*}}$$

for 46, we use the same chebyshev bound to get

$$\Big|\{u_k > 0\}\Big| \leq 2^{q(k+1)}||u_k||_q = (2^{k+1}A_k)^q$$

$$\implies ||u_{k+1}||_q = A_{k+1} \leq ||u_{k+1}||_{2^*}(2^{k+1}A_k)^{1-q/2^*}$$

but

$$q < \frac{d}{d-2} \qquad \frac{1}{2^*} = \frac{d-2}{2d} \implies \frac{1}{2^*} < \frac{1}{2}$$

and so now using the fact that $0 \leq u_{k+1} < u_k$, we get that

$$||u_{k+1}||_{2^*} \leq ||u_k||_{2^*} \leq \rho||u_k||_q^{1/2} = \rho A_k^{1/2}$$

and so combining these inequalities, we get

$$A_{k+1} \leq \rho 2^{(k+1)(1-q/2^*)} A_k^{1+(1/2-q/2^*)}$$

at this point, we have a similar enough recurrence relationship (because $1/2 - q/2^* > 0$) that we can conclude that $A_k \to 0$ for $||u||_2 = \delta_0$ sufficiently small.

Repeat the theorem for $-u$ and we get the lemma. And thus we conclude that

$$||u||_{L^\infty} \leq C||u||_2 \leq C'||f||_p$$

for constants depending only on ellipticity, dimension, and potentially $||f||_p$.

To show Holder continuity, we want the following lemma

Lemma 4.3.

$$\forall u \in L^p(B_1), \qquad \fint_{B_r(x)\cap B_1} |u(x)-u(y)|^p dx \le C r^{\alpha p} \iff u \in C^\alpha$$

Pf: Assume the latter, then

$$\fint |u(x)-u(y)|^p dx \le C^p \fint |x-y|^{\alpha p} \le C^p \fint r^{\alpha p} \le C r^{\alpha p}$$

Now assume the former, then

$$\forall\ |x-z|<r, \quad |u(x)-u(z)| = \fint_{B_r(x)\cap B_1} |u(x)-u(y)|dy + \fint_{B_r(x)\cap B_1} |u(y)-u(z)|dy$$

$$\fint_{B_r(x)\cap B_1} |u(x)-u(y)|dy \le \left[\fint_{B_r(x)\cap B_1} |u(x)-u(y)|^p dy\right]^{1/p} \left[\int_{B_r(x)\cap B_1} 1^q\right]^{1/q} \le C r^{\alpha p}\mu(B_1) \le C' r^{\alpha p}$$

$$\fint_{B_r(x)\cap B_1} |u(y)-u(z)|dy \le 2^d \fint_{B_{2r}(z)\cap B_1} |u(y)-u(z)|dy \le C r^{\alpha p}$$

because $B_r(x) \subseteq B_{2r}(z)$ and increasing the ball of integration will dilate the volume by at least 1 and at most 2^d. □

From this, we continue with the proof as follows

$$\fint_{B_r(x_0)\cap B_1} |u(x_0)-u(y)|^p \le \frac{K}{r^d} \int_{B_r(x_0)\cap B_1} |u(x_0)-u(y)|^p$$

which basically says that $|B_r(x_0)\cap B_1|/|B_r| \le C$ for all choices of x_0 and r. Now

$$\int_{B_r(x_0)\cap B_1} |u(x_0)-u(y)|^p \le C r^d \int_{B_1\cap B_{1/r}(-x_0)} |u(x_0)-u(x_0+ry)|^p dy$$

$$\int_{B_1\cap B_{1/r}(-x_0)} |u(x_0)-u(x_0+ry)|^p dy \le$$

$$2^p \int_{B_{1/r}(-x_0)\cap B_1} |v(0)-v(y)|^p dy + 2^p \int_{B_{1/r}(x_0)\cap B_1} |[v(0)-u(x_0)] - [v(y)-u(x_0+ry)]|^p$$

where

$$v : B_1 \cap B_{1/r}(-x_0) \to \mathbb{R}, \qquad \partial_i[a_{ij}(rx+x_0))\partial_j v] = r^2 f(rx+x_0)) \qquad v\Big|_{\partial(B_1\cap B_{1/r}(-x_0))} \equiv 0$$

yet $u(rx+x_0)$ is also a solution to the above elliptic PDE, and so their difference is a solution. From which we know that

$$g(y) = v(y) - u(x_0+ry)$$

is a solution on the relatively nice domain of $B_{1/r}(x_0) \cap B_1$, and so we have a holder bound, and immediately we get that

$$\int_{B_{1/r}(x_0)\cap B_1} |[v(0) - u(x_0)] - [v(y) - u(x_0 + ry)]|^p \leq C_1 r^{\alpha p + d}$$

Now using the first part, and choosing a representative of v such that $|v(0)| < ||v||_{L^\infty}$, we get that

$$\int_{B_{1/r}(-x_0)\cap B_1} |v(0) - v(y)|^p dy \leq \int_{B_{1/r}(-x_0)\cap B_1} |2Cr^2||f_{r,x_0}||_p|^p dy$$

$$f_{r,x_0}(x) = f(rx + x_0)$$

$$\implies \int_{B_{1/r}(-x_0)\cap B_1} |v(0) - v(y)|^p dy \leq Cr^{2p}|B_1|\ ||f_{r,x_0}||_p^p \leq C_2 r^{2p-d}||f||_p$$

where we used a change of variables to get

$$||f_{r,x_0}||_{L^p(B_{1/r}(-x_0)\cap B_1)} \leq Cr^{-d}||f||_{L^p(B_1)}$$

Combining these inequalities, we get

$$⨍_{B_r(x_0)} |u(x_0) - u(y)|^p \leq C \int_{B_1 \cap B_{1/r}(-x_0)} |u(x_0) - u(x_0 + ry)|^p dy$$

$$\leq C_1 r^{\alpha p + d} + C_2||f||_p r^{2p-d}$$

now let $\alpha p + d = p(\alpha + d/p)$ and $2p - d = p(2 - d/p)$ and $\beta = \min(\alpha + d/p, 2 - d/p)$, both option of which are positive so that

$$⨍_{B_r(x_0)} |u(x_0) - u(y)|^p \leq C_3 r^{\beta p}$$

where we use the fact that $r < 1$ so that we can take the minimum. This implies that u is holder continuous with that convoluted holder exponent, and the holder constant depending on the dimension and $||f||_p$. □

Question 57: *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Here $d \geq 3$. Let us consider the operator $S : L^2(\Omega) \to H_0^1(\Omega)$ defined as $Sf := u$ where*

$$\begin{aligned} \partial_i[a_{ij}(x)\partial_j u] &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Here a_{ij} are symmetric uniformly elliptic coefficients as usual. Prove that there exists a function $G : \Omega \times \Omega \to \mathbb{R}$ such that

$$Sf(x) = \int_\Omega G(x, y) f(y)\, dy.$$

Needless to say, this function G is the Green function.

First define the analogous operators S_p from $L^p(\Omega) \to L^\infty(\Omega)$ for $p > \frac{d}{2}$ using Question 56. We know that we can apply Question 56 here since the result still holds for Ω bounded with Lipschitz boundary (our proof only uses these facts). Therefore, $S_p f$ is Hölder continuous for all $f \in L^p(\Omega), p > d/2$. It follows that the functional $f \to S_p f(x)$ is well-defined for any fixed $x \in \Omega$. In fact, this is a bounded linear functional, since Question 56 gives us C such that

$$|S_p f(x)| \leq \|S_p f\|_\infty \leq C\|f\|_p.$$

It follows that there exists $G_p(x, \cdot) \in L^q(\Omega)$ such that

$$S_p f(x) = \int_\Omega G_p(x, y) f(y)\, dy \quad \forall\, f \in L^p(\Omega), x \in \Omega.$$

We know that the operators $S_p, p \in (\frac{d}{2}, \infty)$ agree on $C_c(\Omega)$. Since $C_c(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty)$, it follows that there exists $G(x, \cdot)$ such that $G(x, \cdot) = G_p(x, \cdot)$ for all $p \in (\frac{d}{2}, \infty)$. Since $q = \frac{p}{p-1}$, we see that $p > \frac{d}{2}$ if and only if $q \in (1, \frac{d}{d-2})$. It follows that $G(x, \cdot) \in L^q(\Omega)$ for all $q \in (1, \frac{d}{d-2})$. Since Ω is bounded, we conclude that $G(x, \cdot) \in L^1(\Omega)$ as well.

We now turn our attention so the operator $S : L^2(\Omega) \to H_0^1(\Omega)$. We know that

$$Sf(x) = \int_\Omega G(x, y) f(y)\, dy \quad \forall\, f \in \bigcup_{p > \frac{d}{2}} L^p \cap L^2, x \in \Omega.$$

Since this equality holds on a dense set, we want to use some sort of continuity to extend the equality to all of $L^2(\Omega)$. If Ω is unbounded, then S will not be bounded and we will need to use weaker notions of continuity. However, we are assuming for this part that Ω is bounded, so S will be continuous. In fact, it will be compact.

Since $Sf \in H_0^1(\Omega)$ and Ω is bounded, Poincaré's inequality gives us C such that for all $f \in L^2(\Omega)$, we get

$$\|\nabla Sf\|_2^2 \leq \frac{1}{\lambda} \int_\Omega \langle A(x) \nabla Sf(x), \nabla Sf(x) \rangle\, dx = -\frac{1}{\lambda} \int_\Omega Sf(x) f(x)\, dx \leq \frac{1}{\lambda} \|Sf\|_2 \|f\|_2$$

$$\leq \frac{C}{\lambda} \|\nabla Sf\|_2 \|f\|_2.$$

$$\Rightarrow \|\nabla Sf\|_2 \leq \frac{C}{\lambda} \|f\|_2.$$

It follows (again from Poincaré's inequality) that $S \in \mathcal{L}(L^2(\Omega), H_0^1(\Omega))$. Since $\partial\Omega$ is Lipschitz, it follows from Rellich-Kondrachov that $S : L^2(\Omega) \to L^2(\Omega)$ is compact. Since S is self-adjoint (see part (c) below, which is proven using only the existence of G as we have currently defined it), it follows that there exists an orthonormal eigenbasis ψ_n of $L^2(\Omega)$ such that

$$Sf = \sum_n \lambda_n \langle f, \psi_n \rangle \psi_n \quad \forall\, f \in L^2(\Omega).$$

Since $S\psi_n = \lambda_n \psi_n$, each ψ_n either is Hölder continuous and in $H_0^1(\Omega)$ or is in ker$S$. We quickly see, however, that if $Sf = 0$, then $f = \nabla \cdot 0 = 0$, so S is injective and all of its eigenvalues are nonzero. In fact, we can quickly compute that

$$\lambda_n \int_\Omega \langle A(y)\nabla\psi_n(y), \nabla\psi_n(y)\rangle \, dy = -\int_\Omega \psi_n^2 \, dy = -1.$$

$$\implies \lambda_n = -\left(\int_\Omega \langle A(y)\nabla\psi_n(y), \nabla\psi_n(y)\rangle \, dy\right)^{-1} < 0 \,\forall\, n.$$

So all of the eigenvalues are negative. Since the above series has a pointwise a.e. convergent series, we conclude that for any $f \in L^2(\Omega)$ Sf has a pointwise a.e. representative

$$Sf(x) = \sum_n \lambda_n \langle f, \psi_n \rangle \psi_n(x) = \sum_n \lambda_n \langle f, \psi_n \rangle \int_\Omega G(x,y)\psi_n(y)\, dy = \int_\Omega G(x,y) f(y)\, dy.$$

This last bit lacks some details, but they can be filled in with relative ease. $\square$

(a) *For every fixed $x \in \Omega$, $G(x,\cdot) \in L^q(\Omega)$ for every $q \in [1, d/(d-2))$.*

This was already shown above. $\square$

(b) *The map $x \to G(x,\cdot)$ is continuous from Ω to $L^q(\Omega)$.*

It follows from Question 56 that there exists $C = C(\Lambda, \lambda, d, \Omega, \|f\|_p)$ such that

$$\|G(x,\cdot) - G(z,\cdot)\|_q = \max_{\|f\|_{L^p(\Omega)}=1} \int_\Omega [G(x,y) - G(z,y)] f(y)\, dy = \max_{\|f\|_{L^p(\Omega)}=1} |S_p f(x) - S_p f(z)|$$

$$\leq C|x-z|^\alpha. \quad \square$$

(c) *We have $G(x,y) = G(y,x)$ and $G \leq 0$.*

Recall that Sf is the unique solution to

$$\min\left\{ \int_\Omega \langle A\nabla u, \nabla u\rangle + 2fu \, dx : u \in H_0^1(\Omega)\right\}.$$

If $f \leq 0$, then since $|u| \in H_0^1(\Omega)$ we have

$$\int_\Omega \langle A\nabla|u|,\nabla|u|\rangle + 2f|u|\,dx = \int_\Omega \langle A\nabla u,\nabla u\rangle + 2f|u|\,dx \le \int_\Omega |\nabla u|^2 + 2fu\,dx \quad \forall\, u\in H^1_0(\Omega).$$

It follows that $Sf \ge 0$ whenever $f \le 0$. Therefore, $Sf \le 0$ whenever $f \ge 0$. Another quick proof of this fact comes from noticing that

$$\|\nabla Sf\|_2^2 \le \frac{1}{\lambda}\int_\Omega \langle A(x)\nabla Sf(x),\nabla Sf(x)\rangle\,dx = -\frac{1}{\lambda}\int_\Omega Sf(x)f(x)\,dx,$$

so it follows that $f\ge 0 \Rightarrow Sf \le 0$. We can conclude that

$$\int_\Omega G(x,y)f(y)\,dy = Sf(x) \le 0 \quad \forall\, f\in L^2(\Omega) \text{ s.t. } f\ge 0, x\in\Omega.$$

Therefore, $G\le 0$.☺

In order to show that $G(x,y)=G(y,x)$, we want to show that

$$\iint_{\Omega^2} G(x,y)f(x)g(y)\,dxdy = \iint_{\Omega^2} G(x,y)f(y)g(x)\,dxdy \quad \forall\, f,g\in L^2(\Omega).$$

Indeed we see that

$$\iint_{\Omega^2} G(x,y)f(x)g(y)\,dxdy = \int_\Omega Sg(x)f(x)\,dx,$$

so it is equivalent to show that S is self-adjoint as on operator on $L^2(\Omega)$. Since $Sg\in H^1_0(\Omega)$ we see that

$$\int_\Omega Sg(x)f(x)\,dx = -\int_\Omega \langle A(x)\nabla Sf(x),\nabla Sg(x)\rangle\,dx = \int_\Omega Sf(x)g(x)\,dx. \quad \square$$

(d) *The function G satisfies the equation*

$$\partial_{x_j}[a_{ij}(x)\partial_{x_j}G(x,y)] = 0 \qquad \textit{for } (x,y)\in\Omega\times\Omega\setminus\ \{x=y\}.$$

Fix $y_0\in\Omega$. Define $f_\delta = \omega_d^{-1}\delta^{-d}\chi_{B_\delta(y_0)}$ and $u_\delta = Sf_\delta$.It follows that

$$u_\delta(x) = Sf_\delta(x) = ⨍_{B_\delta(y_0)} G(x,y)\,dy \quad \forall\, x\in\Omega.$$

Since $f_\delta\ge 0$ we know that $u_\delta \le 0$. Let $x_0\in\Omega\setminus\{y_0\}$ and fix $r>0$ such that $y_0\notin B_r(x_0)$. We know that $\nabla\cdot A\nabla u_\delta = 0$ in $B_r(x_0)$ for all $\delta\le d(x_0,y_0)-r$. Since $-u_\delta$ is a nonnegative solution in $B_r(x_0)$, we can conclude the following:

1. $\|\nabla u_\delta\|_{L^2(B_{r/2}(x_0))} \le C_1\|u_\delta\|_{L^2(B_r(x_0))}$ $\forall\,\delta$ small enough (Cacciopoli's Inequality).

2. $[u_\delta]_{C^\alpha(B_{r/2}(x_0))} \le C_2\|u_\delta\|_{L^2(B_r(x_0))}$ $\forall\,\delta$ small enough (DeGiorgi-Nash).

3. $|B_{r/2}|^{1/p} \min_{B_{r/2}(x_0)} -u_\delta \leq \|u_\delta\|_{L^p(B_r(x_0))} \leq C_3 |B_{r/2}|^{1/p} \min_{B_{r/2}(x_0)} -u_\delta \quad \forall\, p \in [1,\infty), \delta$ small enough (Harnack Inequality).

Additionally, we know that

$$\lim_{\delta\to 0^+} u_\delta(x) = G(x,y_0) \qquad \text{almost everywhere.}$$

Fix a point $x_0 \neq y_0$ where $u_\delta(x_0) \to G(x_0,y_0)$. We know that $\{u_\delta(x_0)\}_\delta$ is bounded, so Harnack tells us that $\|u_\delta\|_{L^\infty(B_{r/2}(x_0))}$ is uniformly bounded by $C_3 \sup_\delta -u_\delta(x_0)$, with $r < \min\{d(x_0,y_0), d(x_0,\partial\Omega)\}$. It follows that $\|u_\delta\|_{L^2(B_{r/2}(x_0))} \leq |B_{r/2}|^{1/2}\|u_\delta\|_{L^\infty(B_{r/2}(x_0))} \leq |B_{r/2}|^{1/2} \sup_\delta \|u_\delta\|_{L^\infty(B_{r/2}(x_0))}$ for all $\delta > 0$ sufficiently small. Therefore, both $\|\nabla u_\delta\|_{L^2(B_{r/4}(x_0))}$ and $[u_\delta]_{C^\alpha(B_{r/4}(x_0))}$ are uniformly bounded in δ small enough. Since u_δ is uniformly Hölder continuous on $B_{r/4}(x_0)$, u_δ is equicontinuous on $\overline{B_{r/4}(x_0)}$. It follows from Arzelà-Ascoli that $u_\delta \to G(x,y_0)$ uniformly on $\overline{B_{r/4}(x_0)}$. Picking any $x_1 \in \overline{B_{r/4}(x_0)}$ and repeating this process, or just picking another point where u_δ converges pointwise and repeating this process, we see that $u_\delta(x) \to G(x,y_0)$ locally uniformly on $\Omega \setminus \{y_0\}$. It follows that $G(x,y_0)$ is continuous with respect to x for $x \in \Omega \setminus \{y_0\}$. In fact, for every $B \subset\subset \Omega \setminus \{y_0\}$ sufficiently far from y_0 and $x, z \in B$ we have

$$|G(x,y_0) - G(z,y_0)| \leq \limsup_{\delta\to 0^+} |u_\delta(x) - u_\delta(z)| \leq \sup_\delta [u_\delta]_{C^\alpha(B)} |x-z|^\alpha.$$

Therefore, $G(x,y_0)$ is also locally Hölder continuous in x away from $x = y_0$.

We also know that $\|u_\delta\|_{H^1(B_{r/4}(x_0))}$ is uniformly bounded, so every subsequence of u_δ has an $H^1(B_{r/4}(x_0))$-weakly convergent subsequence. But since $u_\delta \to G(\cdot,y_0)$ uniformly, and therefore in L^2, on $B_{r/4}(x_0)$, we see that each $H^1(B_{r/4}(x_0))$-weakly convergent subsequence of u_δ must be converging weakly to $G(\cdot,y_0)$ in $B_{r/4}(x_0)$. So $u_\delta \rightharpoonup G(\cdot,y_0)$ in $H^1(B_{r/4}(x_0))$. This allows us to conclude that $G(\cdot,y_0) \in H^1(B_{r/4}(x_0))$, which we did not previously know.

In fact, for any open set $V \subset\subset \Omega \setminus \{y_0\}$ we can cover $\overline{V}$ in an open cover of balls of the form $B_{r/4}(x_0)$ and reduce to a finite subcover. Since the $H^1(V)$ norms of u_δ are uniformly bounded by the sum over the finite subcover of the uniform bounds, and $u_\delta \to G(\cdot,y_0)$ uniformly on V, we see that $G(\cdot,y_0) \in H^1(V)$ and $u_\delta \rightharpoonup G(\cdot,y_0)$ on V.

Let (U,ϕ) be a Lipschitz patch of $\partial\Omega$, i.e. $\phi : B_1 \to \Omega$ is a bi Lipschitz map such that $U \cap \Omega = \phi(B_1^+)$. We are assuming that $y_0 \notin \overline{U}$ because these are the patches we care about, and these patches can cover $\partial\Omega$. Then $u_\delta \circ \phi$ is a solution to a divergence form uniformly elliptic pde in B_1^+. Since $u_\delta \circ \phi$ vanishes on $\{x_d = 0\}$, we know from Question 54' that the odd reflection of $u_\delta \circ \phi$ is a solution on B_1. Picking a point $x_0 \in B_1^+$ close to 0 such that $u_\delta \circ \phi(x_0) \to G(\phi(x_0),y_0)$, we can use Harnack again to uniformly bound $\|u_\delta \circ \phi\|_{L^\infty(B_{1/2}}$. This gives us a uniform bound on $\|u_\delta\|_{L^\infty(\phi(B^+_{1/2})}$. Since ϕ is bi Lipschitz, we also can repeat the above steps in B_1 to get uniform bounds on $[u_\delta]_{C^\alpha(\phi(B^+_{1/4}))}$ and $\|\nabla u\|_{L^2(\phi(B^+_{1/4}))}$ as well.

Now let $V \subset \Omega$ be any set such that $y_0 \notin \overline{V}$. Create the following open cover of $\overline{V}$: around every point of $\partial V \cap \partial\Omega$ assign an open set of the form $\phi(B^+_{1/4})$. Now to every point x of V not covered by these neighborhoods, assign a ball of the form $B_{r/4}(x)$. Reduce this to a finite subcover. Using this finite subcover, we know that $\|u_\delta\|_{H^1(V)}$ is uniformly bounded and $u_\delta \to G(\cdot, y_0)$ uniformly on V. Repeating the arguments above, we see that $u_\delta \rightharpoonup G(\cdot, y_0)$ in $H^1(V)$.

Let $\varphi \in H^1(\Omega)$ with $\operatorname{supp}\nabla\varphi = \omega$ and $y_0 \notin \omega$, and consider the bounded linear functional

$$u \to -\int_V \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle\, dx$$

on $H^1(V)$ where $\omega \subset \overline{V} \subset \overline{\Omega} \setminus \{y_0\}$. Since

$$-\int_V \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle\, dx = -\int_\Omega \langle A(x)\nabla u(x), \nabla\varphi(x)\rangle\, dx$$

for all $u \in H^1(\Omega)$, we know that

$$-\int_\Omega \langle A(x)\nabla u_\delta(x), \nabla\varphi(x)\rangle\, dx \to -\int_\Omega \langle A(x)\nabla G(x, y_0), \nabla\varphi(x)\rangle\, dx.$$

To summarize,

$$\int_\Omega \langle A(x)\nabla u_\delta(x), \nabla\varphi(x)\rangle dx \to \int_\Omega \langle A(x)\nabla G(x, y_0), \nabla\varphi(x)\rangle dx \quad \forall\varphi \in H^1(\Omega) \text{ s.t. } y_0 \notin \operatorname{supp}\nabla\varphi.$$

We also know from the definition of u_δ that

$$-\int_\Omega \langle A(y)\nabla u_\delta(y), \nabla\varphi(y)\rangle\, dy = \fint_{B_\delta(y_0)} \varphi(y)\, dy \quad \forall\, \varphi \in H_0^1(\Omega).$$

It follows that

$$\lim_{\delta\to 0^+} -\int_\Omega \langle A(y)\nabla u_\delta(y), \nabla\varphi(y)\rangle\, dy = \varphi(y_0) \quad \forall\, \varphi \in C(\Omega) \cap H_0^1(\Omega).$$

Therefore, we see that

$$-\int_\Omega \langle A(x)\nabla G(x, y_0), \nabla\varphi(x)\rangle\, dx = 0 \quad \forall\, \varphi \in C_c^1(\Omega \setminus \{y_0\}).$$

By the symmetry of G, we conclude that

$$-\int_\Omega \langle A(y)\nabla_y G(x_0, y), \nabla\varphi(y)\rangle\, dx = 0 \quad \forall\, \varphi \in C_c^1(\Omega \setminus \{x_0\}). \quad \square$$

($\star$) *Let $\varphi \in H_0^1(\Omega)$ such that $x \notin \operatorname{supp}\nabla\varphi$. Then*

$$\varphi(x) = -\int_\Omega a_{ij}(y)\partial_i\varphi(y)\partial_{y_i} G(x, y)\, dy.$$

This follows immediately from the previous proof. Notice that the statement of this result makes sense because φ is smooth on $\Omega \backslash \operatorname{supp} \nabla\varphi$ so there is a natural choice of $\varphi(x)$. $\square$

Remark: The use of "support" in reference to $\nabla\varphi$ in the past two proofs is made in reference to the distributional support of $\nabla\varphi$. See Rudin's functional analysis, chapter 6.

($\star\star$) *For every fixed $x \in \Omega$, $G(x, \cdot)$ vanishes on $\partial\Omega$.*

u_δ converges uniformly to $G(\cdot, y_0)$ on a neighborhood of the boundary. $\square$

(e) *For every fixed $x \in \Omega$, $\nabla_y G(x, \cdot) \in L^1(\Omega)$.*

[Jared's solution] We want the following lemma to start

Lemma 4.4.

$$u \in H^1_0(\Omega), \quad \partial_i[a_{ij}\partial_j u] = \nabla \cdot F \qquad F \in [L^p(\Omega)]^d \quad \& \quad \nabla \cdot F \in L^p(\Omega) \ s.t. \ p > d$$

$$\implies u \in L^\infty(\Omega)$$

with $||u||_{L^\infty} \leq C||F||_p$.

Once we have proven this lemma, then since $G(x, \cdot)$ vanishes on $\partial\Omega$ we get

$$S_p(\nabla \cdot F)(x) = u(x) = \int G(x,y)[\nabla \cdot F(y)] = \sum_{i=1}^{d} \int_{\partial\Omega} G(x,y)F_i(y) - \int \nabla_y G(x,y) \cdot F(y)$$

$$= - \int \nabla_y G(x,y) \cdot F(y).$$

It follows that

$$\sup_{||F||_p=1} \int \nabla_y G(x,y) \cdot F(y) = ||\nabla_y G(x, \cdot)||_q \leq C,$$

but of course this means that

$$||\nabla_y G(x, \cdot)||_1 < \infty$$

because the domain is bounded and $q > 1$. Further, if we note that

$$[C^\infty(\Omega)]^d \subseteq \{F \in [L^p(\Omega)]^d \mid \nabla \cdot F \in L^p\} \subseteq [L^p(\Omega)]^d,$$

$$\overline{[C^\infty(\Omega)]^d} = [L^p(\Omega)]^d,$$

where the closure is taken w.r.t. the $|| \cdot ||_p$ norm, then it suffices to take a supremum over

$$F \in T = \{||F||_p = 1 \mid \nabla \cdot F \in L^p\} \subseteq [L^p(\Omega)]^d$$

Begin Proof
Since $u \in H_0^1(\Omega)$ and Ω is bounded we know that u extends to $H_0^1(\mathbb{R}^d)$ when made identically zero outside Ω. It follows from the SGN inequality that

$$||u||_{2^*} \leq C||\nabla u||_2.$$

Let q be dual to $p > \max(d/2, 2)$. Since $q \leq \min(2, \frac{d}{d-2}) \leq \frac{2d}{d-2} = 2^*$, Hölder's inequality now gives us

$$\int_\Omega |u|^q \, dx \leq \|u\|_{2^*}^q |\Omega|^{1-\frac{q}{2^*}}.$$

Therefore

$$(*) \quad \|u\|_q \leq |\Omega|^{1/q - \frac{1}{2^*}} \|u\|_{2^*} \leq C||\nabla u||_2.$$

Since we are given $\nabla \cdot F \in L^p$ and $u \in H_0^1$, Cauchy-Schwarz gives us

$$\|\nabla u\|_2^2 \leq \frac{1}{\lambda} a_{ij} \partial_i u \partial_j u = -\frac{1}{\lambda} \int (\nabla \cdot F) u = \frac{1}{\lambda} \int F \cdot \nabla u \leq \frac{1}{\lambda} \|F\|_2 \|\nabla u\|_2.$$

$$\implies \|\nabla u\|_2 \leq \frac{1}{\lambda} \|F\|_2.$$

(Note that we're allowed to perform integration by parts via trace theory because $u \in H_0^1(\Omega)$.) Since Ω is bounded and $p > 2$ Hölder's inequality gives us

$$\implies \sum_{i=1}^d \|\Phi_i\|_2 \leq |\Omega|^{1/2 - 1/p} \sum_{i=1}^d \|\Phi_i\|_p \quad \forall \, \Phi \in [L^p(\Omega)]^d.$$

Since $\sum_{i=1}^d \| \cdot \|_r \sim \| \cdot \|_r$ on $[L^r(\Omega)]^d$ for $r \in [1, \infty]$, it follows that there exists C independent of our initial choice of F such that

$$(**) \quad \|\nabla u_2\| \leq C\|F\|_p.$$

With this in mind, we can repeat question 56 with the set up of

$$\partial_j [a_{ij} \partial_i u] = \nabla \cdot F \qquad u\Big|_{\partial\Omega} \equiv 0$$

and then show that

$$||u||_{L^\infty} \leq C||u||_q \leq C||F||_p$$

via the exact same procedure as before. The only other time $||f||_p$ needs to be replaced is in the analogy of 44, when

$$||u_{k+1}||_{L^{2^*}(B_1)} \leq C||\nabla u_{k+1}||_{L^2(B_1)} \leq \sqrt{||f||_p/\lambda} \sqrt{||u_{k+1}||_q} = K||u_{k+1}||_q^{1/2}$$

but from the proceeding equations, we in fact have that

$$||u_{k+1}||_{L^{2^*}(B_1)} \leq C||\nabla u_{k+1}||_{L^2(B_1)} \leq C\sqrt{||\nabla \cdot F||_p ||u||_q} \leq K||u||_q^{1/2}$$

which is an analogous bound. $\square$

(f) *For any $\varphi \in C_c^1(\Omega)$, the following identity holds*

$$\varphi(x) = -\int_\Omega a_{ij}(y)\partial_i\varphi(y)\partial_{y_i}G(x,y)\,dy.$$

For $\epsilon > 0$, define φ_ϵ to be a smooth flattening of φ such that $\nabla\varphi_\epsilon = 0$ in $B_\epsilon(x)$. It is not hard to prove that $\|\nabla\varphi_\epsilon\|_\infty$ is uniformly bounded by some C. We conclude that the function $\Lambda C|\nabla G(x,y)|$ dominates $A(y)\nabla G(x,y)\cdot\nabla\varphi_\epsilon(y)$, and by dominated convergence theorem we get

$$-\int_\Omega \langle A(y)\nabla G(x,y), \nabla\varphi_\epsilon(y)\rangle\,dy \to -\int_\Omega \langle A(y)\nabla G(x,y), \nabla\varphi(y)\rangle\,dy.$$

It follows that

$$-\int_\Omega \langle A(y)\nabla G(x,y), \nabla\varphi(y)\rangle\,dy = \lim_{\epsilon\downarrow 0}\varphi_\epsilon(x) = \varphi(x). \quad \square$$

(g) *Prove that*

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla_y G(x,y)|^2\,dy \geq Cr^{2-d}.$$

Here C depends on Λ and d only.

We know that $G(x,\cdot) \in H^1(\Omega\setminus B_r(x))$. Therefore, for any $\varphi \in C_c^1(\Omega)$ with $\overline{B_r(x)} \prec \varphi \prec B_{2r}(x)$ we have

$$1 = \varphi(x) = \int_\Omega \langle A(y)\nabla G(x,y), \nabla\varphi(y)\rangle\,dy = \int_{B_{2r}(x)\setminus B_r(x)} \langle A(y)\nabla G(x,y), \nabla\varphi(y)\rangle\,dy$$

$$\leq \left(\int_{B_{2r}(x)\setminus B_r(x)} \langle A(y)\nabla G(x,y), \nabla G(x,y)\rangle\right)^{1/2}\left(\int_{B_{2r}(x)\setminus B_r(x)} \langle A(y)\nabla\varphi(y), \nabla\varphi(y)\rangle\right)^{1/2}$$

$$\leq \Lambda\left(\int_{B_{2r}(x)\setminus B_r(x)} |\nabla G(x,y)|^2\,dy\right)^{1/2}\left(\int_{B_{2r}(x)\setminus B_r(x)} |\nabla\varphi|^2\right)^{1/2}.$$

$$\implies \int_{B_{2r}(x)\setminus B_r(x)} |\nabla G(x,y)|^2\,dy \geq \frac{1}{\Lambda^2}\left(\int_{B_{2r}(x)\setminus B_r(x)} |\nabla\varphi|^2\right)^{-1}.$$

Consider the continuous function

$$\varphi_\delta(y) = \begin{cases} 1 & y \in \overline{B_{r+\delta}(x)} \\ \frac{2r-\delta}{r-2\delta} - \frac{1}{r-2\delta}|x-y| & y \in B_{2r-\delta} \setminus \overline{B_{r+\delta}(x)} \\ 0 & \text{elsewhere} \end{cases} .$$

$$\nabla\varphi_\delta(y) = \begin{cases} -\frac{1}{r-2\delta}\frac{y-x}{|y-x|} & y \in B_{2r-\delta} \setminus \overline{B_{r+\delta}(x)} \\ 0 & \text{elsewhere} \end{cases}$$

We know that

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla\varphi_\delta|^2 \, dx = \frac{1}{(r-2\delta)^2}|B_{2r-\delta} \setminus B_{r+\delta}| = \omega_d \frac{(2r-\delta)^d - r^d}{(r-2\delta)^2}.$$

If ρ is a mollifier, then for all $\epsilon \in (0, r-\delta)$ we have $\overline{B_r(x)} \prec \rho_\epsilon * \varphi_\delta \prec B_{2r}(x)$. It follows that

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla G(x,y)|^2 \, dy \geq \lim_{\epsilon\to 0^+} \frac{1}{\Lambda^2}\left(\int_{B_{2r}(x)\setminus B_r(x)} |\nabla(\rho_\epsilon * \varphi_\delta)|^2\right)^{-1} = \frac{1}{\omega_d\Lambda^2}\frac{(r-2\delta)^2}{(2r-\delta)^d - r^d}.$$

Letting $\delta \downarrow 0$ we see that

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla G(x,y)|^2 \, dy \geq \frac{1}{\omega_d \Lambda(2^d-1)} r^{2-d}. \quad \square$$

(h) T*here is a constant C (depending on the uniform ellipticity assumption only) such that for every $r > 0$,*

$$\sup\{-G(x,y) : y \in B_{5r/2}(x) \setminus B_{r/2}(x)\} \leq C \inf\{-G(x,y) : y \in B_{2r}(x) \setminus B_r(x)\}.$$

Provided that $B_{3r}(x) \subset \Omega$.

Let $V_r = B_{3r}(x) \setminus B_{r/4}(x)$ and equip V_r with the adjacency metric $\rho(\cdot,\cdot)$ (see Lawler's notes on harmonic functions). $B_{5r/2}(x) \setminus B_{r/2}(x)$ is a bounded subset of V_r with respect to this metric. There clearly exists $M \in \mathbb{Z}^+$ such that

$$\max_{x,y \in B_{5r/2}(x)\setminus B_{r/2}(x)} \rho(x,y) = M \quad \forall\, r \text{ s.t. } V_r \subset \Omega.$$

$\nabla \cdot A(y)\nabla G(x,y) = 0$ in V_r, and $-G(x,\cdot) \geq 0$, so Harnack tells us that there exists a universal constant C such that

$$\sup\{-G(x,y) : y \in B_{5r/2}(x) \setminus B_{r/2}(x)\} \leq C^M \inf\{-G(x,y) : y \in B_{2r}(x) \setminus B_r(x)\}. \quad \square$$

(i) *Let $m = \inf\{-G(x,y) : y \in B_{2r}(x) \setminus B_r(x)\}$. Assume $B_{3r}(x) \subset \Omega$. Prove that*

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla_y G(x,y)|^2 \, dy \leq C m^2 r^{d-2}.$$

Here C is a constant that depends only on the ellipticity constants and d.

Define V_r as in the previous part. Since $G(x,\cdot)$ is a solution in V_r, we know from Caccioopoli's inequality that

$$\int_{V_r} \varphi^2 |\nabla G(x,y)|^2 \, dy \le \frac{4\Lambda^2}{\lambda^2} \int_{V_r} |G(x,y)|^2 |\nabla \varphi(y)|^2 \, dy \quad \forall \, \varphi \in H_0^1(V_r).$$

Let φ be the radially symmetric function such that

$$\varphi(te_1) = \begin{cases} 0 & t \in (0, r/2] \\ \frac{2}{r}(t - r/2) & t \in (r/2, r) \\ 1 & t \in [r, 2r] \\ 1 - \frac{2}{r}(t-2r) & t \in (2r, 5r/2) \\ 0 & t \in [5r/2, +\infty) \end{cases}.$$

Then since

$$\int |\nabla \varphi|^2 = 4\omega_d [(\frac{5}{2})^d - 2^d + 1 - 2^{-d}] r^{d-2},$$

Cacciopoli's inequality gives us

$$\begin{aligned} \int_{B_{2r}(x)\setminus B_r(x)} |\nabla G(x,y)|^2 \, dy &\le \int_{V_r} \varphi(y)^2 |\nabla G(x,y)|^2 \, dy \\ &\le 16 \frac{\Lambda^2 \omega_d}{\lambda^2} [(\frac{5}{2})^d - 2^d + 1 - 2^{-d}] r^{d-2} \max_{y \in \overline{B_{5r/2}(x) \setminus B_{r/2}(x)}} |G(x,y)|^2 \\ &\le 16 \frac{\Lambda^2 \omega_d}{\lambda^2} [(\frac{5}{2})^d - 2^d + 1 - 2^{-d}] C^{2M} r^{d-2} m^2 \\ &= C r^{d-2} m^2. \quad \square \end{aligned}$$

(j) *Prove that if $|x-y| < \frac{2}{3} d(x, \partial\Omega)$ then*

$$-G(x,y) \ge C|x-y|^{2-d}.$$

Here C is a constant that depends only on the ellipticity constants and d.

We know from parts (g) and (i) that there exist constants C_1, C_2 (dependent only on ellipticity and d, not on r or Ω) such that

$$C_1 r^{2-d} \le \int_{B_{2r}(x)\setminus B_r(x)} |\nabla G(x,y)|^2 \, dy \le C_2 r^{d-2} m(r)^2 \quad \forall \, r \in (0, \frac{1}{3} d(x, \partial\Omega)).$$

Here $m(r) = \min\{-G(x,y) : y \in \overline{B_{2r}(x) \setminus B_r(x)}\}$. It follows that

$$\sqrt{\frac{C_1}{C_2}}r^{2-d} \le m(r) \quad \forall\, r \in (0, \frac{1}{3}d(x, \partial\Omega)).$$

Inserting $|x-y| = 2r$ we see that

$$\sqrt{\frac{C_1}{C_2}}2^{d-2}|x-y|^{2-d} \le m(\frac{1}{2}|x-y|) \le -G(x,y) \quad \forall\, y \text{ s.t. } |x-y| < \frac{2}{3}d(x,\partial\Omega). \quad \square$$

(k) *Prove the other inequality*

$$-G(x,y) \lesssim |x-y|^{2-d}.$$

Again, we want this to hold with a constant that depends only on ellipticity and d.

[Silvestre's solution] Recall that since $d \ge 3$, Sobolev embedding provides us with a universal constant $C = C(d,2)$ such that for any arbitrary open set Ω we have

$$\|u\|_{L^{2*}(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \quad \forall\, u \in H_0^1(\Omega).$$

See Remark 20 on page 290 of Bresiz if this isn't perfectly clear. Let $p = \frac{2d}{d+2}$, the Hölder conjugate of $2^* = \frac{2d}{d-2}$. Let $f = -\chi_{B_{2r}(x)\setminus B_r(x)}$, and let $u = Sf$. Since u is the unique minimizer of

$$J : H_0^1(\Omega) \to \mathbb{R}, J(v) = \int_\Omega \langle A(y)\nabla v(y), \nabla v(y)\rangle + 2f(y)v(y)\,dy,$$

we know that

$$0 = J(0) > J(u) \ge \lambda\|\nabla u\|_2^2 - 2\|f\|_p\|u\|_{2^*}.$$
$$\implies 2C\|f\|_p\|\nabla u\|_2 \ge \lambda\|\nabla u\|_2^2.$$
$$\implies \frac{2C}{\lambda}\|f\|_p \ge \|\nabla u\|_2.$$

Therefore,

$$(*) \quad \|u\|_{2^*} \le C\|\nabla u\|_2 \le \frac{2C^2}{\lambda}\|f\|_p.$$

We compute that

$$\|f\|_p = \left(\omega_d(2^d-1)r^d\right)^{\frac{d+2}{2d}} = [\omega_d(2^d-1)]^{\frac{d+2}{2d}}r^{\frac{d+2}{2}}.$$

Since G and f are negative and $f = 0$ in $B_r(x)$, u is a positive solution in $B_r(x)$ and Harnack's inequality gives us a universal constant C' depending only on ellipticity and dimension such that

$$C'u(x) \leq C' \sup_{B_{r/2}(x)} u \leq \inf_{B_{r/2}(x)} u.$$

It follows that

$$\|u\|_{2^*} \geq \left(\int_{B_{r/2}(x)} |u|^{\frac{2d}{d-2}} \, dy \right)^{\frac{d-2}{2d}} \geq C'u(x)[\omega_d(r/2)^d]^{\frac{d-2}{2d}} = C'u(x)\omega_d^{\frac{d-2}{2d}} 2^{-\frac{d-2}{2}} r^{\frac{d-2}{2}}.$$

We now conclude from equation $(*)$ that

$$C'u(x)\omega_d^{\frac{d-2}{2d}} 2^{-\frac{d-2}{2}} r^{\frac{d-2}{2}} \leq \frac{2C^2}{\lambda}[\omega_d(2^d-1)]^{\frac{d+2}{2d}} r^{\frac{d+2}{2}}.$$

$$\implies u(x) \leq \frac{\omega_d^{\frac{2}{d}} C^2 2^{\frac{d}{2}} (2^d-1)^{\frac{d+2}{2d}}}{\lambda C'} r^2 = \tilde{C} r^2.$$

Part (h) above now gives us M such that for all $r < \frac{1}{3}d(x,\Omega)$ we get

$$\tilde{C} r^2 \geq u(x) = -\int_{B_{2r}(x)\setminus B_r(x)} G(x,y)\, dy$$

$$\geq \omega_d(2^d-1)r^d m(r) \geq \omega_d(2^d-1)r^d C'^M(-G(x,y)) \quad \forall\, y \in B_{2r}(x) \setminus B_r(x).$$

$$\implies -G(x,y) \leq \frac{\omega_d^{\frac{2}{d}-1} C^2 2^{\frac{d}{2}} (2^d-1)^{\frac{2-d}{2d}}}{\lambda C'^{M+1}} r^{2-d} \quad \forall\, y \in B_{2r}(x) \setminus B_r(x).$$

Inserting $|x-y| = 2r$ we see that

$$|G(x,y)| \leq \frac{\omega_d^{\frac{2}{d}-1} C^2 2^{\frac{3d}{2}-2} (2^d-1)^{\frac{2-d}{2d}}}{\lambda C'^{M+1}} |x-y|^{2-d} \quad \text{when } |x-y| < \frac{2}{3} d(x,\partial\Omega). \quad \square$$

Question 58: *Let $w : B_R \to \mathbb{R}$ be a solution to the obstacle problem* $\min(w, 1-\Delta w) = 0$. *Assume that $w(x) = 0$ for some $x \in B_1$ and $R > 2$. Prove that*

$$\|w\|_{C^{1,1}(B_{R/2})} \leq C,$$

for some universal constant C.

Caffarelli proves this on page 18 of his notes on the obstacle problem. $\square$