Summary of Summer Analysis REU

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1 Introduction

Over the course of the Summer Analysis REU the four of us studied the Obstacle Problem under the supervision of Professor Luis Silvestre and Stephen Cameron, a graduate student studying PDEs. Our work consisted of three parts: (i) reading a set of notes by Professor Caffarelli on the theory of the obstacle problem focused on the study of the free boundary, (ii) solving weekly problem sets given to us by Professor Silvestre with material to supplement our readings, and (iii) discussing the readings and problem sets with Professor Silvestre in weekly meetings.

The reading by Professor Caffarelli was a set of lecture notes delivered in 1998. The main purpose of the notes was to study the regularity of the solutions to the obstacle problem and the regularity of the free boundary (the boundary of the contact set). The notes conclude with a novel result proven by Professor Caffarelli on the set of singular points of the free boundary. Because the lectures were given to people who studied PDEs many of the details are left out and familiarity with the proof techniques used is assumed. Hence, we had to do a lot of work during the reading filling in parts of proofs that Caffarelli omitted. He also used some larger results without proof that we independently derived on problem sets given to us by Professor Silvestre (see the sections on De Giorgi-Nash and LSW Theory below). In addition, our problem sets contained results to give us background on several other topics discussed in the readings including Hölder spaces, the Laplace equation, and variational analysis.

2 The Obstacle Problem

The Obstacle Problem is a variant of the Dirichlet problem for Laplace's equation. In addition to the boundary condition and minimizing the energy functional, there is a height condition that all solutions to the obstacle problem must satisfy. Formally, in the obstacle problem we are given (i) a domain $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, (ii) a continuous function f on $\partial\Omega$ (in the most generality it needs to at least be in the range of the trace operator from $H^1(\Omega) \to L^2(\partial\Omega)$), and (iii) a function $\varphi \in C(\overline{\Omega})$ such that $\varphi \leq f$ on $\partial\Omega$. The obstacle problem is to find a unique function u that minimizes the Dirichlet energy functional:

$$J(u) = \int_\Omega |\nabla u(x)|^2 dx \tag{1}$$

subject to the conditions (i) $u = f$ on $\partial\Omega$ and (ii) $u \geq \varphi$ in Ω. Since the Banach space $H^1(\Omega)$ is reflexive, $K = \{u\big|_{\partial\Omega} = f\} \cap \{u \geq f\}$ is a closed convex set in $H^1(\Omega)$, and $J : K \to \mathbb{R}$ is a coercive, continuous, and convex functional, we know that J attains its minimum. Since J is strictly convex, we know that this minimum is unique.

By looking at the first variation of J at u we can deduce:
(i) u is harmonic in the set $u > f$
(ii) u is superharmonic on Ω
Here we are using the weak (i.e. distributional) definitions of harmonic and superharmonic functions. For example, a function $u \in L^1_{\text{loc}}$ is weakly superharmonic if

$$\int_\Omega u\Delta\phi \leq 0 \qquad \forall\, \phi \in C_c^\infty(\Omega) \text{ s.t. } \phi \geq 0. \tag{2}$$

We stopped to prove that all superharmonic (subharmonic) functions $v \in L^1_{\text{loc}}$ have classically superharmonic (subharmonic) representatives, and went on to prove that the support of the distributional Laplacian of u is contained in the contact set $\{u = \varphi\}$. Since φ is continuous, we deduced that u is continuous everywhere as a consequence of a theorem of Evans:

Theorem 1. *Any superharmonic function u that is continuous on the support of Δu is continuous.*

For supplementary material, we were given exercises that introduced us to (i) basic variational analysis, (ii) the formulation of divergence form uniform elliptic PDEs as variational problems, (iii) a variety of basic yet useful results about Sobolev spaces, and (iv) various results about harmonic and subharmonic (superharmonic) functions. Those of us who had not already read chapters 1, 3, and 9 of Bresiz's *Functional Analysis, Sobolev Spaces, and Partial Differential Equations* or section 2.2 of Craig Evan's *Partial Differential Equations* did so as supplementary reading.

2.1 Regularity of Solutions and the Free Boundary

Next we proved the following theorem:

Theorem 2. *If $\varphi \in C^1(\Omega)$, then $u \in C^1(\Omega)$. If $\varphi \in C^{1,1}(\Omega)$, then $u \in C^{1,1}(\omega)$ for all $\omega \subset\subset \Omega$.*

It is easy to see that this is the best possible result we could hope for. Consider the case where φ has constant Laplacian $\Delta\varphi = -1$. Then across the boundary of the contact set $u = \varphi$ the Laplacian of u "jumps" from 0 to -1. So no matter how smooth φ is, u is not necessarily C^2 on all of Ω.

For supplementary material, we proved several results about Hölder spaces and Lipschitz spaces as exercises.

Problems like the obstacle problem are free boundary problems which means that the solution has some implicitly defined boundary which we are investigating the structure of. In this case the free boundary is the boundary of the contact set $\{u = \varphi\}$. It is not easy to immediately deduce anything about this free boundary, which is why the regularity theory is so important. To investigate this problem we used normalized solutions to the obstacle problem. A normalized solution to the obstacle problem is a function w on the unit ball $B_1 \subset \mathbb{R}^n$ such that:

(i) $w \geq 0$ in B_1 and $w \in C^{1,1}$.

(ii) On the set $\Omega = \{w > 0\}$ we have $\Delta w = 1$.

(iii) The point $0 \in \partial\Omega$.

Intuitively we have taken our solution u to an obstacle problem on B_1 and have subtracted the obstacle to get $w = u - \varphi$. The contact set is now the set $\{w = 0\}$, and we are given that this set is nonempty by condition (iii). Of course, our second constraint on w translates to $\Delta\varphi = -1$ on $\{u > \varphi\}$, which is not true for any arbitrary obstacle φ, but this is a much simpler case that we must start in.

The first thing we proved about the free boundary of normalized solutions is that they have finite $n-1$-dimensional Hasudorff measure. To do this we first proved that Ω has uniform positive density along the free boundary. This implies that the contact set is a set of finite perimeter. This shows that they roughly look like we would expect; that is, they do not have some bizarre fractal structure. In fact, due to a theorem of De Giorgi we know:

Theorem 3. *For every set A of finite perimeter there exists a set called the reduced boundary $\partial^\star A \subset \partial A$ and a vector field $\nu : \partial^\star A \to \partial B_1$ such that for any vector field $\varphi \in (C^1(A))^n$ we have:*

$$\int_A \operatorname{div} \varphi dx = \int_{\partial^\star A} \varphi \cdot \nu dH^{n-1} \tag{3}$$

and almost every point in the boundary is in the reduced boundary with respect to the H^{n-1} measure.

This shows that sets of finite perimeter look "almost" like C^1 manifolds where the divergence theorem can be applied:

$$\int_A \operatorname{div} \varphi dx = \int_{\partial A} \varphi \cdot n dS \tag{4}$$

To further our study of the free boundary we studied global solutions to the obstacle problem, that is to say normalized solutions that are defined in the whole of $\mathbb{R}^n$ not just the unit ball. These are important because given a point x in the free boundary we can define a blow-up solution around x as:

$$w_x^\star(y) = \lim_{t\to 0^+} \frac{w(x+ty)}{t^2} \tag{5}$$

Although this limit is not always guaranteed to exist we would suspect it usually does because we know that w separates approximately quadratically from the free boundary. Note that when this limit exists it defines a global solution to the normalized obstacle problem. Hence, an analysis of these solutions will help us understand the local geometry of the free boundary. For example, for points in the reduced boundary $x \in \partial^\star\Omega$ we know that the blow-up indicator functions:

$$f_R(y) = \chi_{R(\Omega - x)} \tag{6}$$

converge to indicator functions on some half space in L^1_{loc} as $R \to \infty$. Hence, we would assume that for these points the blow-up solutions to the obstacle problem converge to functions that are 0 on a half space and separate quadratically from 0. We proved the following about global solutions, showing that they have a much more well-defined structure than local solutions:

Theorem 4. *A global solution to the normalized obstacle problem is convex.*

We used these results to study the structure of the free boundary and prove the following theorem:

Theorem 5. *If w is a normalized solution there is a universal modulus of continuity $\sigma(r)$ such that if for some r_0 and $x \in \partial\Omega$ the set $\{w = 0\} \cap B_{r_0}(x)$ cannot be enclosed within a strip of width $r_0\sigma(r_0)$ then in a neighborhood of x the free boundary is a $C^{1,\alpha}$ surface for some α.*

The condition on enclosing the contact set within a strip is necessary to distinguish the regular points (around which the free boundary is a $C^{1,\alpha}$ manifold) from the singular points where this fails. The set of singular points has H^{n-1} measure 0 which can be deduced from the fact that they must lie outside the reduced boundary (as we know the blow-up indicator functions for the reduced boundary converge to a half strip). However, we went on to prove that the set of singular points roughly look like a manifold of codimension 1 within the free boundary. This can be proven by a close analysis of the global blow-up solutions near singular points.

3 Geometric Background

Our analysis of the free boundary used several results about sets of finite perimeter. Because none of us had much experience with geometric measure theory our week 3 problem set focused on proving basic results about Hausdorff dimension, Minkowski content, and sets of finite perimeter. We were able to bound the perimeter of sets in terms of their $n-1$-dimensional Hausdorff dimension and bound the m-dimensional Hausdorff dimension of Borel sets in terms of their m-dimensional upper Minkowski content. Thus, proving that the $n-1$ dimensional upper Minkowski content of the free boundary is finite proves that the contact set has finite perimeter as required.

4 Theory of Divergence Form Uniformly Elliptic Equations

During weeks 6-8 of the analysis REU we studied the theory of divergence form uniformly elliptic equations and proved several important results. This was divided into two parts: (i) De Giorgi-Nash theory, which analyzes the regularity of solutions to uniformly elliptic PDEs and allows us to prove a Harnack inequality for positive solutions, and (ii) Littman-Stampacchia-Weinberger theory, which analyzes the behavior of Green's functions and fundamental solutions for divergence form uniformly elliptic operators. These results were entirely derived by the four of us using guidance from exercises given by Professor Silvestre which walked us through the proofs.

A uniformly elliptic partial differential operator acting on functions on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ is given by a symmetric $n \times n$ coefficient matrix $A(x)$ of measurable functions

$a_{ij}(x)$ such that there exist two constants $\Lambda \geq \lambda > 0$ (called the ellipticity constants) such that $\lambda I \leq A(x) \leq \Lambda I$ for all $x \in \Omega$. Given such coefficients, we want to study solutions to the equation:

$$\partial_i(a_{ij}(x)\partial_j u) = 0 \tag{7}$$

$$\text{i.e.} \quad \sum_{ij} \partial_i(a_{ij}(x)\partial_j u) = 0$$

$$\text{i.e.} \quad \nabla \cdot A(x)\nabla u(x) = 0.$$

Since we have not placed any constraints on the smoothness of the coefficients, we primarily concern ourselves with the weak solutions, i.e. functions $u \in H^1(\Omega)$, such that

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla\varphi(x)\, dx = 0 \qquad \forall\, \varphi \in C_c^\infty(\Omega).$$

The existence of such weak solutions is guaranteed by variational analysis: for $f \in L^2(\Omega)$ and F in the range of the trace operator on $H^1(\Omega)$ the solution to

$$\begin{cases} \partial_i(a_{ij}(x)\partial_j u) = f & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega \end{cases}$$

is also unique minimizer of the functional

$$J : \{u \in H^1(\Omega) : u\big|_{\partial\Omega} = F\} \to \mathbb{R}, \quad J(u) = \int_\Omega A(x)\nabla u(x) \cdot \nabla u(x) + 2f(x)u(x)\, dx.$$

4.1 De Giorgi-Nash Theory

Our first important result is the following De Giorgi-Nash theorem:

Theorem 6. *There exist constants $C \geq 0$ and $\alpha \in (0, 1]$ depending only on the ellipticity constants λ, Λ and dimension n such that if u is a solution to $\partial_i(a_{ij}(x)\partial_j u) = 0$ in B_2 then*

$$||u||_{C^\alpha(B_1)} \leq C||u||_{L^2(B_2)}. \tag{8}$$

Here $\|\cdot\|_{C^\alpha}$ denotes the Hölder seminorm.

Given that we have placed no constraints on the regularity of the coefficients $a_{ij}(x)$, this is the best regularity result we can get. As an exercise, for any $\alpha > 0$ our group constructed a solution u to a uniformly elliptic equation $\partial_i(a_{ij}(x)\partial_j u) = 0$ in $B_1 \subset \mathbb{R}^2$ with continuous coefficients such that u was not C^α at the origin.

The outline of our proof for these results was as follows. First, we proved Cacciopoli's inequality which says that for a nonnegative subsolution u in B_2 and a function $\varphi \in C_c^\infty(B_2)$ we have for some constant C:

$$\int_{B_2} \varphi^2 |\nabla u|^2 dx \leq C \int_{B_2} u^2 |\nabla\varphi|^2 dx \tag{9}$$

Using this result and some of the Sobolev inequalities we deduced that for any nonnegative subsolution there exists a constant C such that:

$$\operatorname*{ess-sup}_{B_1} u \leq C||u||_{L^2(B_2)} \tag{10}$$

From here we deduced that for any solution u in a ball B_2 there exists some universal $\theta > 0$ such that

$$\operatorname*{osc}_{B_1} u(x) \leq (1-\theta) \operatorname*{osc}_{B_2} u(x). \tag{11}$$

We then iterated and scaled this result to show that u must be in C^α in B_1 for some α.

Using this theorem, we derived the following form of the Harnack inequality:

Theorem 7. *If u is a nonnegative solution to $\partial_i(a_{ij}(x)\partial_j u) = 0$ in B_1 then there exists a constant C depending only on the ellipticity constants and dimension such that*

$$\sup_{B_{1/8}} u(x) \leq C \inf_{B_{1/8}} u(x). \tag{12}$$

As a result, there exists a constant C depending only on the ellipticity constants and dimension such that for all nonnegative solutions u we have

$$\sup_{B_{1-\delta}} u(x) \leq \delta^{-C} \inf_{B_{1-\delta}} u(x) \quad \forall\, \delta \in (0,1). \tag{13}$$

To prove the Harnack inequality from the regularity result we used a construction by De-Giorgi in which one assumes towards a contradiction that the ratio

$$\frac{\sup_{B_{1/8}} u(x)}{u(0)} \tag{14}$$

is sufficiently large and proves from there that one can construct a sequence of points x_n in the ball $B_{1/4}$ such that $u(x_n)$ goes to infinity, contradicting the boundedness of u in $B_{1/2}$.

It follows from this Harnack inequality that for any domain Ω and any open set $V \subset\subset \Omega$ there exists a constant C depending on V, Ω, λ, and Λ such that if $\partial_i(a_{ij}(x)\partial_j u) = 0$ in Ω then

$$\sup_V u(x) \leq C \inf_V u(x).$$

4.2 LSW Theory

Next we analyzed Green's functions for uniformly elliptic PDEs. The theory here is based off the theory of the Laplacian. Global solutions to Poisson's equation

$$\Delta u = f \tag{15}$$

are given by the form

$$u(x) = \int_{\mathbb{R}^n} G(x,y) f(y) dy. \tag{16}$$

Here we have $G(x,y) = -\Phi(x-y)$, where Φ is the fundamental solution of Laplace's equation. It is known that for $n \geq 3$ we have $G(x,y) = \frac{C_n}{|x-y|^{n-2}}$ where C_n is a normalization constant depending only on dimension. Our goal in studying LSW theory was to show that Green's functions for divergence form uniformly elliptic equations existed and behaved similar to the Green's function for Laplace's equation on $\mathbb{R}^n$.

Green's functions for uniformly elliptic equations are defined similarly to the function $-\Phi(x-y)$ above. Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If u is a solution to the equation

$$\begin{cases} \partial_i(a_{ij}(x)\partial_j u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{17}$$

then u has the form

$$u(x) = \int_\Omega G(x,y) f(y) dy. \tag{18}$$

Here $G(x,y)$ is the Green's function associated with the given uniformly elliptic PDE. The main result that we proved in this section was a result cited in the appendix of Cafferelli's notes without proof:

Theorem 8. *Let $n \geq 3$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $G(x,y)$ be the Green's function for some divergence form uniformly elliptic operator in Ω. Then there exist nonnegative constants C_1, C_2 depending only on λ, Λ, and n such that for $|x-y| < \frac{1}{2}d(x, \partial\Omega)$ we have*

$$C_1|x-y|^{2-n} \leq -G(x,y) \leq C_2|x-y|^{2-n} \tag{19}$$

5 Conclusion

This summer we had three main accomplishments: (i) we gained a good understanding of the theory of the Obstacle Problem, (ii) we had an introductory look at many of the techniques and methods used in the study of PDEs and related free boundary problems, and (iii) we derived through problem sets given to us by Professor Silvestre several important results in the theory of uniformly elliptic equations. Because of the nature of the obstacle problem our group ended up studying variational analysis, geometric measure theory, Sobolev space theory, and more. Many of the techniques we studied can be applied to other standard free boundary problems that we discussed on problem sets. The understanding we gained this summer will be invaluable in continuing our studies of analysis and PDEs specifically.