SURF 2025 Problems + Solutions

Oscar Pan

Introduction

The below is a list of problems (assigned by Jared) from chapters 1-7 of Stein and Shakarchi.

1 Set 1

Problem 1. From the definitions, prove the commutative law $uv = vu$ and the distributive law $u(v + w) = (uv) + (uw)$.

Solution. Suppose $u = x_1 + iy_1$ and $v = x_2 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then, we have

$$\begin{aligned} uv &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned}$$

Since real numbers are commutative, we have

$$\begin{aligned} uv &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \\ &= (x_2x_1 - y_2y_1) + i(x_2y_1 + y_2x_1) \\ &= vu. \end{aligned}$$

Suppose $w = x_3 + iy_3$ with $x_3, y_3 \in \mathbb{R}$. Then, we have

$$\begin{aligned} u(v + w) &= (x_1 + iy_1)((x_2 + iy_2) + (x_3 + iy_3)) \\ &= (x_1 + iy_1)((x_2 + x_3) + i(y_2 + y_3)) \\ &= (x_1(x_2 + x_3) - y_1(y_2 + y_3)) + i(x_1(y_2 + y_3) + y_1(x_2 + x_3)). \end{aligned}$$

Since real numbers are commutative and distributive, we have

$$\begin{aligned} u(v + w) &= ((x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)) + ((x_1x_3 - y_1y_3) + i(x_1y_3 + y_1x_3)) \\ &= (uv) + (uw). \end{aligned}$$

□

Problem 2. Prove that, if $uv = 0$, then either $u = 0$ or $v = 0$.

Solution. Suppose $u = x_1 + iy_1$ and $v = x_2 + iy_2$ with $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then, we have

$$\begin{aligned} |uv| &= |(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)| \\ &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} \\ &= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2} \\ &= \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\ &= |u||v|. \end{aligned}$$

Hence, we have $|uv| = |u||v|$. Suppose $uv = 0$. Then, we have $|uv| = |u||v| = 0$. If $u \neq 0$, then either $x_1 \neq 0$ or $y_1 \neq 0$, so $|u| = \sqrt{x_1^2 + y_1^2} \neq 0$. Assume by contradiction that $v \neq 0$, then $|v| \neq 0$, so $|uv| = |u||v| \neq 0$. Therefore, we have $uv \neq 0$, which is a contradiction. Hence, we have $v = 0$. Therefore, we prove that either $u = 0$ or $v = 0$. □

Problem 3. Compute $(1+i)^k$ for $k \in \mathbb{Z}$.

Solution. We have $1 + i = \sqrt{2}e^{\pi i/4}$. Hence, we have

$$(1+i)^k = (\sqrt{2}e^{\pi i/4})^k = 2^{k/2}e^{k\pi i/4}.$$

When $k = 8n$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{2n\pi i} = 2^{k/2}.$$

When $k = 8n+1$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+\frac{1}{4})\pi i} = 2^{k/2}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = 2^{(k-1)/2}(1+i).$$

When $k = 8n+2$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+\frac{1}{2})\pi i} = 2^{k/2}i.$$

When $k = 8n+3$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+\frac{3}{4})\pi i} = 2^{k/2}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = 2^{(k-1)/2}(-1+i).$$

When $k = 8n+4$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+1)\pi i} = -2^{k/2}.$$

When $k = 8n+5$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+\frac{5}{4})\pi i} = 2^{k/2}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 2^{(k-1)/2}(-1-i).$$

When $k = 8n+6$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+\frac{3}{2})\pi i} = -2^{k/2}i.$$

When $k = 8n+7$ where $n \in \mathbb{Z}$, we have

$$(1+i)^k = 2^{k/2}e^{(2n+\frac{7}{4})\pi i} = 2^{k/2}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 2^{(k-1)/2}(1-i).$$

□

Problem 4. Find the roots $z \in \mathbb{C}$ of $z^8 - 16$ and prove that your list is exhaustive. Hint: Once you have the roots, factor the polynomial.

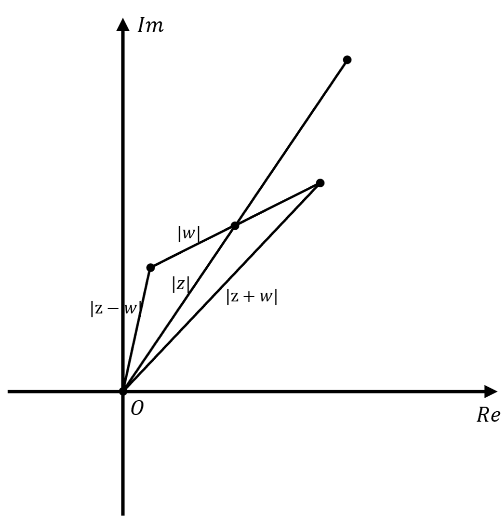
Solution. Suppose $z^8 - 16 = 0$. Then, we have $z^8 = 16 = 2^4 = (\sqrt{2})^8 e^{2k\pi i}$ where $k \in \mathbb{Z}$. Therefore, we have $z = \sqrt{2}e^{k\pi i/4}$.
When $k = 0$, we have $z_1 = \sqrt{2}$.
When $k = 1$, we have $z_2 = \sqrt{2}e^{\pi i/4} = \sqrt{2}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = 1 + i$.
When $k = 2$, we have $z_3 = \sqrt{2}e^{\pi i/2} = \sqrt{2}i$.
When $k = 3$, we have $z_4 = \sqrt{2}e^{3\pi i/4} = \sqrt{2}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = -1 + i$.
When $k = 4$, we have $z_5 = \sqrt{2}e^{\pi i} = -\sqrt{2}$.
When $k = 5$, we have $z_6 = \sqrt{2}e^{5\pi i/4} = \sqrt{2}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = -1 - i$.
When $k = 6$, we have $z_7 = \sqrt{2}e^{3\pi i/2} = -\sqrt{2}i$.
When $k = 7$, we have $z_8 = \sqrt{2}e^{7\pi i/4} = \sqrt{2}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = 1 - i$.
Therefore, we have can factor the polynomial as $z^8 - 16 = (z - z_1)(z - z_2)\cdots(z - z_7)f(z)$. Since the highest term is z^8 and the coefficient is 1, we have $f(z) = 1$. This proves that roots are exhaustive. □

Problem 5. Prove that $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$. Draw a picture to explain why this is called the parallelogram law.

Solution. Suppose $z = x_1 + iy_1$ and $w = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then, we have

$$\begin{aligned} |z + w|^2 + |z - w|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= 2x_1^2 + 2x_2^2 + 2y_1^2 + 2y_2^2 \\ &= 2(x_1^2 + y_1^2) + 2(x_2^2 + y_2^2) \\ &= 2|z|^2 + 2|w|^2. \end{aligned}$$

The figure is shown below.



□

Problem 6. Give a geometric description of the following sets:

(a) $\{z \in \mathbb{C} \colon |z+1| = |z-1|\}$.

(b) $\{z \in \mathbb{C} \colon |z+1| + |z-1| = a\}$ for $a \geq 0$.

(c) $\{z \in \mathbb{C} \colon \operatorname{Re}(u+vz) > 0\}$ for $u, v \in \mathbb{C}$.

Solution. Suppose $z = x+iy$ where $x, y \in \mathbb{R}$.

(a) We have $|z+1| = \sqrt{(x+1)^2+y^2}$ and $|z-1| = \sqrt{(x-1)^2+y^2}$. Since $|z+1| = |z-1|$, we have $x+1 = -x+1$, so $x = 0$. Hence, the set is the y-axis.

(b) We know $|z+1|$ is the distance between (x,y) and $(-1,0)$ and $|z-1|$ is the distance between (x,y) and $(1,0)$. If $a < 2$, the set is empty. If $a = 2$, then the set is the line segment between $(-1,0)$ and $(1,0)$. If $a > 2$, then the set is an ellipse with foci at $(-1,0)$ and $(1,0)$.

(c) Suppose $u = a+ib$ and $v = c+id$ where $a, b, c, d \in \mathbb{R}$. Then, we have $\operatorname{Re}(u+vz) = a+cx-dy > 0$. Therefore, we get

$$dy < cx + a.$$

If $d > 0$, the set is the plane under the line $y = \frac{c}{d}x + \frac{a}{d}$.
If $d < 0$, the set is the plane above the line $y = \frac{c}{d}x + \frac{a}{d}$.
If $d = 0$, then we have $cx + a > 0$. If $c > 0$, then the set is the plane on the right of the vertical line $x = -\frac{a}{c}$. If $c < 0$, then the set is the plane on the left of the vertical line $x = -\frac{a}{c}$. If $c = 0$, then if $a > 0$, the set is entire. If $a < 0$, the set is empty.

□

Problem 7. Prove there are no $u, v \in \mathbb{C}$ such that, for all $z \in \mathbb{C}$, $\overline{z} = u + vz$.

Solution. Assume by contradiction that there are $u, v \in \mathbb{C}$ such that $\overline{z} = u + vz$ for all $z \in \mathbb{C}$. Then, let $z = 0$. We have $0 = u$. Let $z = 1$, we have $1 = 0 + v$, so $v = 1$. Hence, we have $\overline{z} = z$ for all $z \in \mathbb{C}$, which contradicts $i \in \mathbb{C}$. Therefore, there are no $u, v \in \mathbb{C}$ such that, for all $z \in \mathbb{C}$, $\overline{z} = u + vz$. □

2 Set 2

Problem 8. Suppose $f \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ satisfies $|f(z) - f(w)| \leq |z - w|$ for all $z, w \neq 0$. Prove that $\lim_{z \to 0} f(z)$ exists.

Solution. Suppose $\{z_1, z_2, \dots\}$ is a sequence that converges to 0, so we have $\lim_{n\to\infty} z_n = 0$. Then, we have

$$|f(z_n) - f(z_m)| \leq |z_n - z_m|.$$

When $n, m \to \infty$, we have $|z_n - z_m| \to 0$, so $f(z_n)$ is a Cauchy sequence in $\mathbb{C}$. Hence, $f(z_n)$ has a limit in $\mathbb{C}$, so $\lim_{z\to 0} f(z)$ exists. □

Problem 9. Prove that, if $U \subseteq \mathbb{C}$ is open and $f, g \colon U \to \mathbb{C}$ are holomorphic, then the product $fg \colon U \to \mathbb{C}$ is holomorphic and that $(fg)' = f'g + fg'$.

Solution. Suppose $f = u_1 + iv_1$ and $g = u_2 + iv_2$. Then, we have

$$\begin{aligned} fg &= (u_1u_2 - v_1v_2) + i(u_1v_2 + v_1u_2) \\ &= u + iv. \end{aligned}$$

Since f and g are holomorphic, by Proposition 2.3, we have

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= \frac{\partial v_1}{\partial y} \\ \frac{\partial u_1}{\partial y} &= -\frac{\partial v_1}{\partial x} \\ \frac{\partial u_2}{\partial x} &= \frac{\partial v_2}{\partial y} \\ \frac{\partial u_2}{\partial y} &= -\frac{\partial v_2}{\partial x} \end{aligned}$$

Hence, we have

$$\frac{\partial u}{\partial x} = u_2\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_2}{\partial x} - v_2\frac{\partial v_1}{\partial x} - v_1\frac{\partial v_2}{\partial x}.$$

Also, we have

$$\begin{aligned} \frac{\partial v}{\partial y} &= v_2\frac{\partial u_1}{\partial y} + u_1\frac{\partial v_2}{\partial y} + u_2\frac{\partial v_1}{\partial y} + v_1\frac{\partial u_2}{\partial y} \\ &= -v_2\frac{\partial v_1}{\partial x} + u_1\frac{\partial u_2}{\partial x}u_2\frac{\partial u_1}{\partial x} - v_1\frac{\partial v_2}{\partial x} \\ &= \frac{\partial u}{\partial x}. \end{aligned}$$

Then, we also have

$$\frac{\partial u}{\partial y} = u_2\frac{\partial u_1}{\partial y} + u_1\frac{\partial u_2}{\partial y} - v_2\frac{\partial v_1}{\partial y} - v_1\frac{\partial v_2}{\partial y}.$$

And we get

$$\begin{aligned} -\frac{\partial v}{\partial x} &= -v_2\frac{\partial u_1}{\partial x} - u_1\frac{\partial v_2}{\partial x} - u_2\frac{\partial v_1}{\partial x} - v_1\frac{\partial u_2}{\partial x} \\ &= -v_2\frac{\partial v_1}{\partial y} + u_1\frac{\partial u_2}{\partial y} + u_2\frac{\partial u_1}{\partial y} - v_1\frac{\partial v_2}{\partial y} \\ &= \frac{\partial u}{\partial y}. \end{aligned}$$

Therefore, fg satisfies the Cauchy-Riemann equation on U. By Theorem 2.4, we get that fg is holomorphic on U and by Proposition 2.3, we have

$$\begin{aligned}
(fg)' &= \frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y} \\
&= u_2\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_2}{\partial x} - v_2\frac{\partial v_1}{\partial x} - v_1\frac{\partial v_2}{\partial x} - i\left(u_2\frac{\partial u_1}{\partial y} + u_1\frac{\partial u_2}{\partial y} - v_2\frac{\partial v_1}{\partial y} - v_1\frac{\partial v_2}{\partial y}\right) \\
&= u_2\frac{\partial u_1}{\partial x} - iu_2\frac{\partial u_1}{\partial y} + iv_2\frac{\partial u_1}{\partial x} + v_2\frac{\partial u_1}{\partial y} + u_1\frac{\partial u_2}{\partial x} - iu_1\frac{\partial u_2}{\partial y} + iv_1\frac{\partial u_2}{\partial x} + v_1\frac{\partial u_2}{\partial y} \\
&= \left(\frac{\partial u_1}{\partial x} + \frac{1}{i}\frac{\partial u_1}{\partial y}\right)(u_2 + iv_2) + (u_1 + iv_1)\left(\frac{\partial u_2}{\partial x} + \frac{1}{i}\frac{\partial u_2}{\partial y}\right) \\
&= f'g + fg'.
\end{aligned}$$

□

Problem 10. Suppose that $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$ is a polynomial. Show that p and p' have a zero in common if and only if $z_i = z_j$ for some $i \neq j$.

Solution. Suppose p and p' have a zero in common. Let z_k be the zero in common. Then, we have

$$p'(z) = \sum_{i=1}^{n} \frac{p(z)}{z - z_i}.$$

Therefore, we have

$$p'(z_k) = (z_k - z_1)(z_k - z_2)\cdots(z_k - z_{k-1})(z_k - z_{k+1})\cdots(z_k - z_n).$$

Since z_k is a zero of $p'(z_k)$, we have $z_k = z_m$ for some $m \neq k$.
Suppose $z_i = z_j$ for some $i \neq j$. Then, we have $p(z) = (z - z_i)^2 q(z)$. Therefore, we have

$$p'(z) = 2(z - z_i)q(z) + (z - z_i)^2 q'(z).$$

Since $p'(z_i) = 0$, z_i is the common zero.
Therefore, p and p' have a zero in common if and only if $z_i = z_j$ for some $i \neq j$. □

Problem 11. For $m > 0$ an integer, expand $(1-z)^{-m}$ in powers of z.

Solution. By binomial series expansion, we have

$$\begin{aligned}(1-z)^{-m} &= \sum_{n=0}^{\infty} \binom{-m}{n} (-z)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{(-m)(-m+1)\cdots(-m+n-1)}{n!} \right) (-z)^n.\end{aligned}$$

Since $m > 0$, we have

$$\begin{aligned}(1-z)^{-m} &= \sum_{n=0}^{\infty} \frac{(-m)(-m-1)\cdots(-m-n+1)}{n!} (-z)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(m+n-1)!}{(m-1)!n!} (-z)^n \\ &= \sum_{n=0}^{\infty} \binom{m+n-1}{m-1} z^n.\end{aligned}$$

□

Problem 12. What is the radius of convergence of $\sum_{n=0}^{\infty} n^{2016} z^n$?

Solution. By Theorem 2.5, we have that

$$\begin{aligned}\frac{1}{R} &= \limsup \left| n^{2016} \right|^{1/n} \\ &= \limsup n^{2016/n}. \\ &= 1.\end{aligned}$$

Hence, we have $R = 1$. □

Problem 13. Show that, if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = R$, then the series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R.

Solution. Let

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{R}.$$

Suppose $|z| < R$. Then, choose $\epsilon > 0$ such that

$$|z| < R - \epsilon = \frac{1}{L+\delta}.$$

By definition of L, there exists N such that if $n > N$, then $\frac{a_{n+1}}{a_n} < L + \delta$. Therefore, we have

$$\frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \left| \frac{a_{n+1}}{a_n} \right| |z| < (L+\delta)|z| < 1.$$

Therefore, we have the series

$$\sum_{n=N+1}^{\infty} a_n z^n$$

to be convergent. Hence, we have

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n$$

to be convergent.
Suppose $|z| > R$. Similarly, there exist some N' such that for all $n > N'$, we have

$$\left|\frac{a_{n+1}}{a_n}\right| |z| > (L - \delta)|z| > 1.$$

Hence, we have the series

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{N'} a_n z^n + \sum_{n=N'+1}^{\infty} a_n z^n$$

to be divergent.
Therefore, the series has radius of convergence R. □

Problem 14. For which values of z does $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ converge?

Solution. We have

$$\left|\frac{z}{1+z}\right| < 1.$$

Therefore, we have

$$|z| < |1 + z|.$$

Suppose $z = x + iy$ where $x, y \in \mathbb{R}$. Then, we have

$$\begin{aligned} x^2 + y^2 &< (1 + x)^2 + y^2 \\ x^2 &< 1 + 2x + x^2 \\ x &> -\frac{1}{2}. \end{aligned}$$

Hence, we have $\text{Re}(z) > -\frac{1}{2}$ and $z \neq -1$. □

3 Set 3

Problem 15. Find $\sin i, \cos i$, and $\tan(1 + i)$.

Solution. We have

$$\begin{aligned}
\sin i &= \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{e^{-1} - e}{2i} = i\frac{e - e^{-1}}{2} \\
\cos i &= \frac{e^{i^2} + e^{-i^2}}{2} = \frac{e + e^{-1}}{2} \\
\tan(1+i) &= \frac{e^{i(1+i)} - e^{-i(1+i)}}{e^{i(1+i)} + e^{-i(1+i)}}
\end{aligned}$$

□

Problem 16. Find the real and imaginary parts of $\cos(a+bi)$ in terms of $a, b \in \mathbb{R}$.

Solution. We have

$$\begin{aligned}
\cos(a+bi) &= \frac{e^{i(a+bi)} + e^{-i(a+bi)}}{2} \\
&= \frac{e^{ai}e^{-b} + e^{-ai}e^{b}}{2} \\
&= \frac{(\cos a + i\sin a)e^{-b} + (\cos(-a) + i\sin(-a))e^{b}}{2} \\
&= \frac{\cos a(e^{-b} + e^{b})}{2} + i\frac{\sin a(e^{-b} - e^{b})}{2}.
\end{aligned}$$

□

Problem 17. Prove that $3 < \pi < 2\sqrt{3}$. Hint: use the power series of $\mathrm{Re}(e^{ti})$ to estimate the location of its first positive zero.

Solution. □

Problem 18. Under what conditions is $\log(zw) = \log(z) + \log(w)$?

Solution. Let $z = r_1 e^{i\theta_1}$ and $z = r_2 e^{i\theta_2}$. Then, we should have $\theta_1 + \theta_2 < \pi$. □

Problem 19. Define the "angles" of a triangle and prove that their sum is π.

Solution. □

Problem 20. Sketch the image of the curve $\gamma\colon [0,\infty) \to \mathbb{C}$ given by $\gamma(t) = e^{-t+t^2 i}$.

Solution. We have

$$\gamma(t) = e^{-t}e^{t^2 i} = e^{-t}(\cos(t^2) + i\sin(t^2)).$$

Hence, we have

$$\begin{cases} x = e^{-t}\cos(t^2) \\ y = e^{-t}\sin(t^2) \end{cases}$$

□

Problem 21. Sketch the image of the curve $\gamma\colon \mathbb{R} \to \mathbb{C}$ given by $\gamma(t) = e^{-t^2 + t/\sqrt{1+t^2}i}$. What angle is formed at the origin?

Solution. □

4 Set 4

Problem 22. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = e^{ti}$. Evaluate $\int_\gamma z^n dz$ for all $n \in \mathbb{Z}$.

Solution. When $n \neq -1$, we have

$$\begin{aligned} \int_\gamma z^n dz &= \int_0^{2\pi} e^{nti} \cdot ie^{ti} dt \\ &= \int_0^{2\pi} ie^{(n+1)ti} dt \\ &= \left. \frac{e^{(n+1)ti}}{n+1} \right|_0^{2\pi} \\ &= 0. \end{aligned}$$

When $n = -1$, we have

$$\begin{aligned} \int_\gamma z^{-1} dz &= \int_0^{2\pi} e^{-ti} \cdot ie^{ti} dt \\ &= \int_0^{2\pi} i dt \\ &= 2\pi i. \end{aligned}$$

□

Problem 23. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = 2e^{ti}$. Evaluate $\int_\gamma \frac{1}{z(z-3)} dz$ and $\int_\gamma \frac{1}{z^2-1} dz$.

Solution. We have

$$\begin{aligned} \int_\gamma \frac{1}{z(z-3)} dz &= \int_0^{2\pi} \frac{2ie^{ti}}{2e^{ti}(2e^{ti}-3)} dt \\ &= \int_0^{2\pi} \frac{i}{2e^{ti}-3} dt \\ &= -\frac{1}{3} i \int_0^{2\pi} \frac{1}{1 - \frac{2}{3}e^{ti}} dt \\ &= -\frac{1}{3} i \int_0^{2\pi} \sum_{n=0}^{\infty} \left(\frac{2}{3} e^{ti} \right)^n dt \\ &= -\frac{1}{3} i \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n \int_0^{2\pi} e^{nti} dt. \end{aligned}$$

When $n \neq 0$, we have

$$\int_0^{2\pi} e^{nti} dt = \left. \frac{1}{ni} e^{nti} \right|_0^{2\pi} = 0.$$

When $n = 0$, we have

$$\int_0^{2\pi} 1 dt = 2\pi.$$

Hence, we have $\int_\gamma \frac{1}{z(z-3)} dz = -\frac{2\pi i}{3}$. ☐

Problem 24. Suppose that $f, g \colon D(0,1) \to \mathbb{C}$ are holomorphic and $f' = g'$ is continuous. Prove that $f - g$ is constant.

Solution. By Proposition 3.1, we have $(f-g)' = f' - g' = 0$. By Corollary 3.4, we have $f - g$ is constant. ☐

Problem 25. Suppose $U \subseteq \mathbb{C}$ is open, $f \colon U \to \mathbb{C}$ is holomorphic, and f' is continuous. Show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary for all closed paths γ in U.

Solution. We have

$$\int_\gamma \overline{f(z)} f'(z) dz = \int_a^b \overline{f(\gamma(t))} f'(\gamma(t)) \gamma'(t) dt.$$

Then, consider

$$\begin{aligned}
\mathrm{Re}\Big(\int_a^b \overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t) dt\Big) &= \int_a^b \mathrm{Re}\big(\overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t)\big) dt \\
&= \int_a^b \frac{1}{2}\Big(\overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t) + \overline{\overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t)}\Big) dt \\
&= \int_a^b \frac{1}{2}\Big(\overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t) + f(\gamma(t))\overline{f'(\gamma(t))\gamma'(t)}\Big) dt.
\end{aligned}$$

Let $F(t) = \frac{1}{2} f(\gamma(t))\overline{f(\gamma(t))}$. Then, we have

$$F'(t) = \frac{1}{2}\Big(\overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t) + f(\gamma(t))\overline{f'(\gamma(t))\gamma'(t)}\Big).$$

Hence, we have

$$\begin{aligned}
\mathrm{Re}\Big(\int_a^b \overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t) dt\Big) &= \frac{1}{2}\int_a^b F'(t) dt \\
&= \frac{1}{2}(F(a) - F(b)) \\
&= \frac{1}{2} f(\gamma(a))\overline{f(\gamma(a))} - \frac{1}{2} f(\gamma(b))\overline{f(\gamma(b))}.
\end{aligned}$$

Since γ is closed, we have $\gamma(a) = \gamma(b)$. Hence, we have

$$\mathrm{Re}\Big(\int_a^b \overline{f(\gamma(t))} f'(\gamma(t))\gamma'(t) dt\Big) = \mathrm{Re}\Big(\int_\gamma \overline{f(z)} f'(z) dz\Big) = 0.$$

Therefore, $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary for all closed paths γ in U. ☐

Problem 26. Suppose $U \subseteq \mathbb{C}$ is open, $f\colon U \to \mathbb{C}$ is holomorphic, and $|f(z)-1| < 1$ for all $z \in U$. Show that $\int_\gamma f'(z)/f(z)dz = 0$ for all closed paths γ in U.

Solution. Since f is holomorphic, $f'(z)$ exists. Also, since $|f(z)-1| < 1$ for all $z \in U$, we have $0 < f(z) < 2$. Hence, $\frac{f'(z)}{f(z)}$ is defined for all $z \in U$. Set $F(z) = \log f(z)$. We have $F(z)$ defined for all $z \in U$. Then, we have $F'(z) = \frac{f'(z)}{f(z)}$, so $\frac{f'(z)}{f(z)}$ is primitive. Hence, $\int_\gamma f'(z)/f(z)dz = 0$ for all closed paths γ in U. □

Problem 27. Describe the paths γ for which $\int_\gamma \log z dz = 0$ is meaningful and true.

Solution. We have γ to be a closed path in the domain $\mathbb{C} - r$, where r is a ray at the origin pointing in any direction. □

5 Set 5

Problem 28. Show that $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$.

Solution. We consider the function $f(z) = \frac{e^{iz}}{z}$. Since e^{iz} is entire, when $z \neq 0$, by Proposition 2.2, $f(z)$ is holomorphic. We integrate over the indented semicircle in the upper half-plane positioned on the x-axis, as shown in Figure 1.

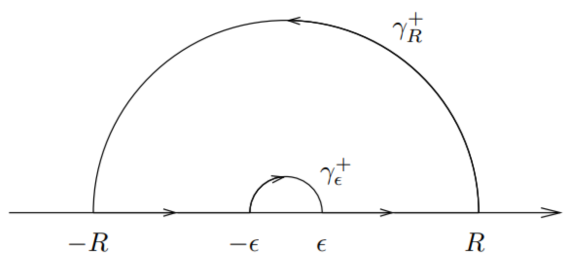


Figure 1: The indented semicircle

If we denote by γ_ϵ^+ and γ_R^+ the semicircles of radii ϵ and R with negative and positive orientations respectively, Cauchy's theorem gives

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz + \int_\epsilon^R \frac{e^{ix}}{x} dx + \int_{\gamma_R^+} \frac{e^{iz}}{z} dz = 0.$$

Consider the integral along γ_ϵ^+. We can parametrize by $z = \epsilon e^{i\theta}$ where θ goes from π to 0. Then, we have

$$\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz = \int_\pi^0 \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} \cdot (i\epsilon) e^{i\theta} d\theta = -i \int_0^\pi e^{i\epsilon e^{i\theta}} d\theta.$$

When $\epsilon \to 0$, we have $e^{i\epsilon e^{i\theta}} \to 1$. Hence, when $\epsilon \to 0$, we have

$$\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz \to -i \int_0^\pi 1 d\theta = -i\pi.$$

Consider the integral along γ_R^+. We can also parametrize by $z = Re^{i\theta}$ where θ goes from 0 to π. Then, we have

$$\int_{\gamma_R^+} \frac{e^{iz}}{z} dz = \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} \cdot (iR)e^{i\theta} d\theta = i \int_0^\pi e^{iRe^{i\theta}} d\theta.$$

Therefore, we get

$$\left| \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \right| \leq \int_0^\pi \left| e^{iR(\cos\theta + i\sin\theta)} \right| d\theta = \int_0^\pi \left| e^{-R\sin\theta} e^{iR\cos\theta} \right| d\theta = \int_0^\pi e^{-R\sin\theta} d\theta.$$

Then, for any $0 < \delta < \pi$, we have

$$\int_0^\pi e^{-R\sin\theta} d\theta = \int_0^\delta e^{-R\sin\theta} d\theta + \int_\delta^{\pi-\delta} e^{-R\sin\theta} d\theta + \int_{\pi-\delta}^\pi e^{-R\sin\theta} d\theta$$

Since $0 \leq \theta \leq \delta < \pi$, we have

$$e^{-R\sin\theta} \leq e^{-R\cdot 0} = 1.$$

Also, for all $\delta \leq \theta \leq \pi - \delta$, we have $\sin\theta \geq \sin\delta > 0$. Therefore, we get

$$\begin{aligned} \int_0^\pi e^{-R\sin\theta} d\theta &= \int_0^\delta e^{-R\sin\theta} d\theta + \int_\delta^{\pi-\delta} e^{-R\sin\theta} d\theta + \int_{\pi-\delta}^\pi e^{-R\sin\theta} d\theta \\ &< \int_0^\delta 1 d\theta + \int_\delta^{\pi-\delta} e^{-R\sin\delta} d\theta + \int_{\pi-\delta}^\pi 1 d\theta \\ &= 2\delta + (\pi - 2\delta)e^{-R\sin\delta}. \end{aligned}$$

When $R \to \infty$, we have $e^{-R\sin\delta} \to 0$. Hence, when $R \to \infty$, we have

$$\left| \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \right| < 2\delta.$$

Since this holds for all $0 < \delta < \pi$, we have $\left| \int_{\gamma_R^+} \frac{e^{iz}}{z} dz \right| = 0$. Therefore, when $\epsilon \to 0$ and $R \to \infty$, we have

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + \int_{\gamma_\epsilon^+} \frac{e^{iz}}{z} dz + \int_\epsilon^R \frac{e^{ix}}{x} dx + \int_{\gamma_R^+} \frac{e^{iz}}{z} dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx + (-i\pi) + \int_\epsilon^R \frac{e^{ix}}{x} dx + 0 = 0.$$

Therefore, we have

$$\text{Im}\left(\int_{-\infty}^\infty \frac{e^{ix}}{x} dx \right) = \int_{-\infty}^\infty \text{Im}\left(\frac{e^{ix}}{x} \right) dx = \int_{-\infty}^\infty \frac{\sin x}{x} dx = \text{Im}(i\pi) = \pi.$$

□

Problem 29. Show that $\int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}$.

Solution. We consider the function $f(z) = e^{-z^2}$. Since e^{-z^2} is entire, we can integrate over the contour as shown in Figure 2.

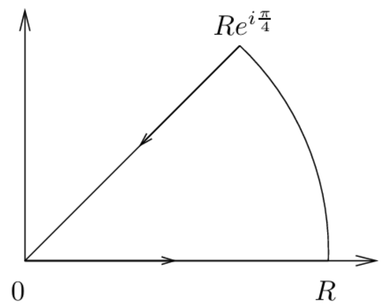


Figure 2: The contour

Let the arc be γ_R and the path along $y = x$ be γ^+. Then, Cauchy's theorem gives

$$\int_0^R e^{-x^2} dx + \int_{\gamma_R} e^{-z^2} dz + \int_{\gamma^+} e^{-z^2} dz = 0.$$

Consider the integral along the x-axis. Since we know $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ and e^{-x^2} is an even function, when $R \to \infty$, we have

$$\int_0^R e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Consider the integral along γ_R. We can parametrize by $z = Re^{i\theta}$ where θ goes from 0 to $\pi/4$. Then, we have

$$\int_{\gamma_R} e^{-z^2} dz = \int_0^{\pi/4} e^{-(Re^{i\theta})^2} (iRe^{i\theta}) d\theta = iR \int_0^{\pi/4} \left(e^{-R^2 \cos 2\theta - iR^2 \sin 2\theta} \right) (e^{i\theta}) d\theta.$$

Hence, we have

$$\left| \int_{\gamma_R} e^{-z^2} dz \right| \leq |iR| \int_0^{\pi/4} \left| e^{-R^2 \cos 2\theta - iR^2 \sin 2\theta} \right| \left| e^{i\theta} \right| d\theta = \int_0^{\pi/4} |Re^{-R^2 \cos 2\theta}| d\theta.$$

Since $0 \leq \theta \leq \pi/4$, we have $0 \leq \cos 2\theta \leq 1$. Therefore, when $R \to \infty$, we have $Re^{-R^2 \cos 2\theta} \to 0$, so $\int_{\gamma_R} e^{-z^2} dz \to 0$.

Consider the integral along γ^+. We can parametrize by $z = te^{i\frac{\pi}{4}}$ where t goes from R to 0. Then, we have

$$\int_{\gamma^+} e^{-z^2} dz = \int_R^0 e^{-(te^{i\frac{\pi}{4}})^2} (e^{i\frac{\pi}{4}}) dt = -e^{i\frac{\pi}{4}} \int_0^R e^{-it^2} dt = -e^{i\frac{\pi}{4}} \int_0^R \left(\cos(t^2) - i \sin(t^2) \right) dt.$$

Therefore, when $R \to \infty$, we have

$$\int_{\gamma^+} e^{-z^2} dz = -e^{i\frac{\pi}{4}} \int_0^\infty \left(\cos(t^2) - i\sin(t^2)\right) dt.$$

Hence, when $R \to \infty$, we have

$$\int_0^R e^{-x^2} dx + \int_{\gamma_R} e^{-z^2} dz + \int_{\gamma^+} e^{-z^2} dz = \frac{\sqrt{\pi}}{2} + 0 - e^{i\frac{\pi}{4}} \int_0^\infty \left(\cos(t^2) - i\sin(t^2)\right) dt = 0.$$

Therefore, we get

$$\int_0^\infty \cos(t^2)dt - i\int_0^\infty \sin(t^2)dt = \frac{\sqrt{\pi}}{2} e^{-i\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i.$$

Hence, we have

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}.$$

□

Problem 30. Evaluate $\int_{\partial D(0,2)} \frac{e^z}{z+1} dz$.

Solution. Since $\frac{e^z}{z+1}$ is holomorphic in an open set containing $\partial D(0,2)$ and its interior, except for a pole at $z = -1$. Then, by Theorem 2.1, we have

$$\int_{\partial D(0,2)} \frac{e^z}{z+1} dz = 2\pi i \operatorname{res}_{-1} f.$$

Since $z = -1$ is a simple pole, we have

$$\operatorname{res}_{-1} f = \lim_{z \to -1} (z+1) \frac{e^z}{z+1} = e^{-1}.$$

Hence, we have

$$\int_{\partial D(0,2)} \frac{e^z}{z+1} dz = 2\pi i e^{-1}.$$

□

Problem 31. Evaluate $\int_{\partial D(0,1)} z^n e^z dz$ for $n \in \mathbb{Z}$.

Solution. If $n \geq 0$, then $\frac{e^z}{z+1}$ is entire. Cauchy's theorem gives

$$\int_{\partial D(0,1)} z^n e^z dz = 0.$$

If $n < 0$, then $z^n e^z$ has a pole of order $-n$ at $z = 0$. Then, by Theorem 2.1, we have

$$\int_{\partial D(0,1)} z^n e^z dz = 2\pi i \operatorname{res}_0 z^n e^z.$$

By Theorem 1.4, we have

$$\begin{aligned}
\operatorname{res}_0 z^n e^z &= \lim_{z\to 0} \frac{1}{(-n-1)!}\left(\frac{d}{dz}\right)^{-n-1} z^{-n} z^n e^z \\
&= \frac{1}{(-n-1)!}\lim_{z\to 0}\left(\frac{d}{dz}\right)^{-n-1} e^z \\
&= \frac{1}{(-n-1)!}\lim_{z\to 0} e^z \\
&= \frac{1}{(-n-1)!}.
\end{aligned}$$

Hence, we have

$$\int_{\partial D(0,1)} z^n e^z dz = \frac{2\pi i}{(-n-1)!}.$$

□

Problem 32. Suppose $f_n \colon U \to \mathbb{C}$ is a sequence of continuous functions that converges uniformly as $n \to \infty$. Show that $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz$ holds for all paths γ in U.

Solution. We can parametrize γ by $z = \gamma(t)$ where t goes from a to b. Then, we have

$$\int_\gamma f_n(z)dz = \int_a^b f_n(\gamma(t))\gamma'(t)dt.$$

Set

$$f(z) = \lim_{n\to\infty} f_n(z).$$

Then, we have

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Hence, we have

$$\begin{aligned}
\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| &= \left|\int_a^b f_n(\gamma(t))\gamma'(t)dt - \int_a^b f(\gamma(t))\gamma'(t)dt\right| \\
&= \left|\int_a^b \left(f_n(\gamma(t)) - f(\gamma(t))\right)\left(\gamma'(t)\right)dt\right| \\
&\le \int_a^b |f_n(\gamma(t)) - f(\gamma(t))|\,|\gamma'(t)|dt \\
&\le \max_{a\le t\le b} |f_n(\gamma(t)) - f(\gamma(t))| \int_a^b |\gamma'(t)|dt.
\end{aligned}$$

Since f_n converges uniformly as $n \to \infty$, for every $\delta > 0$, there exists N such that for all $n \ge N$, we have

$$|f_n(\gamma(t)) - f(\gamma(t))| < \delta.$$

For all $\epsilon > 0$. Let $\delta = \frac{\epsilon}{\int_a^b |\gamma'(t)|dt}$. Then, we have

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| < \frac{\epsilon}{\int_b^a |\gamma'(t)|dt} \cdot \int_b^a |\gamma'(t)|dt = \epsilon.$$

Hence, we have

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz.$$

□

Problem 33. Prove that, if $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f(z)| \leq 1 + |z|^n$ for all $z \in \mathbb{C}$, then f is polynomial of degree at most n.

Solution. Since f is entire, we can have a power-series expansion at the origin that has a infinite radius of convergence. Therefore, we get

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

where $a_k = \frac{f^{(k)}(0)}{k!}$. Then, by Corollary 4.2, we have

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz.$$

Since f is entire, C can be any circle in $\mathbb{C}$. Let C be a circle of radius R centered at the origin, so we have $|z| = R$. Hence, we have

$$\left|\frac{f(z)}{z^{k+1}}\right| \leq \frac{1 + |z|^n}{|z|^{k+1}} = \frac{1 + R^n}{R^{k+1}}.$$

Therefore, we have

$$\begin{aligned}
|a_k| &= \left|\frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz\right| \\
&= \left|\frac{1}{2\pi i}\right| \left|\int_C \frac{f(z)}{z^{n+1}} dz\right| \\
&\leq \frac{1}{2\pi} \int_C \left|\frac{f(z)}{z^{n+1}}\right| dz \\
&\leq \frac{1}{2\pi} (2\pi R) \sup_{z\in C} \left|\frac{f(z)}{z^{n+1}}\right| \\
&\leq R \cdot \frac{1 + R^n}{R^{k+1}} \\
&= \frac{1 + R^n}{R^k}.
\end{aligned}$$

Since f is entire, when $k > n$, we can let $R \to \infty$, so $a_k = 0$. Therefore, f is a polynomial of degree at most n. □

6 Set 6

Problem 34. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic and $\overline{D}(0,1) \subseteq U$. Prove that

$$|f'(0)| \le \frac{1}{2} \max_{z,w\in\overline{D}(0,1)} |f(z) - f(w)|.$$

Solution. Let $g(z) = f(z) - f(-z)$. Then, $g(z)$ is holomorphic on U. Also, we have

$$g'(z) = f'(z) - (-f'(-z)) = f'(z) + f'(-z).$$

Hence, we have

$$g'(0) = f'(0) + f'(0) = 2f'(0).$$

For all $z \in \overline{D}(0,1)$, we have $-z \in \overline{D}(0,1)$. Hence, we have

$$|g(z)| = |f(z) - f(-z)| \le \max_{z,w\in\overline{D}(0,1)} |f(z) - f(w)|.$$

Then, by Corollary 4.2, we have

$$|g'(0)| = \left|\frac{1}{2\pi i}\int_{|z|=1} \frac{g(z)}{z^2}dz\right| \le \frac{1}{2\pi}(2\pi) \sup_{|z|=1} |g(z)| = \sup_{|z|=1} |g(z)| \le \sup_{z\in\overline{D}(0,1)} |g(z)|.$$

Therefore, we have

$$|g'(0)| = 2|f'(0)| \le \sup_{z\in\overline{D}(0,1)} |g(z)| \le \max_{z,w\in\overline{D}(0,1)} |f(z) - f(w)|.$$

Hence, we get

$$|f'(0)| \le \frac{1}{2} \max_{z,w\in\overline{D}(0,1)} |f(z) - f(w)|.$$

□

Problem 35. Show that, if $f\colon U \to \mathbb{C}$ is holomorphic and $\overline{D}(z_0, r) \subseteq U$, then f can be uniformly approximated by polynomials on $\overline{D}(z_0, r)$. That is, there is a sequence of polynomials p_n such that $p_n \to f$ uniformly on $\overline{D}(z_0, r)$ as $n \to \infty$.

Solution. Since U is open, there exists some ϵ such that f is holomorphic on $\overline{D}(z_0, r + \epsilon)$. Let $R = r + \epsilon$. Then, for all $z \in \overline{D}(z_0, R)$, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^{k+1}} dz.$$

Then, we have

$$|a_k| = \left|\frac{1}{2\pi i}\int_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^{k+1}} dz\right| \le \frac{1}{2\pi}(2\pi R) \sup_{|z-z_0|=R} \frac{|f(z)|}{R^{k+1}} = \sup_{|z-z_0|=R} \frac{|f(z)|}{R^k}.$$

Then, for $z \in \overline{D}(z_0, r)$, we have

$$\left|a_k(z-z_0)^k\right| \le \sup_{|z-z_0|=R} \frac{|f(z)|}{R^k} \cdot r^k = \sup_{|z-z_0|=R} |f(z)| \left(\frac{r}{R}\right)^k.$$

Since $r < R$, we have $\frac{r}{R} < 1$. Hence, we have

$$\sum_{k=n+1}^{\infty} a_k(z-z_0)^k \le \sum_{k=n+1}^{\infty} \sup_{|z-z_0|=R} |f(z)| \left(\frac{r}{R}\right)^k = \sup_{|z-z_0|=R} |f(z)| \frac{\left(\frac{r}{R}\right)^{n+1}}{1-\left(\frac{r}{R}\right)}.$$

When $n \to \infty$, we have

$$\sup_{|z-z_0|=R} |f(z)| \frac{\left(\frac{r}{R}\right)^{n+1}}{1-\left(\frac{r}{R}\right)} \to 0.$$

Let

$$p_n(z) = \sum_{k=0}^{n} a_k(z-z_0)^k.$$

Then, for all $z \in \overline{D}(z_0, r)$, we have

$$|f(z) - p_n(z)| \le \sum_{k=n+1}^{\infty} |a_k|(z-z_0)^k \le \sup_{|z-z_0|=R} |f(z)| \frac{\left(\frac{r}{R}\right)^{n+1}}{1-\left(\frac{r}{R}\right)}.$$

Therefore, we conclude that there is a sequence of polynomials p_n such that $p_n \to f$ uniformly on $\overline{D}(z_0, r)$ as $n \to \infty$. □

Problem 36. Is the conclusion of the previous exercise true if f is merely continuous? What about the real variable case $f\colon [-1,1] \to \mathbb{R}$?

Solution. No, since we cannot apply Cauchy's integral formula to f. The real variable case is valid since we use Taylor expansion on differentiable real functions. □

7 Set 7

Problem 37. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and, for every point $z_0 \in \mathbb{C}$, the series expansion

$$f(z) = \sum_{n \ge 0} a_n(z-z_0)^n$$

has at least one coefficient that is zero. Show that f is a polynomial. Hint: there are more points in every disk $D(0,1)$ than there are integers.

Solution. We have

$$f^{(k)}(z_0) = a_k \cdot k!.$$

Therefore, if $a_k = 0$ for some k, then

$$f^{(k)}(z_0) = 0.$$

Define

$$M_n = \{z_0 \in \mathbb{C} \colon f^{(n)}(z_0) = 0\}$$

Then, since $f(z)$ has at least one coefficient that is zero, we have

$$\bigcup_{n=0}^{\infty} M_n = \mathbb{C}.$$

Assume by contradiction that M_n is countable for all n. Then, since n is a nonnegative integer, we have $\bigcup_{n=0}^{\infty} M_n$ to be countable. This contradicts that $\mathbb{C}$ is uncountable. Therefore, we have $|M_n|$ to be uncountable for some n.

Hence, by Theorem 4.8, since $f^{(n)}(z)$ is a holomorphic function that vanishes on a sequence of distinct points with a limit point in $\mathbb{C}$, we have $f^{(n)}(z)$ to be identically 0. Therefore, f is a polynomial. □

Problem 38. Show that

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \text{ for } z \in D(0,1)$$

is holomorphic in $D(0,1)$. Show that f does not have a continuation to any open set strictly containing $D(0,1)$.

Solution. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence such that

$$f_k(z) = \sum_{n=0}^{k} z^{2^n}.$$

Since f_k is a polynomial for all k, we have f_k to be holomorphic on all of $D(0,1)$.

Let $S \subset D(0,1)$ be any compact subset. Hence, we have $K \subset \overline{D}(0,r)$ for some $0 < r < 1$. Then, we have

$$f(z) - f_k(z) = \sum_{n=0}^{\infty} z^{2^n} - \sum_{n=0}^{k} z^{2^n} = \sum_{n=k+1}^{\infty} z^{2^n}.$$

Since $z \in K$, we have

$$\left| \sum_{n=k+1}^{\infty} z^{2^n} \right| \leq \sum_{n=k+1}^{\infty} |z^{2^n}| \leq \sum_{n=k+1}^{\infty} r^{2^n} < \sum_{m=2^{k+1}}^{\infty} r^m = \frac{r^{2^{k+1}}}{1-r}.$$

For all $\epsilon > 0$, since $\epsilon(1-r)$ is constant, we can choose k arbitrarily great such that $r^{2^{k+1}} < \epsilon(1-r)$. Hence, $\{f_k\}$ is a sequence of holomorphic functions that converges uniformly to f in every compact subset of $D(0,1)$. By Theorem 5.2, f is holomorphic in $D(0,1)$.

For any $\epsilon > 0$, there exists n such that $\frac{1}{2^n} < \epsilon$. For any $x \in [0,1]$, let $m = \lfloor x \cdot 2^n \rfloor$. Then, we have

$$m = \lfloor x \cdot 2^n \rfloor \leq x \cdot 2^n \leq m + 1 = \lfloor x \cdot 2^n \rfloor + 1.$$

Hence, we have

$$\frac{m}{2^n} \leq x \leq \frac{m+1}{2^n}.$$

Therefore, we have

$$0 \le x - \frac{m}{2^n} \le \frac{1}{2^n} < \epsilon.$$

Hence, we prove that the rational numbers of the form $\frac{m}{2^n}$ is dense in the interval $[0,1]$. Then, we can extend it to $[0,2\pi]$. For $|z|=1$, we have $z=e^{i\theta}$, where $\theta \in [0,2\pi]$. Then, we have a sequence $\{\theta_k\}_{k=1}^{\infty}$ where

$$\theta_k = \frac{m_k}{2^{n_k}}.$$

Since $\{\theta_k\}$ is dense, we can let

$$\lim_{k\to\infty} \theta_k = \theta.$$

Then, we can have a sequence $\{r_k\}_{k=1}^{\infty}$ such that $r_k = 1 - \frac{1}{k}$ so $\lim_{k\to\infty} r_k = 1$.
Hence, we can define the sequence $\{z_k\}_{k=1}^{\infty}$ where

$$z_k = r_k e^{i\theta_k}.$$

Therefore, we get

$$\lim_{k\to\infty} z_k = z.$$

Assume by contradiction that f has a continuation to an open set containing $D(0,1)$. Then, we have

$$\lim_{k\to\infty} f(z_k) = f(z).$$

Evaluating $f(z_k)$, we have

$$\begin{aligned}
f(z_k) &= \sum_{n=0}^{\infty} z_k^{2^n} \\
&= \sum_{n=0}^{\infty} r_k^{2^n} e^{i2^n \cdot \frac{m_k}{2^{n_k}}} \\
&= \sum_{n=0}^{n_k} r_k^{2^n} e^{i2^n \cdot \frac{m_k}{2^{n_k}}} + \sum_{n=n_k+1}^{\infty} r_k^{2^n} e^{i2^n \cdot \frac{m_k}{2^{n_k}}} \\
&= \sum_{n=0}^{n_k} r_k^{2^n} e^{i2^{n-n_k} m_k} + \sum_{n=n_k+1}^{\infty} r_k^{2^n} e^{i2^{n-n_k} m_k} \\
&= \sum_{n=0}^{n_k} r_k^{2^n} e^{i2^{n-n_k} m_k} + \sum_{n=n_k+1}^{\infty} r_k^{2^n}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\left| \sum_{n=0}^{n_k} r_k^{2^n} e^{i2^{n-n_k} m_k} \right| &\le \sum_{n=0}^{n_k} \left| r_k^{2^n} e^{i2^{n-n_k} m_k} \right| \\
&= \sum_{n=0}^{n_k} \left| r_k^{2^n} \right| \left| e^{i2^{n-n_k} m_k} \right| \\
&= \sum_{n=0}^{n_k} |r_k^{2^n}|.
\end{aligned}$$

Since $r_k \le 1$, we have

$$\left|\sum_{n=0}^{n_k} r_k^{2^n} e^{i2^{n-n_k}m_k}\right| \le \sum_{n=0}^{n_k} |r_k^{2^n}| \le n_k + 1.$$

Therefore, we have

$$\begin{aligned}
|f(z_k)| &= \left|\sum_{n=0}^{\infty} z_k^{2^n}\right| \\
&= \left|\sum_{n=0}^{n_k} z_k^{2^n} + \sum_{n=n_k+1}^{\infty} z_k^{2^n}\right| \\
&\ge \left|\sum_{n=n_k+1}^{\infty} z_k^{2^n}\right| - \left|\sum_{n=0}^{n_k} z_k^{2^n}\right| \\
&\ge \sum_{n=n_k+1}^{\infty} r_k^{2^n} - (n_k+1).
\end{aligned}$$

Since $\lim_{k\to\infty} r_k = 1$, we have $|f(z_k)| \to \infty$. Hence, we prove that f does not have a continuation to any open set strictly containing $D(0,1)$. □

Problem 39. Is Morera's theorem still true if triangles are replaced with disks?

Solution. Yes. □

8 Set 8

Problem 40. Find the poles and residues of $\frac{1}{\sin z}$.

Solution. We have $\sin z = 0$, so poles are $z = k\pi$, where $k \in \mathbb{Z}$. □

Problem 41. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} dx$ for $a > 0$.

Solution. Consider the function

$$f(z) = \frac{z^2}{(z^2+a^2)^3}$$

which is holomorphic in the complex plane except for simple poles at the points ai and $-ai$. Also, we choose the contour γ_R shown in Figure 1. The contour consists of the segment $[-R, R]$ on the real axis and of a large half-circle centered at the origin in the upper half-plane.
Since we may write

$$f(z) = \frac{z^2}{(z-ai)^3(z+ai)^3},$$

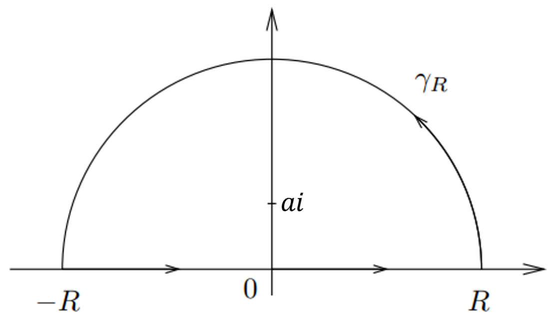


Figure 3: γ_R

by Theorem 1.4, we have

$$\begin{aligned}
\mathrm{res}_{ai}\, f &= \lim_{z\to ai} \frac{1}{2}\left(\frac{d}{dz}\right)^2 (z-ai)^3 \frac{z^2}{(z-ai)^3(z+ai)^3} \\
&= \lim_{z\to ai} \frac{1}{2}\left(\frac{d}{dz}\right)^2 \frac{z^2}{(z+ai)^3} \\
&= \lim_{z\to ai} \frac{1}{2}\frac{d}{dz}\frac{2z(z+ai)^3 - 3z^2(z+ai)^2}{(z+ai)^6} \\
&= \lim_{z\to ai} \frac{1}{2}\frac{d}{dz}\left(\frac{2z}{(z+ai)^3} - \frac{3z^2}{(z+ai)^4}\right) \\
&= \lim_{z\to ai} \frac{1}{2}\left(\frac{2(z+ai)^3 - 6z(z+ai)^2}{(z+ai)^6} - \frac{6z(z+ai)^4 - 12z^2(z+ai)^3}{(z+ai)^8}\right) \\
&= \frac{1}{(2ai)^3} - \frac{3ai}{(2ai)^4} - \frac{3ai}{(2ai)^4} + \frac{6(ai)^2}{(2ai)^5} \\
&= -\frac{i}{16a^3}.
\end{aligned}$$

Therefore, by Corollary 2.3, if R is large enough, we have

$$\int_{\gamma_R} f(z)dz = 2\pi i\left(-\frac{i}{16a^3}\right) = \frac{\pi}{8a^3}.$$

If we denote by C_R^+ the large half-circle of radius R, then we can parametrize C_R^+ by $z = Re^{i\theta}$ where θ goes from 0 to π. Then, we have

$$\int_{C_R^+} \frac{z^2}{(z^2+a^2)^3}dz = \int_0^\pi \frac{(Re^{i\theta})^2}{((Re^{i\theta})^2+a^2)^3}(iRe^{i\theta})d\theta = i\int_0^\pi \frac{(Re^{i\theta})^3}{((Re^{i\theta})^2+a^2)^3}d\theta.$$

Hence, we have

$$\left|\int_{C_R^+} \frac{z^2}{(z^2+a^2)^3}dz\right| \le |i|\int_0^\pi \frac{|Re^{i\theta}|^3}{(|Re^{i\theta}|^2-a^2)^3}d\theta = \int_0^\pi \frac{R^3}{(R^2-a^2)^3}d\theta = \frac{\pi R^3}{(R^2-a^2)^3}.$$

Therefore, as $R \to \infty$, we have $\int_{C_R^+} \frac{z^2}{(z^2+a^2)^3} dz \to 0$.
Since we have

$$\int_{\gamma_R} f(z)dz = \int_{C_R^+} f(z)dz + \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} dx = \frac{\pi}{8a^3},$$

we have

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} dx = \frac{\pi}{8a^3}.$$

□

Problem 42. State and prove a version of the Residue Theorem that allows poles on the boundary of the disk. Hint: be careful about how the integral along the boundary is defined and check that half of the residue is contributed.

Solution. □

Problem 43. How many roots does $z^4 - 6z + 3$ have in the annulus $D(0,2) \setminus \overline{D}(0,1)$?

Solution. Consider the circle $|z| = 1$. We have $|-6z| = 6$ and $|z^4 + 3| \leq |z^4| + 3 = 4$. Hence, we have

$$|-6z| > |z^4 + 3|.$$

By Rouché's theorem, we know that $-6z$ and $z^4 - 6z + 3$ have the same number of zeros inside the circle $|z| = 1$. Since $-6z$ has 1 root $z = 0$ inside the circle $|z| = 1$, we get that $z^4 - 6z + 3$ has 1 root inside the circle $|z| = 1$. Also, $|-6z| > |z^4 + 3|$ guarantees that there is no root on the circle $|z| = 1$. Hence, there is 1 root in $\overline{D}(0,1)$.
Consider the circle $|z| = 2$. We have $|z^4| = 16$ and $|-6z + 3| \leq |-6z| + 3 = 15$. Hence, we have

$$|z^4| > |-6z + 3|.$$

By Rouché's theorem, we know that z^4 and $z^4 - 6z + 3$ have the same number of zeros inside the circle $|z| = 2$. Since z^4 has 4 roots inside the circle $|z| = 2$, we get that $z^4 - 6z + 3$ has 4 roots inside the circle $|z| = 2$, which is $D(0,2)$.
Therefore, we have the number of roots in the annulus $D(0,2) \setminus \overline{D}(0,1)$ to be 3. □