Probability Seminar talk 10/18: Hypercontractivity and Log-Sobolev Inequalities

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Abstract

"Hypercontractivity and the theory of operator semigroups are well-explored techniques in the field of mathematical physics. In particular, these techniques were developed to show the essential self-adjointness of unbounded operators of interest, such as the Hamiltonian in the classical schrodinger equation. In this talk, we define operator semigroups and hypercontractivity, and we explore their relations to log-sobolev inequalities. A few important ramifications of log-sobolev inequalities will be discussed"

Intro to Hypercontractivity

1. **Definition:** Let (Ω, μ_0) be a probability measure space. $H_0 \geq 0$ is a positive, self-adjoint operator on $L^2(\Omega, \mu)$. We say that e^{-tH} (for $t \geq 0$) is a hypercontractive semi-group if and only if

   (a) e^{-tH_0} is a contraction on $L^2(\Omega, \mu)$ for all $t > 0$, i.e.

   $$||e^{-tH_0} f||_2 \leq ||f||_2$$

   (b) For some T, e^{-TH_0} is a bounded map from $L^2(\Omega, \mu_0) \to L^4(\Omega, \mu)$.

2. **Remark** The second condition is non-trivial because on finite measure spaces we have the containment $L^4(\Omega, \mu) \subseteq L^2(\Omega, \mu)$ (and in general $L^p \subseteq L^q$ when $p > q$) but not the other way around

3. **Remark** Sometimes the first condition is reformulated as

   $$||e^{-tH_0} f||_p \leq ||f||_p \qquad \forall 1 \leq p < \infty, \ \forall f \in L^p \cap L^2$$

   and there is some way to pass between the two definitions

4. **Remark** Note that semi-group arises because $[0, \infty)$ is a semigroup under addition and clearly

   $$e^{-t_1 H_0} e^{-t_2 H_0} = e^{-(t_1 + t_2) H_0}$$

   but there is no inverse

5. **Definition:** A family $(P_t)_{t \geq 0}$ of linear operators on a Banach space $(B, \ || \cdot ||)$ is called a semi-group iff it satisfies the following conditions

   (a) $P_0 = I$, the identity on B

   (b) The map $t \mapsto P_t$ is continuous in that for all $f \in B$, the map $t \mapsto P_t f$ is continuous from $\mathbb{R}^+$ to B.

   (c) For any $f \in B$, and $t, s \in \mathbb{R}^+$, we have

   $$P_{t+s} = P_t P_s f$$

6. Ex: Let's consider $H_0 = -\frac{d^2}{dx^2}$ on $L^2([0, 1], dx)$. Then we have that

   $$f(x) = \int \hat{f}(\xi) e^{2\pi i x \lambda} d\lambda \implies e^{-tH_0} = \int e^{-t\lambda^2} \hat{f}(\lambda) e^{i\lambda x} d\lambda = (K_t * f)(x)$$

where

$$K_t = (4\pi t)^{-1/2} e^{-x^2/4t}$$

Then we have the following bounds

$$||K_t * f||_2 \leq ||K_t||_1 \, ||f||_2 \leq ||f||_2$$

because $||K_t||_1 = 1$. But also we have young's convolution inequality of

$$||f * g||_r \leq ||f||_p ||g||_q \qquad p^{-1} + q^{-1} = r^{-1} + 1$$

Letting $r = 4$, $p = 2$ and $q = 4/3$ with $g = K_t$, we see that

$$||f * K_t||_4 \leq ||f||_2 ||K_t||_{4/3}$$

and for any t non-zero, this last norm is finite. Boom Hypercontractivity.

See here for reference

`https://projecteuclid.org/download/pdf_1/euclid.pcma/1416323532`

7. **Definition:** A symmetric, densely defined (potentially unbounded) operator $A : D(A) \subseteq X \to X$ is essentially self-adjoint if either of the following equivalent conditions hold

   (a) It has self-adjoint closure

   (b) It has a unique self-adjoint extension

   Here "self-adjoint" closure means: we can take the closure of the graph of A, $\overline{\Gamma(A)}$, this actually equals the graph of some other operator B, i.e. $\overline{\Gamma(A)} = \Gamma(B)$. Then the self-adjoint closure of A is B. Similarly, to have a unique self-adjoint extension means that if we can extend A from its dense domain $D(A)$ to all of X, then that extension is self-adjoint

8. **Remark** : Proving essential selfadjointness of unbounded operators is **essential** to mathematical physics, especially quantum theory. This is because in quantum, we care about eigenvalues, which correspond to measurable values

9. **Theorem:** Suppose e^{-tH_0} is hypercontractive semigroup on $L^2(\Omega, \mu)$. Let V be a function on Ω and suppose that $e^{-V} \in \cap_{p<\infty} L^p(\Omega)$ and $V \in L^p$ for some $p > 2$. Then $H_0 + V$ is bounded from below and essentially self-adjoint on $D(H_0) \cap D(V)$. The same result is true if we just impose $V \geq 0$ and $V \in L^2$.
   Proof: From Barry Simon's paper, this amounts to showing the following inequality

   $$||e^{A+B}|| \leq ||e^A e^B||$$

   once we have this, then noting that

   $$||e^{-T(V+H_0)}\varphi||_2 \leq ||e^{-TV} e^{-TH_0}\varphi||_2 \leq ||e^{-TV}||_4 ||e^{-TH}||_4 \leq ||e^{-TV}||_4 C ||\varphi||_2$$

   having used that $e^{-V} \in \cap_p L^p(\Omega)$ and the hypercontractivity assumption for some $T > 0$. This bound implies that $H_0 + V$ is bounded from below because it tells us that $||e^{-T(H_0+V)}||_{L^2 \to L^2} < \infty$, which cannot occur if $H_0 + V$ is unbounded below in the operator sense, i.e.

   $$\inf_{||\varphi||=1} (H_0 + V)\varphi = -\infty$$

10. This inspired future work, such as:
    Theorem: : For $V \geq 0$, $V \in L^2(\mathbb{R}^d, e^{-x^2} dx)$, then $\Delta + x^2 + V(x)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$
    Later this was strengthened to:
    (Improved) **Theorem:** $(-\Delta + V)$ is essentially self-adjoint on C_0^∞ if $V \geq 0$ and $V \in L^2_{loc}(\mathbb{R}^d)$

11. **Remark** : This is amazing for solving the schrodinger equation!

    $$i\hbar \frac{\partial}{\partial t} \psi(r, t) = -\frac{\hbar^2}{2m} \left[\Delta - \frac{2m}{\hbar^2} V \right] \psi(r, t)$$

    In particular, for the time independent schrodinger equation, the left side vanishes, and the right side is more easily understood given the essentially self-adjoint condition

Log Sobolev Inequalities + Hypercontractivtiy

1. Let $\mu(dx) = (2\pi)^{-n/2}\exp(-|x|^2/2)dx$ on $\mathbb{R}^n$
2. The prototypical **log sobolev inequality** is
$$\int_{\mathbb{R}^n} |f(x)|^2 \ln|f(x)|dx \le c\int_{\mathbb{R}^n} |\nabla f(x)|^2 d\mu(x) + ||f||_2^2 \ln ||f||_2$$
3. **Remark** On H^1, Jensen's and the sobolev inequality imply the log sobolev inequality
4. It was first derived from hypercontractivity inequalities of the form
$$||e^{t\mathcal{L}}||_{p,q} \le \exp\left(2d(p^{-1} - q^{-1})\right)$$
for all $1 \le p < \infty$ and $p \le q \le q(t, p, c)$.
5. In fact, we have a more general sobolev inequality due to Gross of the form
$$\int |f(x)|^2 \ln\left(\frac{|f(x)|}{||f||_2}\right) d\mu \le c\int \Gamma_1(f,f)d\mu + d||f||^2_{L^2(\mu)}$$
for some $c, d \in [0, \infty)$. Here μ is some probability measure and
$$\Gamma_1(f,f) = \frac{1}{2}\left(\mathcal{L}f^2 - 2f\mathcal{L}f\right)$$
is the carré du champ, and $\mathcal{L}$ is the infinitesimal generator of some semigroup. I've seen this stuff in physics before where $\mathcal{L} = \frac{1}{i}\frac{d}{dx}$, but let's take
$$\mathcal{L} = \Delta \implies \Gamma_1(f,f) = \text{ some work ... } = ||\nabla f||_2^2$$
Then if we further take $d = 0$, we get
$$\int f^2 \ln(|f|) \le ||f||_2^2 \ln ||f||_2 + c||\nabla f||_2^2$$
which wowee that's our original log sobolev inequality.
6. Notation: when the integral is defined, denote
$$\mu(f) = \int f d\mu$$
7. Let $P_t = e^{t\mathcal{L}}$ be a markov semigroup and μ an invariant measure on the semi group, i.e. $\int P_t f d\mu = \int f d\mu$.
8. Define
$$q(t) = q(t,p,c) = 1 + (p-1)e^{2t/c} \quad \text{s.t.} \quad t \in \mathbb{R}^+$$
9. To prove the hypercontractivity inequality from soblev, we proceed as follows:
Assume that
$$\mu\left(f^q \ln\frac{f}{||f||_q}\right) \le \frac{2c}{q}\mu\Gamma_1(f^{q/2}, f^{q/2}) + \frac{2d}{q}\mu(|f|^q)$$
for any $q \in [2, \infty)$ for f non-negative.

I'll sketch the proof that our assumption implies a bound of the form
$$||P_t||_{L^p \to L^q} \le \exp\left(2d\left(\frac{1}{p} - \frac{1}{q}\right)\right)$$

To start, we differentiate the quantity $\ln ||f_t||_{q(t)} = \frac{1}{q(t)} \ln \left(\int |f_t(x)|^{q(t)} d\mu \right)$, which gives

$$\partial_t \ln ||f_t||_{q(t)} = -\frac{\partial_t q(t)}{q(t)^2} \ln \mu(|f_t|^{q(t)}) + \frac{1}{q(t)\mu|f|^{q(t)}} \mu |f_t|^{q(t)} \partial_t(q(t)) \ln f_t + q(t)|f_t|^{q(t)-1} \mathcal{L} f_t)$$
$$= \frac{\partial_t q(t)}{q(t)\mu(f^{q(t)})} \left(\mu \left[f_t^{q(t)} \ln \left(\frac{|f_t|}{||f_t||_{q(t)}} \right) \right] + \frac{cq(t)}{2(q(t)-1)} \mu \left[|f_t|^{q(t)-1} \mathcal{L} f_t \right] \right)$$

stop proof here and save this equation having used that

$$\partial_t q(t) = \frac{2}{c}(q(t) - 1)$$

Through some trickery, we can deduce that the second summand in the parenthesis is bounded in the following form

$$\frac{cq(t)}{2(q(t)-1)} \int |f_t|^{q(t)-1} \mathcal{L} f_t \le -2\frac{c}{q(t)} \mu \Gamma_1(f_t^{q(t)/2}, f_t^{q(t)/2})$$

restart proof here Under our hypothesis, we get

$$\partial_t \log ||f_t||_{q(t)} \le 2d \frac{\partial_t q(t)}{q^2(t)}$$

now because $q(0) = p$, we have

$$||f_t||_{q(t)} \le ||f_t||_p \exp \left(2d(p^{-1} - q(t)^{-1}) \right)$$

and there exists some time t_0 such that $q = q(t_0)$ so that

$$||f_t||_q = ||P_{t_0} f_{t-t_0}||_{q(t_0)} \le ||f_{t-t_0}||_p \exp(2d(p^{-1} - q^{-1})) \le ||f||_p \exp(2d(p^{-1} - q^{-1}))$$

here, this use time invariance of the measure/integral. Make sure time invariance is ok □

10. To see the other direction, we assume that

$$||P_t||_{p,q} \le \exp \left(2d(p^{-1} - q^{-1}) \right)$$

holds for all $p \in [1, \infty)$ and $p \le q \le q(t, p, c)$. We define

$$\phi_f(t) : t \mapsto \exp \left(-2d(p^{-1} - q(t)^{-1}) \right) ||f_t||_{q(t)}$$

for some reason $\phi_f(t)$ is decreasing, which should come from

$$\phi_f(t+s) \le \exp \left(-2d(p^{-1} - q(t+s)^{-1}) \right) ||P_t||_{q(s),q(t+s)} \, ||f||_{q(s)}$$
$$\le \exp \left(-2d(p^{-1} - q(t+s)^{-1}) \right) ||P_t||_{q(s),q(t+s)} \, ||f||_{q(s)} = \phi_f(s)$$

where somehow we're able to conclude $||P_t||_{q(s),q(t+s)} \le 1$, probably by some interpolation of P_t being a contraction.

The above tells us that $\log(\phi_f(t))$ is decreasing, differentiating it, we get

$$\log(\phi_f(t)) = \left(-2d(p^{-1} - q(t+s)^{-1}) \right) + \log(||f_t||_{q(t)})$$
$$\implies \partial_t \left(-2d(p^{-1} - q(t+s)^{-1}) \right) + \partial_t \log(||f_t||_{q(t)}) = \partial_t \log(\phi_f(t)) \le 0$$

rearranging and doing out the algebra, we get

$$\partial_t \ln(||f_t||_{q(t)}) \le \partial_t \left[2d \left(p^{-1} - q(t)^{-1} \right) \right] = 2d \frac{\partial_t q(t)}{q(t)^2}$$

Now looking back at the previous equation

$$\partial_t \ln ||f_t||_{q(t)} = \frac{\partial_t q(t)}{q(t)\mu(f^{q(t)})} \left(\mu \left[f_t^{q(t)} \ln \left(\frac{|f_t|}{||f_t||_{q(t)}} \right) \right] + \frac{cq(t)}{2(q(t)-1)} \mu \left[|f_t|^{q(t)-1} \mathcal{L} f_t \right] \right)$$
$$\implies \mu \left[f_t^{q(t)} \ln \left(\frac{|f_t|}{||f_t||_{q(t)}} \right) \right] + \frac{cq(t)}{2(q(t)-1)} \mu \left[|f_t|^{q(t)-1} \mathcal{L} f_t \right] \le 2d \frac{\mu(f_t^{q(t)})}{q(t)}$$

now we plug in $t = 0$ and $p = 2$ (which forces $q = 2$) to get

$$\mu \left[f^2 \ln \left(\frac{f}{||f_t||_2} \right) \right] \le c\mu\Gamma_1(f, f) + \frac{2d}{q} ||f||_2^2$$

11. In general, there's a nice correspondence between inequalities of the form

$$||e^{-tH}||_{L^2 \to L^\infty} \leq c(t) < \infty \leftrightarrow \int_\Omega |f|^2 \ln |f| d\mu \leq \epsilon ||H^{1/2} f||_2^2 + \beta(\epsilon)||f||_2^2 + ||f||_2^2 \log ||f||_2$$

12. Log-Sobolev quantities are useful for obtaining sharp constants, e.g.

    https://www.ams.org/journals/proc/1998-126-10/S0002-9939-98-04406-2/S0002-9939-98-04406-2.pdf
    https://arxiv.org/pdf/1009.1410.pdf

Here are all of the links I used

https://www.jstor.org/stable/pdf/1997121.pdf?refreqid=excelsior%3A60111daef10d8c092b43a61dba67c0f1
https://www.encyclopediaofmath.org/index.php/Hypercontractive_semi-group#References
https://en.wikipedia.org/wiki/Self-adjoint_operator#Essential_self-adjointness
https://www.jstor.org/stable/pdf/2044957.pdf?refreqid=excelsior%3Afc906ce0b35d55af07f1f83832f24c1a
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