Rice Geometry Seminar: Minimal Surfaces, Allen–Cahn, and Balanced Energy

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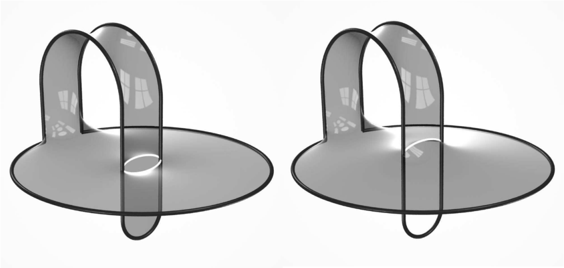
Stanford University

Sept. 20, 2023

Minimal Surfaces



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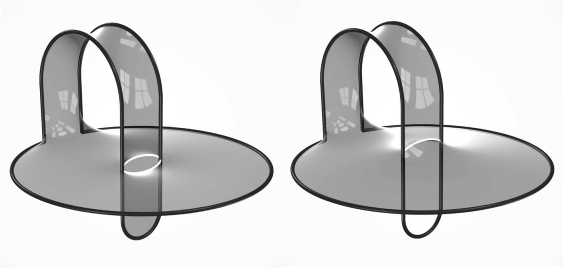

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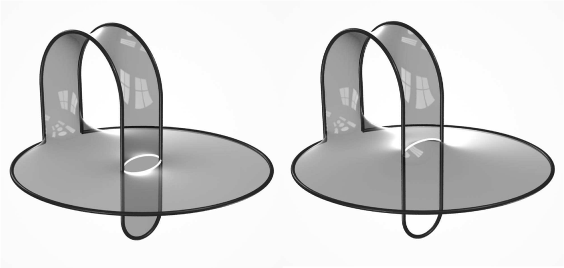

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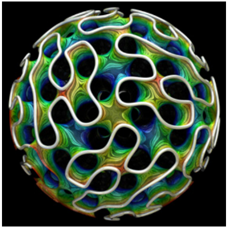

Figure: Gyroid (Alan Schoen)

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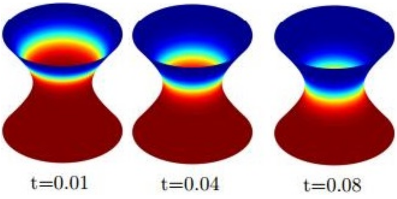

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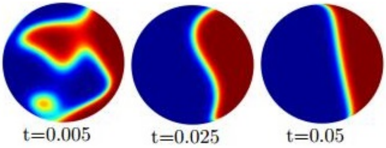

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- Interface in phase separations/transitions

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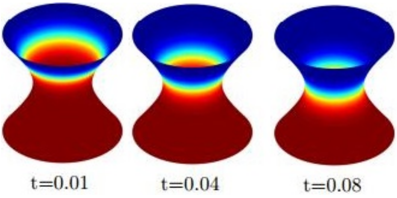

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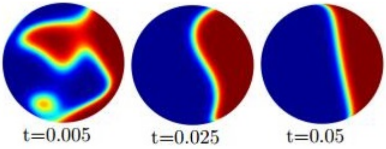

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- Minimize configuration energy $\leftrightarrow$ transition interface is small

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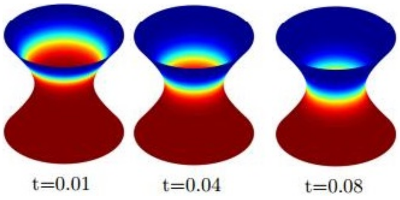

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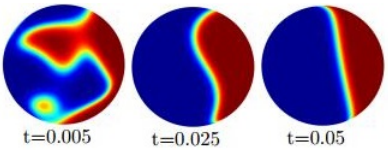

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- Minimize configuration energy $\leftrightarrow$ transition interface is small
- Allen–Cahn equation models phase transitions

Allen–Cahn Background

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$$E_\epsilon(u) = \int_M \epsilon \frac{|\nabla^g u|^2}{2} + \frac{W(u)}{\epsilon} \tag{2}$$

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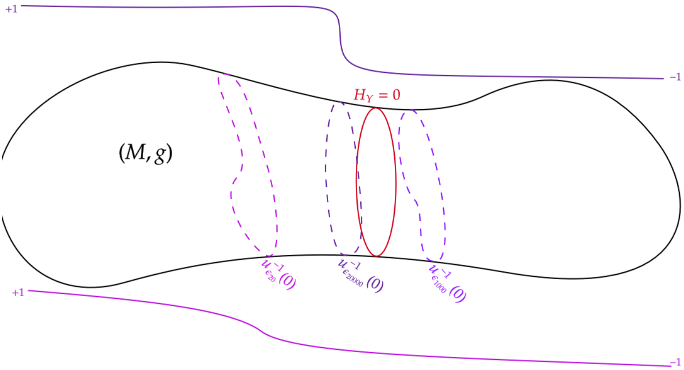

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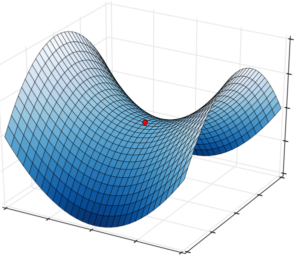
- Index and Nullity bounds:



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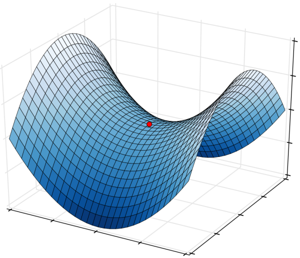

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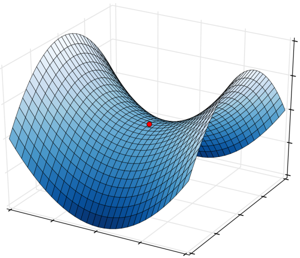

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$$(\textit{Gaspar, Hiesmayr, Le})$$
$$\mathrm{Ind}_{\mathrm{AC},\epsilon}(u_\epsilon) \geq \mathrm{Ind}(Y)$$

$$(\textit{Chodosh–Mantoulidis})$$
$$\mathrm{Ind}_{\mathrm{AC},\epsilon}(u_\epsilon) + \mathrm{Null}_{\mathrm{AC},\epsilon}(u_\epsilon) \leq \mathrm{Ind}(Y) + \mathrm{Null}(Y)$$

Motivation

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- Only interested in Allen–Cahn in connection to minimal surfaces
  - only look at $u \in H^1$ vanishing on hypersurfaces?

BE Set up

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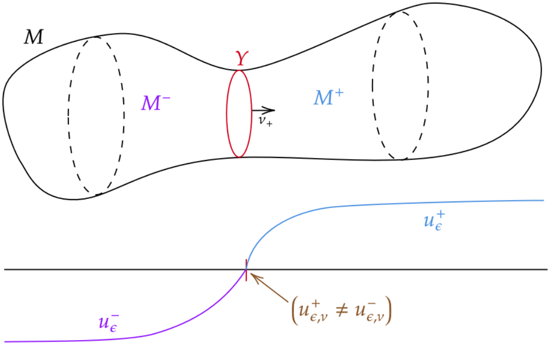

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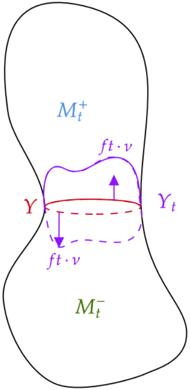
1st Variation



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Theorem (MK, Silva)

The first variation is given by

$$\frac{d}{dt} BE_\epsilon(Y_t)\Big|_{t=0} = \frac{\epsilon}{2} \int_Y f[(u^+_{\epsilon,\nu})^2 - (u^-_{\epsilon,\nu})^2]$$

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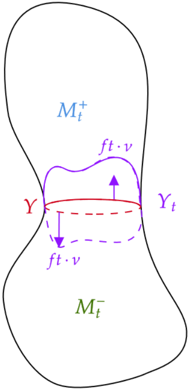

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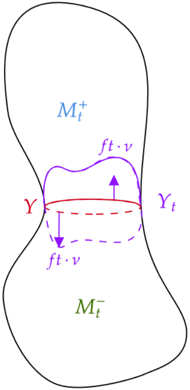

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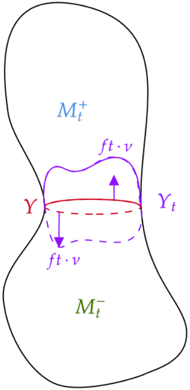

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- For Y satisfying mild geometric assumptions

$$\frac{\epsilon}{2}(u^+_{\epsilon,\nu})^2 - (u^-_{\epsilon,\nu})^2 = \frac{1}{2\sqrt{2}}[H_Y + O(\epsilon)]$$

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If Y satisfies mild geometric assumptions, then

$$\frac{d^2}{dt^2} BE_\epsilon(Y_t)\Big|_{t=0} = D^2 A|_Y(f) + E(f)$$

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- $\dot{u}^\pm_\epsilon$ satisfies linearized Allen–Cahn system on $M^\pm$
- Error bound relies on invertibility of $\epsilon^2 \Delta_g - W''(u) : H^1_0(M^+) \to H^{-1}_0(M^-)$

Applications: Fischer-Colbrie-Schoen Mimic

Let M^3 complete 3-manifold with $R \geq 0$ and $Y^2 \subseteq M^3$, compact.

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Theorem (MK, Silva)

If Y is a stable critical point for BE_ϵ (satisfying mild geometric constraints) then Y is either conformally equivalent to (S^2, g_{round}) or Y is topologically a torus and

$$||A_Y||^2_{L^2(Y)} \leq K\epsilon^{1/2}$$

for K independent of ϵ.

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  (kernel is in $H^1(M)$)
- Theorem says we can compute index/nullity on *smaller space* of

  $$W = \{\dot{w}(f) \in H^1(M) \mid f \in H^1(Y), \epsilon^2\Delta_g \dot{w} = W''(u)\dot{w}, \dot{w}\Big|_Y = -fu_\nu\}$$

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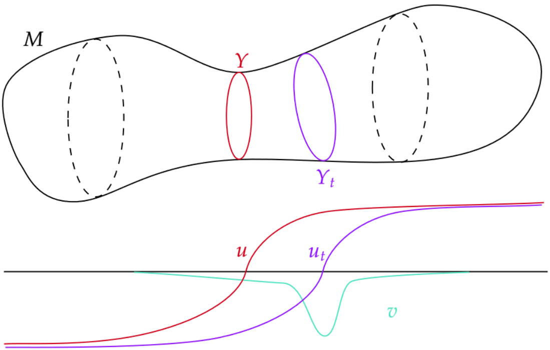

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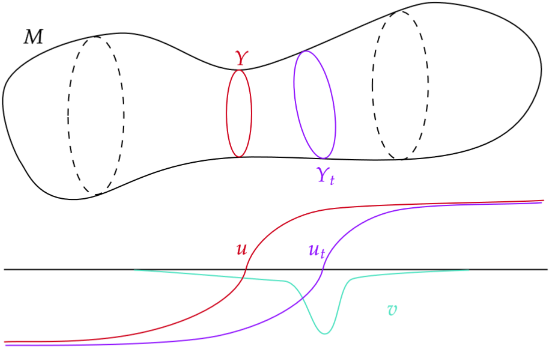

- Rewrite $u + tv = u_t + \psi_t$, u_t is a minimizer on $M_t^{\pm}$

- Let $\dot{\psi} = \partial_t \psi\Big|_{t=0}$, then

$$\frac{d^2}{dt^2} E_\epsilon(u + tv) \overset{!}{=} \frac{d^2}{dt^2} \mathrm{BE}_\epsilon(Y_t)\Big|_{t=0} + Q(u)(\dot{\psi}, \dot{\psi})$$

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- $\dot{\psi}\Big|_Y = 0$ and u_ϵ is a minimizer gives:

$$\begin{aligned} Q(\dot{\psi}, \dot{\psi}) &\geq 0 \\ \implies \frac{d^2}{dt^2} E_\epsilon(u + tv)\Big|_{t=0} - \frac{d^2}{dt^2} \mathrm{BE}_\epsilon(Y_t)\Big|_{t=0} &\geq 0 \\ \implies \mathrm{Ind}_{AC}(u_\epsilon) - \mathrm{Ind}_{\mathrm{BE}_\epsilon}(Y) &\geq 0 \end{aligned}$$

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Let $u_{\epsilon,2p} : S^1 \to \mathbb{R}$ be the unique Allen–Cahn solution on S^1 vanishing on D_{2p}-symmetric points:

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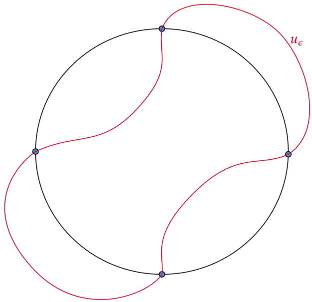

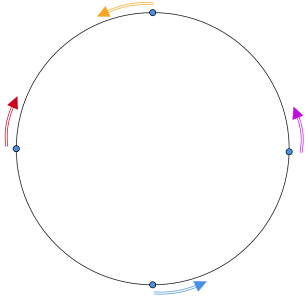

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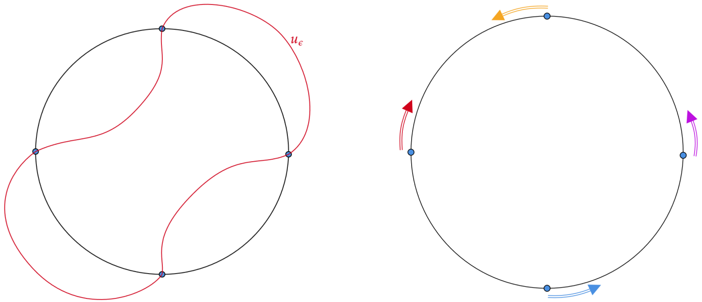

Theorem

Fix $p > 0$. There exists ϵ_p such that for all $\epsilon < \epsilon_p$, $u_{\epsilon,2p}$ has Allen–Cahn Morse index $2p - 1$ and nullity 1. The nullity is realized by rotations and every other variation produces a strictly negative variations.

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$$
\begin{aligned}
\frac{d^2}{dt^2}\mathrm{BE}_\epsilon(Y+tf) &= \sum_{i=0}^{2p-1} f\left(\frac{i}{2p}\right) u_\nu\left(\frac{i}{2p}\right)\left[\dot{u}_{i,x}^{+} - \dot{u}_{i,x}^{-}\right]\left(\frac{i}{2p}\right) \\
&= \epsilon c \sum_{i=0}^{2p-1} f\left(\frac{i}{2p}\right) \dot{u}_{i,x}\left(\frac{i}{2p}\right) \\
&\quad + f\left(\frac{i+1}{2p}\right) \dot{u}_{i,x}\left(\frac{i+1}{2p}\right) \\
&\overset{!}{=} \epsilon c^2 v(\epsilon) \sum_{i=0}^{2p-1}\left[f\left(\frac{i}{2p}\right) - f\left(\frac{i+1}{2p}\right)\right]^2
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where $v(\epsilon) < 0$ - relies on explicit computation of $\dot{u}_{i,x}$

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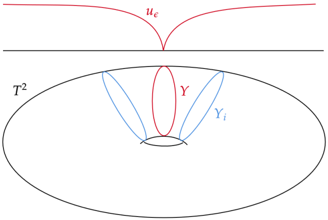

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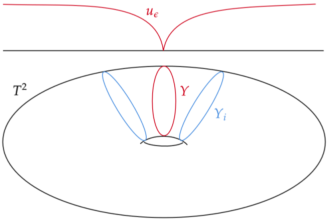

- Development of BE_ϵ-surface flow

$$\partial_t x = [u_\nu^+(x)]^2 - [u_\nu^-(x)]^2$$