SURIM 2020 Analysis Bootcamp

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Contents

1 Introduction

This is a growing collection of reading assignments and problem sets for the SURIM 2020 project entitled "Constructing solutions to the Allen-Cahn equation." The students are Wenqi Li, George Nakayama, and Gautam Manohar. The mentor and author of this document is Jared Marx-Kuo.

Problems that I think are *particularly* difficult are marked with "(h)" - of course, this does not mean that the unmarked problems are easy

2 Week 0: Preliminary Problem Set

2.1 Problem 1

a) Given $f_\alpha(x) = x^{-\alpha}$ for $\alpha > 0$, find all $1 \le p \le \infty$ such that $f \in L^p(1,\infty)$ and all $1 \le k \le \infty$ such that $f \in L^k(0,1)$.

b) Show that

$$||f||_p = \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p\right)^{1/p} & p < \infty \\ \inf\{c \ge 0 \mid |f(x)| < c \text{ almost everywhere}\} & p = \infty \end{cases}$$

is a norm **for** $p = 1$ **and** $p = \infty$ (If "almost everywhere" is confusing, assume f is smooth and replace with "everywhere" for which the statement is true). This is true for all $1 \le p \le \infty$, but the intermediary values require some convexity arguments.

Recall that a norm, denoted by $|| \cdot ||$ satisfies

$$\begin{aligned} ||f|| \ge 0 \qquad & \text{nonnegativity} \\ ||f+g|| \le ||f|| + ||g|| \qquad & \text{triangle inequality} \\ a \in \mathbb{R} \implies ||af|| = |a| \; ||f|| \qquad & \text{scaling} \\ ||f|| = 0 \implies f \equiv 0 & \end{aligned}$$

for all f and g in the space to which we apply the norm.

2.2 Problem 2

a) Holdder's inequality is quite useful. It states that for $1 < p < \infty$ given and q such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$f \in L^p, \qquad g \in L^q \implies ||fg||_1 \le ||f||_p \; ||g||_q$$

Using Holder's inequality, show that $L^p(a,b) \subseteq L^r(a,b)$ for (a,b) a finite interval, and $r < p$. (Hint: you can do this without holder's inequality by decomposing (a,b) appropriately based on f)

b) What if I replaced (a,b) with $\mathbb{R}$?

2.3 Problem 3

a) Formally show that for $f(x) = |x|$,

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} =: H(x)$$

as a weak derivative. (Note: The function $H(x)$ is called the heaviside function. Because we care about weak derivatives, we don't need to define $H(x)$ at 0, as any two functions which differ at just one point give the same integral operator. More specifically, if $g_1(x)$ and $g_2(x)$ are riemann integrable which agree everywhere except at a point, then

$$I_{g_k} : C^\infty(\mathbb{R}) \to \mathbb{R} \text{ s.t. } I_{g_k}(f) = \int_{\mathbb{R}} g_k(x) f(x), \qquad I_{g_1} = I_{g_2} \text{ as operators on smooth functions}$$

b) What is the weak derivative of $H(x)$? Can it be represented as I_g for some g, riemann integrable function?

2.4 Problem 4

In calculus of variations, we often have a functional $I : C^\infty(\Omega) \to \mathbb{R}$ for Ω an open subset of a manifold with nice boundary or an open subset of $\mathbb{R}^n$. Really, $C^\infty(\Omega)$ is replaced by a larger set of functions, but we'll restrict to smooth functions for ease. A function $f \in C^\infty(\Omega)$ is said to be a critical point of I if it has zero derivative under perturbation by a compact function, i.e.

$$\frac{d}{dt} I(f + t\varphi)\Big|_{t=0} = 0, \qquad \forall \varphi \in C_c^\infty(\Omega)$$

Here, $C_c^\infty(M)$ denotes smooth functions with support that is a compact subset of Ω. For

$$I(u) = E_\epsilon(u) = \int_\Omega \frac{\epsilon}{2}||\nabla u||^2 + \frac{1}{\epsilon}W(u)$$

show that critical points satisfy the Allen-Cahn equation. Recall that

$$||\nabla u||^2 = \nabla u \cdot \nabla u = \sum_{i=1}^n (\partial_i u)^2$$

(Hint: integration by parts, make sure boundary values vanish!)

2.5 Problem 5

Suppose $u(x)$ is a smooth solution to Allen-Cahn at scale $\epsilon = 1$. Using the maximum principle and **assuming that** $|u| \le 1$ **everywhere**, show that in fact $|u(x)| < 1$ everywhere. (Hint: If $u(x) = \pm 1$, find a differential equation satisfied by $u - v$ for $v(x) \equiv 1$ or $v(x) \equiv -1$. Apply the maximum principle).

For reference, here is the set up

$$Lu := \sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j u + \sum_{k=1}^n b_k(x)\partial_k(x) + c(x)u = f$$
$$a_{ij}, b_k, c, f : \Omega \to \mathbb{R}$$
$$\exists \lambda, \Lambda > 0 \text{ s.t. } \Lambda Id - \{a_{ij}\}, \{a_{ij}\} - \lambda Id \text{ are positive definite}$$

Maximum principle

$$[\forall x \in N \subseteq \Omega, \qquad Lu(x) \ge 0 \text{ and } c(x) \le 0] \implies \left[\sup_{x\in N} u(x) = \sup_{x \in \partial N} u(x)\right]$$

3 Week 1: Measure theory

3.1 Sources

1. Bass' Real Analysis - free to download
2. Geometric Features of the Allen-Cahn Equation - free to download

3.2 Perspectives on Measure theory and L^p spaces

Read the short posts below on the history and purpose of measure theory and L^p-spaces. I'll ask you to summarize one such post you read during the next meeting

1. Basics of lebesgue integral and why we define it
2. Physical meaning of the lebesgue integral and some of its uses
3. Lebesgue vs. Riemann integration abstractly and pedagogically
4. Integration as area under the graph
5. Usefulness of L^p spaces and measure theory in PDE

3.3 Problems

Allen-Cahn

1. Read up to section 2.3 in Otis' notes
2. Read exercise 2.1 and feel free to use it's results (we won't understand how to do this until later)

3. Complete exercises 2.2 (h), 2.3, and 2.4

Bass

1. Read Chapters 1 - 4
2. Ch. 2: 2.3, 2.5
3. Ch. 3: 3.1 3.3
4. Ch. 4: 4.2, 4.4, 4.6 (h), 4.10 (h)

4 Week 2: Measure functions + L^p spaces

4.1 Sources: Bass

1. Bass' Real Analysis - free to download

4.2 Perspectives on Measure theory and Banach Tarski

1. Lusin's theorem - turns out this is less powerful than it appears to be
2. Egorov's theorem - usually taught along side Lusin's theorem. Both end up not being very useful, though I've used Egorov's theorem more times than I have Lusin's
3. A topological game - an interesting topic now that we're talking about measurable sets
4. Video of the banach tarski paradox - classic Vsauce video
5. Proof of the Banach Tarski paradox - a 10 page write up. If there are things that are unclear, feel free to skim and just try to extract the general gist of the paradox and proof. Alternatively, we can talk about this paper next time

4.3 Readings and Problems

Bass

1. Read Chapters 5,6,7, and 15
2. Ch 5: 5.1, 5.4, 5.9
3. Ch. 6: 6.4, 6.5, 6.7
4. Ch. 7: 7.10, 7.16, 7.26 (h)
5. Ch. 15: 15.2, 15.4, 15.6, 15.25 (h)

5 Weeks 3-4: Sobolev spaces and Regularity

5.1 Sources

1. Bass' Real Analysis
2. Geometric Features of the Allen-Cahn Equation
3. Craig Evan's PDE book second edition

5.2 Perspectives

1. Evans Chapter 1 (Introduction chapter, just read no problems)
2. Introduction to Harmonic functions - these guide our intuition for solutions to second order elliptic PDEs
3. Rayleigh quotient and first eigenvalue - alludes to future work in calculus of variations, but first eigenvalues also show up in this chapter
4. Importance of first eigenvalues - "spectral gap" in particular is something that will continue to show up if you study graph theory/combinatorics

5.3 Readings and Problems

Bass

1. Read Bass 16, Exercises: 16.1, 16.4, 16.5, 16.8
2. Read Bass 18, Exercises: 18.3, 18.4, 18,14, 18.15 (h)

Allen-Cahn

1. Read the short blurb below Exercise 4.1 on the definition of stable solutions to Allen-Cahn
2. Potentially after rescaling u, i.e. $u(x) \to \alpha u(\beta x) = \tilde{u}(x)$ and the potential function $W(t) \to aW(bt) = \tilde{W}(t)$, find k such that $\tilde{u}$ satisfies the equation for a Jacobi elliptic function with the modified potential
3. Show that the Jacobi elliptic function is not a stable solution?
4. Try to find a bound for how this family of Jacobi functions converges to 0 as λ changes as in section 2. This might amount to figuring out by what factor do we need to rescale by in order to make u look like a Jacobi Elliptic function

Evans (h)

1. Read Chapter 5, Problems: 1, 4,5, 7, 15, 17, 18, 19, 21
2. Read Chapter 6, Problems: 2, 4, 7, 10, 12, 16

6 Week 5

Students prepared for the midterm presentation they were giving this week

7 Week 6

7.1 Readings and Problems

Evans

1. Section 8.1-8.3
2. Problems 1, 4, 5

Gilbarg and Trudinger

1. Section 6.1 and 6.2 on Schauder estimates
2. Problems 6.1, 6.2
3. Read Section 4.5, and just make a note of Theorem 4.16 under the extension to (4.44)

Allen-Cahn

1. Complete Exercise 2.5 - sketch

(a) Chapter 8. To argue sign, write $u = u_+ - u_-$, what happens if we switch one of these signs? Maximum principle is useful

(b) u is a minimizer, so find a function which has linear energy, then u will have to have less than linear energy

(c) See the third page for Otis' proof sketch

(d) Apply Arzela ascoli both to the functions and their derivatives. Take some sort of diagonal sequence by doing this first on a sequence of increasing balls, and then increasing the radius of the balls and the diagonal sequence.

(e) Apply the maximum principle to show that the vanishing set is just the axes, else we get a contradiction with energy. There's a technical issue here, which I have a hint from Otis on the second page

2. Read Chapter 5

3. Complete Exercises: 5.3, 5.4 (try to show u is a minimizer, but if you get stuck don't spend too much time on this), 5.5, 5.6, 5.8

7.2 Otis' Hint for (e)

"There's a technical issue with applying monotonicity that I am glad to explain, but I don't think it works directly (I made this mistake since for minimal surfaces, this would be OK, but there's a funny feature with the Allen—Cahn monotonicity that's not present for minimal surfaces and causes a headache).

Here is an alternative approach to finishing this part that does the trick, I think:

1. The functions u_R have the property that for any ball B contained in the first quadrant and if R is large enough so that $B \subset \Omega_R$, then u_R minimize on B. Namely, if w has the same value on ∂B then $E(w; B) \geq E(u_R; B)$. This is clear, since replacing u_R by w inside of B does not change the boundary conditions and would be admissible in the minimization problem.

2. This property passes to the limit: for any ball B in the first quadrant, the limit of u_R, u, minimizes $E(\cdot)$ on B.

3. The function $u = 0$ does not minimize $E(\cdot)$ on large B.

You already have given the argument for 3 (just cut off from 0 on ∂B to -1 on a smaller ball; this has less energy as long as B is large). The fact 2 seems obvious until you start to prove it, and then it seems wrong. But it's actually true. The reason it will seem wrong is that if u was not minimizing on B, there is some function w so that $w = u$ on ∂B and $E(w; B) \leq E(u; B) - \delta$ for some $\delta > 0$.

Now you want to glue w into u_R and decrease the energy. But, u_R and w don't exactly agree on ∂B (they're just close). However, you can still make it work because "w drops energy by a definite (independent of R amount)". Namely, if you choose a bump function ϕ that's supported in B, then

$$(1 - \phi)u_R + \phi w$$

Is smooth. Moreover, as $R \to \infty$, u_R limits to u, so this will limit to $(1 - \phi)u + \phi w$. You can check that as ϕ limits to χ_B, this function limits in H^1 to the function that's u outside of B and w inside of B (this uses the fact that the values of the functions agree at ∂B; otherwise this function could not be in H^1). So, by balancing ϕ and R carefully, we can arrange that

$$E((1 - \phi)u_R + \phi w; \Omega_R) = E(u_R; \Omega_R \setminus B) + E(w; B) + o(1)$$

As $R \to \infty$. (This takes some checking and its a bit annoying but it's worth trying if you don't see why it's true). But, then you get

$$\begin{aligned}
E(u_R; \Omega_R) &\leq E((1 - \phi)u_R + \phi w; \Omega_R) && \text{since } u_R \text{ minimizes} \\
&= E(u_R; \Omega_R \setminus B) + E(w; B) + o(1) && \text{(by the above construction/claim)} \\
&\leq E(u_R; \Omega_R \setminus B) + E(u; B) - \delta + o(1) && \text{(by the property assumed about w)} \\
&= E(u_R; \Omega_R \setminus B) + E(u_R; B) - \delta + o(1) && \text{(since } u_R \to u \text{ in } C^\infty \text{ on } B). \\
&= E(u_R; \Omega_R) - \delta + o(1) && \text{(putting the pieces back together).}
\end{aligned}$$

This is then a contradiction since $\delta \leq o(1)$ can't hold. Sorry about my confusion! I guess strictly speaking, we did not really need to prove the $E \leq CR$ bound proved in the earlier part of the problem, since we have to use the minimizing property again after passing $R \to \infty$ anyways... "

7.3 Otis' Hint for (c)

From Otis: "Consider the set up of $\Delta u = f$ (Weakly) in $B_1 \setminus \{0\}$, that is, for any $\varphi \in C_0^\infty(B_1 \setminus \{0\})$, we have

$$\int \nabla u \cdot \nabla \varphi + f\varphi = 0$$

Then, interior estimates imply that $u \in C^\infty_{loc}(B_1 \setminus \{0\})$ and u solves the PDE in the classical sense at any point besides 0. Now, we want to extend things across 0 (we need some further assumptions on u, f, otherwise $\Delta \log |x| = 0$ is a counterexample). Let's take $\varphi \in C_0^\infty(B_1)$ and choose a sequence of cutoff functions χ_i with $\mathrm{supp}\chi_i \Subset B_1 \setminus \{0\}$ but $\chi_i \to 1$ near 0 point wise. Then we can plug $\chi_i \cdot \varphi$ into the weak formulation to get

$$\int \chi_i(\nabla u \cdot \nabla \varphi) + \varphi \nabla u \cdot \nabla \chi_i + f\chi_i\varphi.$$

So, e.g. if $u \in H^1(B_1)$, then we would have that the first term tends to $\nabla u \cdot \nabla \varphi$ (by dominated convergence) and the second term tends to zero (by Holder), i.e. it satisfies

$$\leq ||\varphi||_\infty (\int |\nabla u|^2)^{1/2} \cdot (\int |\nabla \chi_i|^2)^{1/2}$$

As long as we can ensure that $\int |\nabla \chi_i|^2 \to 0$. This is possible by a common function known as the log-cutoff trick (taking χ_i cutting off $\sim$ linearly just barely fails since we get a gradient of $\frac{1}{r}$ integrated over a ball of radius $\sim r$, i.e. we get $O(1)$ instead of $o(1)$). Take instead

$$\chi_r(x) = 2 - (\log |x|)/\log r$$

For $|x|$ in (r^2, r) (And 0 for $|x| < r$ and 1 for $|x| > r$). I'll leave it for you to check that the Dirichlet energy goes to zero as $r \to 0$. Thus, as long as u $\in H^1$ and f in L^2, say, we get that u is a weak solution across 0. Then, we can use elliptic regularity (assuming, say $f \in C^\infty$) to get that u is smooth and solves the PDE across 0. Of course, in the Allen—Cahn setting, you have to be a bit more careful, since it's non-linear. But I think the proof should go reasonably along these lines."

8 Week 7-8

8.1 Sources

1. Reference for Laplace-Beltrami Operator i.e. the laplacian with a metric
2. Guaraco's notes

8.2 Readings and Problems

Gilbarg and Trudinger

1. Finish problems from last week

Allen-Cahn (Otis)

1. Finish assigned problems from last week
2. Exercise 2.1
   (a) Do it out in full by iterating the (+2) version of elliptic regularity to increase sobolev norm by 2, then use Morrey's inequality, then use schauder estimates and repeat
3. Exercise 5.6
   (a) Prove a stronger version: $|\partial^\alpha u| \leq C_k$ where $k = |\alpha|$ is some constant. Do this with both of the Schauder norms (make sure to explain why we can get information about $||u||_{C^{1,\alpha}}$ and $||u||_{C^{2,\alpha}}$, despite the original schauder estimates having weightings with distance functions)

Allen-Cahn (Guaraco)

1. General Allen-Cahn Exercises: 1,6,7, 10, 11
2. Solution Construction Exercises: 14, 15, 16, 19, 21

9 Week 9

Continued work on Guaraco problems + preparation for final presentation! SURIM was 10 weeks total, but the first week was spent choosing problems, so this is the last week of group project work