Student Analysis Talk 11/5: Introduction to Renormalized Volume

1 Motivation

- Suppose you're in $\mathbb{H}^3$ ball model
$$g_{\mathbb{H}^3} = \frac{dr^2 + r^2 dg_{S^2}}{(1-r^2)^2}$$
and you want to compute the volume of some $Y^2 \hookrightarrow \mathbb{H}^3$ (Draw picture!)
- If Y^2 closed, then its fine because
$$g_{\mathbb{H}^3} \leq K(dr^2 + r^2 d_{S^2})$$
- If Y^2 is noncompact, then consider $\overline{Y}^2 \subseteq \mathbb{H}^3$, then $g_{\mathbb{H}^3}$ has unbounded coefficients
- Example: $\mathbb{H}^2 \subseteq \mathbb{H}^3$ represented as the geodesic disk. The restricted metric on $\mathbb{H}^2$ is
$$h := g\Big|_{\mathbb{H}^2} = \frac{4}{(1-r^2)^2}[dr^2 + r^2 d\theta^2] \implies \int_{\mathbb{H}^2} dVol_h = \int_0^1 \int_0^{2\pi} \frac{4}{(1-r^2)^2} r dr d\theta = \infty$$
(you can check that this integral diverges, because near the boundary it tends like $(1-r)^{-2}$
- Despite our foolish idea, suppose we wanted to still extract some information related to the volume computation
- Let $\rho = \frac{2(1-r)}{1+r}$. Notice that $\rho^{-1}(0) = 1$, and also
$$\overline{g} := \rho^2 g_{\mathbb{H}^2} = \frac{16}{(1+r)^2}[dr^2 + r^2 d\theta]$$
is defined everywhere on $\mathbb{H}^3$. And finally
$$||d\log(\rho)||_g^2 = ||d\rho||_{\overline{g}}^2 = 1$$
so ρ is *like* a distance function. (These conditions actually determine ρ, though its unclear at the moment why we'd want this)
- Consider the expansion of
$$\int_{\rho > \epsilon} dA = \int_{\rho > \epsilon} \frac{4r}{(1-r^2)^2} dr d\theta$$
$$= 4\pi \int_{r=0}^{(2-\epsilon)/(2+\epsilon)} \frac{d}{dr} \frac{1}{1-r^2} dr$$
since $\rho > \epsilon \leftrightarrow \frac{2-\epsilon}{2+\epsilon} > r$. Integrating, we get
$$\int_{\rho > \epsilon} dA = 4\pi \left[(1-r^2)^{-1}\right]_{r=0}^{(2-\epsilon)/(2+\epsilon)} = 4\pi \left[\frac{4 + 4\epsilon + \epsilon^2}{8\epsilon} - 1\right] = 4\pi \left[\frac{1}{2\epsilon} - \frac{1}{2} + \frac{\epsilon}{8}\right]$$

Taking the constant term in ϵ then yields

$$\underset{\epsilon \to 0}{FP} \int_{\rho > \epsilon} dA = 4\pi \cdot \frac{-1}{2} = \boxed{-2\pi}$$

($\underset{\epsilon \to 0}{FP}$ means "finite part as $\epsilon \to 0$)

- We can do this much more generally
  - Instead of $\mathbb{H}^3$, consider M^{n+1}, **asymptotically hyperbolic, conformally compact** (i.e. has a compact boundary with some given conformal class of metrics, $[k]$), and $Y^m \subseteq M^{n+1}$ minimal
  - Instead of $\rho = \frac{4(1-r)}{1+r}$, consider ρ_Y
  - Do the same process and define

$$\boxed{\mathcal{V}(Y) := \underset{\epsilon \to 0}{FP} \int_{\rho_Y > \epsilon} dA_Y}$$

  - We call $\mathcal{V}(Y)$ the **Renormalized Volume**

2 Physical Motivation (Briefly)

- Need to change labels, e.g. let M be X, and then $N = \partial M$ and $\gamma = N$ instead... Make it consistent with Rafe's and my notation
- Source: Graham-Witten "Conformal Anomaly of Submanifold Observables..." (1999)
- String theory: happens (in one old model) on $M \times S^5$, M Einstein and conformally compact, $\partial M = N$. (Draw M at least, ball model is good choice)
- "Wilson Loop Operator", $W(\gamma)$ for $\gamma \subseteq N = \partial M$, $\to$ find "string" whose "world sheet", $Y \subseteq \overline{M}$, $\partial Y = \gamma$
- In "supergravity approximation", we only care about when Y is minimal, and we have

$$\langle W(N) \rangle \approx \exp(-TA(Y))$$

  (T is some constant, called the String tension)
  - Aside: I would hypothesize that

$$\langle W(N) \rangle = \int_{Y \text{ s.t. } \partial Y = \gamma} \exp(-TA(Y)/\hbar) d\mu = \int_{Y \text{ s.t. } \partial Y = \gamma} \exp(i(iA(Y))(T/\hbar)) d\mu$$

  - where μ is some measure on the set of extensions Y with $\partial Y = N$. Ok this would give $\exp(iA(Y))$, but then maybe a wick rotation saves it and you can get $\exp(A(Y))$?
  - Stationary phase approximation (Quals haha) says that we only care about the integral near critical points of $-TV(Y)$ (assume T constant), i.e. local volume minimizers
  - Also, stationary phase approximation is how physicists get around not having to define the measure on the set of extensions Y with $\partial\gamma$, which could be quite complicated
  - So it suffices to compute the volume of Y minimal surfaces with $\partial Y = N$ - usually just 1 (nondegenerate gives uniqueness vs degenerate situation)
- Again, $A(Y)$ not defined, but turns out (**somehow**) that

$$\langle W(N) \rangle \approx \exp(-T\mathcal{V}(Y))$$

  i.e. the divergent terms probably cancel out in the integral ("ultraviolet divergences"), maybe again due to stationary phase approximation
- **Remark** Alexakis, Mazzeo (2008): critical points of Renormalized volume in $\mathbb{H}^3$ are minimal
  - so stationary phase argument with $W(N) = \int_Y \exp(-T\mathcal{V}(Y))$ still makes sense

3 Mathematical Motivation

- Renormalized Volume is a conformal invariant for even dimensional manifolds
  - i.e. if we change the metric on $N = \partial M$ by a conformal factor, then $\mathcal{V}(M)$ stays the same
  - Same holds if we change metric on $\gamma \subseteq N$ and compute $\mathcal{V}(Y)$ for Y a minimal extension
- $\mathcal{V}(Y)$ reflects topological/geometric information

  Proposition (Alexakis, Mazzeo 2008)**.** Suppose (M, g) Einstein with conformally compact boundary. Suppose $\gamma \subseteq \partial M$ and $Y^2 \hookrightarrow M$ with $\partial Y = \gamma$ and Y intersecting the boundary orthogonally, then

  $$\mathcal{V}(Y) = -2\pi\chi(Y) + \frac{1}{2}\int_Y 2|H|^2 - |\hat{k}|^2 dA + \int_Y W_{1212} dA$$

  Some remarks:

  - This formula is very specific for Y two dimensional
  - $\int |H|^2$ is the Willmore energy and is conformally invariant in two dimensions
  - $\int_Y |\hat{k}|^2$, where $\hat{k}$ is trace-free second fundamental form is also conformally invariant
  - W_{1212} is the Weyl curvature, and also conformally invariant (note: this vanishes when $Y \subseteq \mathbb{H}^{n+1}$)
  - "Intersecting the boundary orthogonally" - seems strong, but automatically satisfied by minimal surfaces. It says that in some graphical asymptotic expansion

    $$Y = \text{graph}(u) \implies u(s, \rho) = u_0(s) + u_2(s)\rho^2 + \dots$$

    then u is quadratic as we approach the boundary (Quadratic in what? Some boundary defining function like ρ)
  - Intersecting the boundary orthogonally is guaranteed when Y is minimal!

4 Formal Background

- Given an ambient space, M^n, with boundary $N = \partial M$, and a conformal class of metrics $[k]$, the **conformal infinity**, for N
- (M, g) is Einstein if

  $$\text{Ric}_g = kg$$

  for some $k \in \mathbb{R}$
- **Definition:** M is conformally compact, if $\overline{M}$ is a manifold with compact boundary and

  $$\exists\, \rho : \overline{M} \to \mathbb{R}^{\geq 0}, \quad \text{s.t.} \quad \{\rho = 0\} = \partial M$$

  Moreover, we define

  $$\overline{g} := \rho^2 g$$

  require that $\overline{g}$ is a metric on $\overline{M}$ and

  $$\nabla^{\overline{g}}\rho\Big|_{\partial M} \neq 0$$
- **Definition:** ρ as above is called a boundary definition
- **Remark** If $\varphi : \overline{M} \to \mathbb{R}^+$ smooth, then $\rho^* = \varphi\rho$ is a bdf
- **Definition:** A bdf is **special** if

  $$||d\log(\rho)||_g^2 = ||d\rho||_{\overline{g}}^2 = 1$$

  (So special bdfs are most like distance functions to the boundary)

Proposition. For M conformally compact and a choice of representative $k_0 \in [k]$, there exists a unique special bdf for M such that

$$\overline{g}\Big|_N = k_0$$

Proof: Requires more machinery than I have time for

4.1 Example

- Poincare Ball model of hyperbolic space $\mathbb{H}^3$

$$g = \frac{4}{(1-r^2)^2}\left[dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi d\theta^2\right]$$

is Einstein.

- Want special bdf, ρ, for $\mathbb{H}^3$. Assume rotational symmetry and enforce

$$\begin{aligned} 1 = ||d\log(\rho)||_g^2 = \frac{\rho_r^2}{\rho^2} g^{rr} &= \partial_r(\log(\rho))^2 \frac{(1-r^2)^2}{4} \\ \implies \partial_r(\log(\rho)) &= \frac{-2}{1-r^2} \\ \implies \rho &= A\frac{1-r}{1+r} \end{aligned}$$

- Note: $A \neq 0 \implies \rho^{-1}(0) = \{r = 1\} = S^2 = \partial\overline{\mathbb{H}^3}$.
- Suppose we want to prescribe the standard metric on this boundary i.e. we want

$$k_0(\theta) = \sin^2\phi d\theta^2 + d\phi^2$$

Enforce this

$$k_0 = \rho^2 h\Big|_{r=1} = \frac{4A^2}{(1+r)^4}[dr^2 + r^2 d\phi^2 + r^2\sin^2\phi d\theta^2]\Big|_{r=1} = \frac{4A^2}{16}[d\phi^2 + \sin^2\phi d\theta^2]$$

choose $A = 2$ so that ρ is positive.

- Set $\overline{g} := \rho^2 h$

$$\implies \overline{g}\Big|_{r=1} = \overline{g}\Big|_{\partial M} = k_0$$

also

$$\overline{\nabla}\rho = \overline{g}^{ij}(\partial_i\rho)\partial_j = \overline{g}^{rr}(\partial_r\rho)\partial_r = \frac{(1+r)^4}{16}\cdot\frac{-4}{(1+r)^2}\partial_r$$

$$\overline{\nabla}\rho\Big|_{r=1} = -\partial_r$$

4.2 Conformal Invariance of Renormalized Volume

- With our set up, we can now give meaning to conformal invariance of Renormalized Volume
- If we have $k_0 \in [k] \leftrightarrow \rho_0$, $k_1 \in [k] \leftrightarrow \rho_1$, then

$$\underset{\epsilon\to 0}{FP}\int_{\rho_0>\epsilon} dA_M = \underset{\epsilon\to 0}{FP}\int_{\rho_1>\epsilon} dA_M$$

i.e. if we expand

$$\begin{aligned} dA_M &= d\rho_0 \wedge ({\rho_0}^{-n}\tau_n + {\rho_0}^{-n+1}\tau_{n-1} + \cdots + {\rho_0}^{-1}\tau_1 + \tau_0) \\ &= d\rho_1 \wedge ({\rho_1}^{-n}\tilde{\tau}_n + {\rho_1}^{-n+1}\tilde{\tau}_{n-1} + \cdots + {\rho_1}^{-1}\tilde{\tau}_1 + \tilde{\tau}_0) \end{aligned}$$

then the result is the same

- Because of Poincare-Einstein condition, g actually has an even expansion in terms of special bdfs, i.e. (Modify previous equation, don't write below)

$$dA_M = \begin{cases} d\rho \wedge (\rho^{-n}\tau_n + \rho_0^{-n+2}\tau_{n-2} + \cdots + \rho_0^{-1}\tau_1 + \tau_0) & n \text{ odd} \\ d\rho \wedge (\rho^{-n}\tilde{\tau}_n + \rho_0^{-n+2}\tilde{\tau}_{n-2} + \cdots + \rho_0^{-2}\tilde{\tau}_2 + \tilde{\tau}_0) & n \text{ even} \end{cases}$$

so that

$$\int_{\rho>\epsilon} dA_M = \begin{cases} c_0\epsilon^{-n+1} + \cdots + c_{n-3}\epsilon^{-2} + d\ln(\epsilon) + c_n + o(1) & n \text{ odd} \\ c_0\epsilon^{-n+1} + \cdots + c_{n-2}\epsilon^{-1} + c_n + o(1) & n \text{ even} \end{cases}$$

- **Proposition:** For n odd, d is independent of the special bdf (i.e. independent of the choice of $k_0 \in [k]$). If n even, then c_n is conformally invariant

5 Results Overview

- Care about critical points for renormalized volume (via stationary phase approximation) $\to$ want information about Y minimal
- For convenience, we work in $M = \mathbb{H}^{n+1}$ the half space model with

$$g_{\mathbb{H}^{n+1}} = \frac{dx^2 + dy_1^2 + \cdots + dy_n^2}{x^2}$$

where x is *almost a special bdf* for $\mathbb{H}^{n+1}$, and

$$\overline{g} = x^2 g_{\mathbb{H}^{n+1}} = g_{Euc}$$

I'll explain technicalities of why x is not a special bdf but we can still use it. Moreover, we can treat x like a special bdf for Y, despite it not satisfying the "special" condition of $||dx||_{\overline{g}}\big|_Y = 1$

5.1 Graphicality

- $Y^m \subseteq \mathbb{H}^{n+1}$ minimal, conformally compact with boundary $\gamma = \partial Y = Y \cap \partial\mathbb{H}^{n+1}$. We require that Y be embedded in some neighborhood of the boundary γ.
- Consider cylinder over the boundary: (Draw this in Half-space model)

$$\Gamma = \gamma \times \mathbb{R}^+ = \{(x, s) \mid s \in \gamma\}$$

- Describe Y near the boundary as a graph over Γ via the exponential map (Draw Y!)

$$Y \cap \{x \leq \epsilon\} = \{\overline{\exp}_\Gamma(u(s, x))\}$$

where $\overline{\exp}$ denotes the exponential map taken with respect to the Euclidean metric, restricted to elements of $N(\Gamma)$.

- u satisfies a degenerate elliptic equation coming from Y being minimal, and just like the metric, is even to high order

Proposition: For $u(s,x) = u^i(s,x)\overline{N}_i(s)$ with $\{\overline{N}_i\}$ ONB for Γ,

$$\boxed{u^i(s,x) = \begin{cases} u_2^i(s)x^2 + u_4^i(s)x^4 + \cdots + u_m^i(s)x^m + u_{m+1}^i(s)x^{m+1} + \ldots & \text{m even} \\ u_2^i(s)x^2 + u_4^i(s)x^4 + \cdots + u_{m+1}^i(s)x^{m+1} + U^i(s)x^{m+1}\log(x) + u_{m+2}(s)x^{m+2} + \ldots & \text{m odd} \end{cases}}$$

for smoothly varying coefficients $u_k(s)$ and $U(s)$.

Remarks:

- Even expansion hypothesized for a while by physicists
- PDE can be thought of as an ODE in x, which is essentially

$$(x\partial_x)(x\partial_x - (m+1))u(s,x) = R$$

- Getting regularity is difficult: requires geometric arguments with maximum principle, and microlocal analysis (edge operators)
- Above expansion is **asymptotic**, not convergent (i.e. can give partial series with remainder vanishing to next order)

- **Corollary:** Renormalized Volume is well defined mathematically

5.2 Variations

- Consider variations of Y. Describe smooth family of minimal submanifolds as

$$Y_t = \exp_Y(S_t)$$

for $S_t \in N(Y)$ a smooth

- $\dot{S} := \partial_t S_t \Big|_{t=0}$ satisfies the Jacobi equation

$$J(X) = \Delta^{\perp}(X) - \tilde{A}(X) + \mathrm{Tr}(\mathrm{Ric}(X,\cdot)) = 0$$

- As a result, $\dot{S}$ satisfies a regularity theorem. In codimension 1 we can write

$$\dot{S} = \dot{\phi}(s,x)\nu(s,x)$$

for ν a normal to Y.

$$\boxed{\dot{\phi}(s,x) = \begin{cases} \dot{\phi}_0(s) + \dot{\phi}_2(s)x^2 + \cdots + \dot{\phi}_m(s)x^m + O(x^{m+1}) & \text{m even} \\ \dot{\phi}_0(s) + \dot{\phi}_2(s)x^2 + \cdots + \dot{\phi}_m(s)x^{m+1} + \Phi(s)x^{m+1}\log(x) + O(x^{m+2}) & \text{m odd} \end{cases}}$$

i.e. $\dot{\phi}$ is even in x to high order

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Theorem 5.1. First variation of Renormalized volume in codimension 1:

$$\boxed{\begin{aligned} n \text{ even} &\implies \frac{d}{dt}\mathcal{V}(Y_t)\Big|_{t=0} = -(n+1)\int_\gamma \dot{\phi}_0(s)u_{n+1}(s)\, dA_\gamma(s) \\ n \text{ odd} &\implies \frac{d}{dt}\mathcal{V}(Y_t)\Big|_{t=0} = -(n+1)\int_\gamma \Big[\dot{\phi}_0(s)u_{n+1}(s) + F(\dot{\phi}_0, u_2)(s)\Big]\, dA_\gamma(s) \end{aligned}}$$

and the second variation:

$$\boxed{\begin{aligned} n \text{ even} &\implies \frac{d^2}{dt^2}\mathcal{V}(Y_t)\Big|_{t=0} = \int_\gamma \Big((1-n)\dot{\phi}_0(s)\dot{\phi}_{n+1}(s) + \dot{\phi}_0(s)^2\left[(n-1)(n-2) - 8nu_2u_{n+1}(s)\right]\Big)\, dA_\gamma(s) \\ n \text{ odd} &\implies \frac{d^2}{dt^2}\mathcal{V}(Y_t)\Big|_{t=0} = \int_\gamma \Big[(1-n)\dot{\phi}_0(s)\dot{\phi}_{n+1}(s) + \dot{\phi}_0(s)^2\left[(n-1)(n-2) - 8nu_2u_{n+1}(s)\right] \\ &\qquad - \dot{\phi}_0(s)\Big[4(n+2)\dot{\phi}_0(s)u_2(s)U(s) + \dot{\Phi}(s)\Big] + F_2(\dot{\phi}_0, u_2)\Big]\, dA_\gamma(s) \end{aligned}}$$