Math 257a: Intro to Symplectic Geometry with Umut Varolgunes

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1. The plan for the course is:
   (a) First 8 lectures from Alan Weinstein's "Lectures on Symplectic Manifolds"
   (b) Gromov Non-squeezing (follow Gromov's paper on J-holomorphic curves)
   (c) Symplectic Rigidity: Lagrangian manifolds (see Audin-Lalonde-Potterovich)
2. Some other resources are
   (a) Mc-Duff-Salamon (good reference but large)
   (b) Da Silva: Lectures on Symplectic Manifolds
   (c) Arnold: Methods of Classical Mechanics
3. So what is a symplectic manifold? It is M, a manifold, equipped with ω, a non-degenerate closed two-form
4. ωi is a smoothly varying anti-symmetric bilinear form/pairing on each T_pM with $p \in M$
   (a) $v, v' \in T_pM \implies \omega(v, v') \in \mathbb{R}, \qquad \omega(v, v') = -\omega(v', v)$
   (b) Non-degenerate:
   $$\forall p \in M, \quad \forall v \in T_pM, \quad \exists v' \in T_pM \ \text{ s.t. } \ \omega(v, v') \neq 0$$
   (c) Closed: $d\omega = 0$
5. The above three conditions give us some *magic*: around each point $p \in M$, there exists coordinates s.t. $x_1, \dots, x_n, y_1, \dots, y_n$ and
   $$\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n \quad \forall p \in TU \subseteq TM$$
   which in matrix coordinates looks like
   $$\begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & 0 & \dots \\ 0 & \dots & A & 0 \\ 0 & \dots & 0 & A \end{pmatrix} \text{ s.t. } A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1}$$
   contrast this with a metric, which is a 2-form but has no such expression
6. Why do we study symplectic forms?
   (a) Answer 1: They arise naturally in many different areas of mathematics
      i. Classical (Hamiltonian) Mechanics
         A. Stuff happens in T^*X, e.g. $H : T^*X \to \mathbb{R}$, $H =$ hamiltonian
         B. Energy function, H, leads to time evolution via the symplectic structure
      ii. Complex Geometry
         A. Y a complex manifold: Assume that it is an affine variety (or more generally, a stein manifold), i.e. it is a solution set of a list of polynomial equations in $\mathring{A}^n_{\mathbb{C}} = \mathbb{C}^n$ as a complex manifold. Y then has a natural symplectic structure up to isomorphism
      iii. Lie Groups and Algebras → "coadjoint orbits"

        iv. Representation Varieties, which are important in low dimensional topology
    (b) Answer 2: Symplectic Manifolds are worth studying for their own sake
        i. No local invariants
        ii. Interesting (highly dependent on ω!) but ∞-dimensional (large!) symmetry groups. Symmetries are diffeomorphisms of M
        iii. Beautiful and Complicated global invariants
        iv. Lots of input from other fields

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There are 3 levels of symplectic structures:

Symplectic Vector Spaces

1. V a symplectic $\mathbb{R}$-vector space (finite dim)

2. A symplectic structure is a bilinear pairing

$$\Omega : V \times V \to \mathbb{R} \leftrightarrow \tilde{\Omega} : V \to V^* \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$$

3. Anti-symmetry tells us that $\Omega(v, w) = -\Omega(w, v)$

4. Non-degeneracy tells us $\tilde{\Omega}$ is an isomorphism

5. Ex:Let U be any vector space over $\mathbb{R}$. $U \oplus U^*$ has a natural symplectic structure given by

$$\Omega(u \oplus u^*, v \oplus v^*) = v^*(u) - u^*(v)$$

6. If you pick a basis $\{e_1, \ldots, e_n\}$ for U, this has a dual basis, $\{e_1^*, \ldots, e_n^*\}$ for U^*

7. The matrix of Ω is then given by 1

8. Note that every symplectic vector space is of the form $U \oplus U^*$, and we can find some decomposition using linear algebra

Symplectic vector bundles over manifolds

1. X a smooth manifold, $E \to X$ (this should be a vertical arrow to denote a vector bundle, but I'm too lazy and will be using this notation from hereon)

2. Symplectic structure:= smoothly varying symplectic pairing Ω_x on E_x for every $x \in X$

$$\begin{tikzcd} E \otimes E \arrow[r, "\Omega"] \arrow[dr] & X \times \mathbb{R} \arrow[d] \\ & X \end{tikzcd}$$

or equivalently

$$\begin{tikzcd} E \arrow[r, "\Omega"] \arrow[dr] & E^* \arrow[d] \\ & X \end{tikzcd}$$

3. Here Ω is a section of $\mathrm{Hom}(E, E^*) \to X$, and smoothness is formulated in consider this bundle

4. Ex: P is any vector bundle, $P \oplus P^* \to X$ has a symplectic structure (whitney sum of vector bundles)

5. **Question**: Is any symplectic vector bundle of this form? No consider TS^2 which has a symplectic structure not induced by a Whitney sum

Symplectic Manifolds

1. X a smooth manifold
2. Symplectic structure is given by a symplectic structure on $TX \to X$ as a vector bundle, i.e. $\omega \in \Omega^2(X)$ nondegenerate and closed
3. Ex: V a vector space, thought of as a manifold
   (a) There exists a natural/canonical form on T^*V
   (b) $T^*V \cong V \times V^*$ where the $\cong$ denotes diffeomorphism so $T_{v\oplus v^*} = V \oplus V^*$ which is naturally symplectic
   (c) We have our original example of a symplectic structure, i.e.
   $$\omega(u \oplus u^*, v \oplus v^*) = u^*(v) - v^*(u)$$
   (d) Note that this form is immediately closed because it is translation invariant
   (e) V has a basis $\{e_1, \ldots, e_n\} \to \{q_1, \ldots, q_n\}$, which can be thought of as position coordinates
   V^* has a basis $\{f_1, \ldots, f_n\} \to \{p_1, \ldots, p_n\}$, which can be thought of as momentum coordinates.
   (f) Then
   $$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$
4. Another example is for M a manifold, T^*M has a natural symplectic structure
   (a) Construction 1: $M = \cup_\alpha U_\alpha \subseteq \mathbb{R}^n$, an open cover by coordinate charts, then
   $$T^*M = \cup_\alpha T^*U_\alpha \subseteq T^*\mathbb{R}^n \cong T\mathbb{R}^{2n}$$
   The idea is to glue charts on T^*M together, where each chart has the natural symplectic structure for an open set in $\mathbb{R}^{2n}$
   (b) Construction 2: There exists a "tautological" one form on T^*M given by
   $$\lambda \in \Omega^1(T^*M) \;\text{ s.t. }\; \omega = d\lambda$$
   consider the exact sequence
   $$\begin{tikzcd} 0 \arrow[r] & T_X^*M \arrow[r, "\text{inclusion as the vertical subbundle}"] & T_{p_x}(T^*M) \arrow[r, "d\pi_*"] & T_xM \arrow[r] & 0 \end{tikzcd}$$
   which motivates
   $$\lambda_{p_x}(v) = p_x(d\pi_* v) \quad \forall x \in M,\; p_x \in T_x^*M,\; v \in T_{p_x}(T^*M)$$
   Clearly this is a one form on T^*M as it takes in elements of the tangent space to T^*M
   (c) Claim: $d\lambda$ is symplectic, we have that
   $$d(d\lambda) = 0$$
   so $d\lambda$ is closed. Non-degeneracy follows because for
   $$T^*M = \cup_\alpha T^*U_\alpha \quad T^*\mathbb{R}^n \cong \mathbb{R}^{2n} = \mathbb{R}^n_q \times \mathbb{R}^n_p \implies \lambda \in \Omega^1(T^*M) \;\text{ s.t. }\; \lambda\Big|_{T^*U_\alpha} = \sum_{i=1}^{n} p_i dq_i$$
   This computation can be done in coordinates, and its fairly straightforward, because
   $$d\pi_* : T_{p_x}T^*M \to T_xM$$
   is given by
   $$d\pi_*(v) = d\pi_*(a_x, b_x) = a_x$$
   where a_x lies in the horizontal component of $T_{p_x}T^*M$ corresponding to T_xM, and b_x lies in the vertical component corresponding to T_x^*M. Then we also have that locally
   $$p_x = \sum_{i=1}^{n} p_i de_i \;\text{ s.t. }\; p_x(a_x) = \sum_{i=1}^{n} p_i a_i$$

where $a_x = \sum_i a_i e_i$ locally. Thus

$$p_x(d\pi_*(v)) = \sum_i p_i a_i$$

because dq_i denotes the spanning set for $\mathbb{R}^n_q$, the latter n-dimensions of $T^*\mathbb{R}^n\Big|_{x=(q,p)}$ which act on elements of T_qM (remember that q denotes the position part, not p), and so we have that

$$p_x(d\pi_*) = \sum_i p_i dq_i$$

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1. Recall: T^*M for M a smooth manifold, is naturally a symplectic manifold with $\omega_M = d\lambda_M$ the tautological one form

2. If we have $M \xrightarrow{\varphi} N$ a diffeomorphism, then we get

$$\begin{tikzcd} T^*N \arrow[r, "T\varphi"] \arrow[d, "\pi"] & T^*M \arrow[d, "\pi"] \\ N \arrow[r, "\varphi^{-1}"] & M \end{tikzcd}$$

   with $T\varphi = d\varphi$

3. Claim: $(T\varphi)^*\lambda_M = \lambda_N$
   Proof: Consider $v \in T_{\xi_n}(T^*N)$, then we want to show

$$((T\varphi)^*(\lambda_m))_{\xi_n}(v) = (\lambda_n)_{\xi_n}(v)$$

   Note that the left hand side is equivalent to

$$(\lambda_m)_{d\varphi^*\xi_n}(dT\varphi_* v) = d\varphi^*\xi_n(d\pi_x \circ dT\varphi_*(v)) = \xi_n(d\varphi_* \circ d\pi_* \circ dT\varphi_* v)$$
$$= \xi_n(d\varphi_* \circ (d\varphi^{-1})_* \circ d\pi_*(v)) = \xi_n(d\pi_*(v))$$

4. This all works because φ is a diffeo

5. In coordinates, this amounts to

$$p^T \cdot dq = \sum_i p_i dq_i = (d\varphi^T \circ \tilde{p})^T (d\varphi^{-1}) d\tilde{q} = \tilde{p}^T d\varphi d\varphi^{-1} d\tilde{q} = \tilde{p}^T d\tilde{q}$$

   where $\tilde{p}$ and $\tilde{q}$ denote the momentum and position coordinates on N.

6. **Corollary:**

$$T\varphi^*\omega_M = \omega_N$$

   i.e. $T\varphi$ is a symplectomorphism

7. **Proposition:** Let $T^*N \xrightarrow{\Phi} T^*M$ be a diffeomoprhism, with $\Phi^*\lambda_M = \lambda_N$. Then $\Phi = T\varphi$ for some $\varphi : M \to N$.
   Proof: The key object here is the Liouville vector field. For any T^*X, we define a canonical vector field implicitly by

$$\iota_{\mathcal{L}_x}(\omega_x) = \omega_x(\mathcal{L}_x, \cdot) = \lambda_x$$

   here $\iota_{\mathcal{L}_x}$ is the interior derivative, which normally has a more complicated form than the above form, but it works out because we're working with a two form.
   In coordinates, we have

$$\omega_x = \sum_i dp_i \wedge dq_i \qquad \lambda_x = \sum_i p_i dq_i \implies \mathcal{L}_x = \sum_i p_i \frac{\partial}{\partial p_i}$$

   which can be thought of as the fiberwise radial vector field (remember that p stands for momentum and hence the $\{\partial/\partial p_i\}$ span the fiber).

Given that $\Phi^*\lambda_m = \lambda_n$, it follows that $\Phi_*\mathcal{L}_N = \mathcal{L}_M$. Note that Φ_* is not usually well defined acting on vector fields, but it is here because Φ is a diffeomorphism.
Now let φ^t_N be the flow of $-\mathcal{L}_N$ for all $t \in \mathbb{R}$ and $\Phi \circ \varphi^t_N = \varphi^t_M \circ \Phi$. We claim that Φ sends the zero section of TN to the zero section of TM, but this follows immediately because

$$\Phi\left(Z(\mathcal{L}_N)\right) = Z(\mathcal{L}_M)$$

where $Z(X)$ denotes the points on M where X vanishes as a vector field. This follows because Φ is a diffeo.
Now we claim: Φ sends cotangent fibers to cotangent fibers. The proof is that

$$\lim_{t\to\infty} \varphi^t_N(\xi_N) = 0_N \in T_N N$$

i.e. because the flow φ^t_N is generated by the radial vector in vertical fiber, then letting $t \to \infty$ pushes any vector in the fiber to 0. Now let

$$\psi := \Phi\Big|_{Z(\Phi)} : N \to M$$

i.e. ψ is the restriction of Φ to its zero section, which is N because Φ is a diffeo. We also have

$$\Phi \circ \varphi^t_N(\xi_N) = \varphi^t_M(\Phi(\xi_N)) \quad \forall t \in \mathbb{R}$$

by nature of the flow commuting. Taking $t \to \infty$ on both sides, we get

$$\lim_{t\to\infty} \varphi^t_N(\xi_N) = 0_N$$

In coordinates, this amounts to

$$\{p^N_i\},\ \{q^N_i\} \qquad \{p^M_i\},\ \{q^M_i\} \ \text{ s.t. } \ q^M = \psi(q^N) \quad p^M = \rho(q^N, p^N)$$

for an unknown ρ. Then because $\lambda_M = \lambda_N$ we have

$$p^N_i dq^N_i = p^M_i dq^M_i = \rho(q^N, p^N) d\psi(q^N) = \rho(q^N, p^N)(d\psi) dq^N$$

but also

$$(p^N)^T dq^N = (p^M)^T dq^M = (\rho(q^N, p^N))^T d\psi(dq^N)$$

Thinking of these as matrices, we have

$$(p^M)^T = (\rho(q^N, p^N)^T) d\psi \implies \rho(q^N, p^N) = (d\psi^T)^{-1} p^M$$

and so $\Phi = d\psi$ for ψ our restriction of Φ to its zero section. ☐

8. Remark: $\Phi^*\omega_M = \omega_N$ would not have lead to the same conclusion. The counterexample is the Dehn Twist, which is a symplectomorphism

$$T^*S^1 = \mathbb{R}_q/\mathbb{Z} \times \mathbb{R}_p, \qquad \omega = dp \wedge dq$$

The Dehan Twist takes a vertical fiber and the twists a full 2π about 0 in the fiber

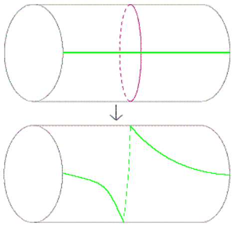

Figure 1: Dehn Twist twisting the fibers a full rotation about 0

In this case, the fibers are **not sent** to fibers, and so we can't have that the above symplectomorphism is $T\varphi = d\varphi$ for some φ. However. $dp \wedge dq$ **is** mapped to $dp \wedge dq$ somehow, while $pdq \not\mapsto pdq$

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1. From last time: note that the Dehn Twist can be made into a compactly diffeomorphism, i.e. the twist is accomplished in a finite amount of time. This means that we can "implant" the twist in other surfaces, e.g. the torus which locally looks like $S^1 \times [0,1]$, which gives a symplectomorphism on most surfaces which is not of the form $T\varphi = d\varphi$

2. Note that for a symplectomorphism of the form $T\varphi : T^*M \to T^*M$, fibers move via a linear map, and so not compact supported

3. Three levels of important subobjects

Symplectic Vector Spaces

(a) V a symplectic vector space with Ω. Let $W \subseteq V$ be a subspace, and define

$$W^\Omega = \{v \in V \mid \Omega(v,w) = 0, \qquad \forall w \in W\}$$

which is called the symplectic complement

(b) Claim: $\dim W^\Omega + \dim W = \dim V$
Proof: $f : V \to W^*$ such that $v \mapsto \Omega(v, \cdot)$. Rank-nullity tells us that

$$\ker(f) = W^\Omega$$

but f is surjective because Ω is non-degenerate, so

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "\tilde{\Omega}"] & V^* \arrow[r, two heads] & W^* \arrow[r] & 0 \end{tikzcd}$$

I think. ☐

(c) Claim: $W \subseteq (W^\Omega)^\Omega$. Proof is left as an exercise

(d) **Definition:** W is

  i. Coisotropic if $W^\Omega \subseteq W$ (e.g. when W is codim 1)
  ii. isotropic if $W \subseteq W^\Omega$ (e.g. when W is dim 1)
  iii. Lagrangian if $W = W^\Omega$
  iv. Symplectic if $W \cap W^\Omega = \{0\}$

Subbundles

(a) A subbundle $P \subseteq E \to M$ for E a symplectic vector bundle is called (coisotropic, isotropic, lagrangian, or symplectic) if for all $y \in M$, $P_y \subseteq E_y$ is one of those, respectively

Submanifolds

(a) $Y \subseteq M$ a submanifold is called (coisotropic, isotropic, lagrangian, or symplectic) if $TY \subseteq TM\Big|_Y$ if $TY \subseteq TM\Big|_Y$ is (coisotropic, isotropic, lagrangian, or symplectic) respectively

4. Ex: $Y \subseteq M$ is lagrangian if $\omega_y\Big|_{T_yY} = 0$ for all $y \in Y$ and Y is half the dimension of M

5. Examples of Lagrangian objects: Lagrangian subspaces of $(V \oplus V^*, \Omega_{st})$, as for any V a real vector space, $V, V^* \subseteq V \oplus V^*$ are Lagrangian

6. When is the graph of a linear map $V \to V^*$ lagrangian? We want that

$$\forall v, w \in F \qquad \Omega(v \oplus f(v), w \oplus f(w)) = f(w)v - f(v)w = 0$$

i.e. $\tilde{f} : V \otimes V \to \mathbb{R}$ is linear and symmetric

7. **Lemma:** $W \subseteq V$ is a lagrangian subspace iff it is isotropic + half dimensional

8. Not every lagrangian subspace is graphical over V

Lagrangian subbundles

(a) If a bundle is of the form $E \oplus E^* \to X$ for some $E \to X$ an $\mathbb{R}$-vector bundle, then $E, E^* \subseteq E \oplus E^*$ are lagrangian subbundles

(b) Claim: Let $P \to X$ be a symplectic vector bundle, then $P \to X$ is symplectically isomorphic to some $E \oplus E^* \to X$ iff P has a Lagrangian subbundle.
Proof: Only if is easy, if will be proved Wedn. □

(c) Fact: There exist symplectic vector bundles which admit half-dimensional subbundles which are not Lagrangian

9. Lagrangian Manifolds in symplectic manifolds

   (a) Surfaces (2-dim) manifolds with an area form because any 1-dim submanifold, S, is lagrangian b/c at a point p, we have

   $$v_p, w_p \in T_pS \implies \omega(v_p, w_p) = K\omega(v_p, v_p) = 0$$

   as $w_p = Kv_p$ for some K and this is an area form so it is zero on linearly dependent vectors

   (b) Cotangent bundles T^*X with $d\lambda_x$ are also lagrangian manifolds

   (c) The zero section $\subseteq T^*X$ is lagrangian because $\sum_i dp_i \wedge dq_i = 0\Big|_{p=0}$

   (d) Also every cotangent fiber because $d\lambda_x$ is of the form

   $$d\lambda_x = \sum_i dp_i \wedge dq_i\Big|_{q=c} = 0$$

   as all of the dq_i terms will be vanish on this fiber

   (e) Claim: Let $s : X \to T^*X$ be a section, and let η_s be the corresponding one form given the isomorphism between $TX \cong T^*X$. Then

   $$s^*(\lambda_x) = (\eta_s)_x$$

   for λ_x the "tautological" one form evaluated at a point x.

10/02

1. Examples of lagrangian manifolds: Curves in surfaces

2. Claim: if $\alpha \in \Omega^1(X)$, then for $s_\alpha : X \to T^*X$ we have $s_\alpha^*(\lambda_x) = \alpha$

3. Last time, we defined a Lagrangian submanifold $L^n \subseteq M^{2n}$ as a submanifold such that $TL \subseteq TM\Big|_L$ is a Lagrangian subbundle.

4. Generalizing this:
L^n is a smooth manifold. $L^n \xrightarrow{f} (M^{2n}, \omega)$ is an embedding. We will call this a Lagrangian embedding if $f^*\omega = 0$

5. Check: $f^*\omega = 0$ is equivalent to $\text{Im}(f)$ being a lagrangian submanifold. This follows directly, as we want $W^\Omega = W$, but then $\omega(v, \cdot) = 0$ for all $v \in W$. If $W = \text{Im}(f) = f(L)$, then we have $\omega(f(\ell), \cdot) = 0$ for all $\ell \in L$, and so $f^*\omega = 0$ by definition.

6. **Corollary:** (follows by the claim) If α is closed then $\mathrm{Im}(s_\alpha)$, called "the graph of α" is a Lagrangian.
   Proof: We want to show
   $$s_{\alpha^*}(d\lambda_x) = 0 \implies s_\alpha^*(d\lambda_x) = ds_\alpha^*(\lambda_x) = d\alpha = 0$$
   by nature of $s_\alpha^*(\lambda_x) = \alpha$

7. Important examples of closed 1-forms: Exact 1-forms $\alpha = df$, for then
   $$s_{\alpha^*}(\lambda_x) = \alpha = df$$
   meaning that $\lambda_x\Big|_{\mathrm{Im}(s_\alpha)}$ is exact (note that $\mathrm{Im}(s_\alpha)$ is the graph α). This gives us "an exact Lagrangian"

8. Recall: For V a real vector space yielding $(V \oplus V^*, \Omega)$ symplectic, and $f : V \otimes V \to \mathbb{R}$ a symmetric map, then we get a graphical lagrangian over V inside $V \oplus V^*$, where
   $$f : V \to \mathbb{R} \text{ s.t. } v \mapsto q(v, v)$$
   is a quadratic form arising from the symmetric map f.

9. A separate type of lagrangian manifold arises from $(X_1, \omega_1) \xrightarrow{\varphi} (X_2, \omega_2)$ where φ is a symplecticomorphism

10. A symplectic manifold can be created from the above via $(X_1 \tilde{\times} X_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ where the $\tilde{\times}$ is just to denote the extra symplectic structure and we have
    $$\begin{tikzcd} & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

11. Claim: $\mathrm{Graph}(\varphi) \subseteq X_1 \tilde{\times} X_2$ is a Lagrangian submanifold.
    Proof: Consider
    $$\Phi : X_1 \to X_1 \tilde{\times} X_2 \text{ s.t. } x \mapsto (x, \varphi(x))$$
    then
    $$\Phi^*(\pi_1^*\omega_1 - \pi_2^*\omega_2) = \omega_1 - \varphi^*\omega_2 = 0$$
    and so by our corollary about an immersed submanifold being lagrangian iff $f^*(\omega) = 0$, we have the claim $\square$

12. The converse is also true. If $X_1 \xrightarrow{\varphi} X_2$ a diffeomorphism, then φ is a symplectomorphism iff $\mathrm{graph}(\varphi) \subseteq X_1 \tilde{\times} X_2$ is Lagrangian

13. There's a famous quote by Alan Weinstein "Symplectomorphisms are special cases of Lagrangian Manifolds" which references the above

Symplectic Linear Algebra (More of It)

(a) What we know: If F an $\mathbb{R}$-v.s. then $F \oplus F^*$ symplectic

(b) V^{2n}, Ω symplectic vector space ($n \geq 1$)

(c) We can always find a lagrangian subspace via the following method

   i. Start with a 1-dimensional W, then $W \subseteq W^\Omega$ because
   $$W = \mathrm{span}(v_0) \implies \forall k_1, k_2 \in \mathbb{R}\Omega(k_1 v, k_2 v) = k_1 k_2 \Omega(v, v) = -k_1 k_2 \Omega(v, v) = 0$$
   by anti-symmetry. If $W = W^\Omega$ then we're done, else if $W \subsetneq W^\Omega$, then choose $w_0 \in W^\Omega \backslash W$ and repeat and form $W' = \mathrm{span}\{v_0, w_0\}$

   ii. Note that W' will again satisfy the condition $W' \subseteq (W')^\Omega$ because
   $$\Omega(v_0, w_0) = \Omega(v_0, v_0) = 0 \implies \Omega(av_0 + bw_0, cv_0 + dw_0) = 0$$
   by definition of W and such

    iii. We can repeat this process until our updated W' is n-dimensional and $W' \subseteq (W')^{\Omega}$, i.e. W' is isotropic. Then by our lemma that half dimensional plus isotropic is equivalent to Lagrangian, we're done

  (d) Let $W \subseteq V$ be a Lagrangian subspace, then $\exists W' \subseteq V$ Lagrangian with $W \oplus W' = V$
  Proof: Similar to as the previous claim:

  (e) Claim:

$$\exists \text{basis } \{e_1, \dots, e_n, f_1, \dots, f_n\} \subseteq V \;\text{ s.t. }\; \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j), \quad \Omega(e_i, f_j) = \delta_{ij}$$

  (f) Note: if $V = F \oplus F^*$, then we already have such a basis, namely

$$e_i = v_i \qquad f_i = v_i^*$$

  where the $\{v_i\}$ span V and v_i^* are the corresponding dual basis vectors.

  (g) To prove our claim, it suffices to show that there exists such a W' such that $W \oplus W' = V$

  (h) Claim: $\varphi : V toW \oplus W^*$ s.t. $w \oplus w' \mapsto w \oplus \Omega(\cdot, w')$ is a symplectomorphism, so just need to show that the map $f : W' \to W^*$ given by $w' \mapsto \Omega(\cdot, w')$ is an isomorphism + a symplectomorphism (i.e. the pull back maps the symplectic two forms to each other).
  Clearly, $w' \mapsto \Omega(\cdot, w')$ is an isomorphism of vector space. We now want to show

$$\Omega(w_1 \oplus w_1', w_2 \oplus w_2) \overset{?}{=} \Omega(w_1, w_2') - \Omega(w_2, w_1')$$

  but this is true by definition of Ω. On second thought, I think the above should be

$$\Omega(w_1 \oplus w_1', w_2 \oplus w_2) \overset{?}{=} w_2'(w_1) - w_1'(w_2)$$

  (i) The upshot is $V = W \oplus W'$ corresponds to a symplectomorphism $V \to W \oplus W^*$. In fact, fix $W \subseteq V$, then

$$\left\{\text{Symplectic Isomoprhisms } V \to W \oplus W^* \text{which restrict to } W \xrightarrow{Id} W\right\} \overset{1-1}{\leftrightarrow}$$

$$\left\{\text{Lagrangian complements to W inside of V}\right\}$$

  (j) We can understand the above correspondence as follows:
    i. Choose W' s.t. $V = W \oplus W' \cong W \oplus W^*$, then

$$\begin{aligned}\left\{\text{Lagrangian complements to W inside of V}\right\} &= \left\{\text{Lagrangian complements of W inside of } W \oplus W^*\right\} \\ &= \left\{\text{graphical Lagrangians over } W^* \subseteq W \oplus W^*\right\} \\ &= \left\{\text{Maps } W^* \oplus W^* \xrightarrow{q} \mathbb{R} \;\text{ s.t. }\; q\text{symmetric}\right\}\end{aligned}$$

    ii. Note that $W' \neq W^{\perp}$, because we can things like $\mathbb{R}^2 = \mathbb{R}(1,0) \oplus \mathbb{R}(1,1)$
    iii. Note: This collection of symmetric maps forms a convex set, i.e. q, q' are symmetric maps $W^* \oplus W^* \to \mathbb{R}$ then so is $tq + (1-t)q'$, so the space of all symmetric maps of this form is **contractible**

10/04

1. Last Time

  (a) $W \subseteq V$ a Lagrangian subspace of a symplectic vector space, then there exists a $1-1$ correspond (in fact a diffeomorphism)

$$\left\{\begin{matrix}\text{Symplectic isomorphisms } V \to \\ W \oplus W^* \text{ extending } W \xrightarrow{Id} W\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Lagrangian complement } W' \text{ to } W, \\ \text{i.e. } W \oplus W' = V\end{matrix}\right\}$$

$$\overset{\text{for fixed complement W'}}{\leftrightarrow} \left\{\begin{matrix}\text{graphical Lagrangian subspaces} \\ \text{over } W^* \text{ inside } W \oplus W^*\end{matrix}\right\}$$

  Note that the last collection of lagrangian subspaces are each contractible

2. **Corollary:** $P \to X$ a symplectic vector bundle. If P admits a Lagrangian subbundle $\mathcal{L} \subseteq P$ then $P = \mathcal{L} \oplus \mathcal{L}^*$, Ω is a symplectic pairing.
   Proof: Finding an iso $\mathcal{L} \oplus \mathcal{L}^* = P$ is equivalent to finding a complementary Lagrangian subbundle $\mathcal{L}' \oplus \mathcal{L} = P$. At every point $x \in P$, we have **a contractible space of choices of Lagrangian complements** in P_x. More precisely, we have a bundle

   $$\begin{tikzcd} \bigcup_{x \in X} \{\mathcal{L}'_x \subseteq P_x \mid \mathcal{L}'_x \text{ is lagrangian complement to } \mathcal{L}_x\} \arrow[d] \\ X \end{tikzcd}$$

   We want to find a continuous section of the above. To do this, choose a CW complex on X by building it up, i.e. $X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(n)}$. Such a construction works because our fibres are contractible.
   We can now approximate our continuous section by a smooth section. ☐

3. Really not sure about the above proof

4. One more corollary of the classification of non-degenerate skew-symmetric bilinear pairings on a symplectic vector space is that we can classify all skew-symmetric bilinear pairings:
   (a) Let V be a vector space, Ω a skew-symmetric bilinear map
   (b) Let $W := \{w \in V \mid \tilde{\Omega}(w) = 0\}$
   (c) Choose an arbitrary complement to W inside V, with $\tilde{V}$ such that $V = W \oplus \tilde{V}$. Then $\Omega\Big|_{\tilde{V}}$ is non-degenerate
   (d) In particular, there exists a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, w_1, \ldots, w_k\}$ of V s.t.

   $$\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j),\ \Omega(e_i, f_j) = \delta_{ij} \quad \Omega(w_i, x) = 0 \quad \forall x$$

5. **Corollary:** We can think of Ω, a skew-sym bilinear pairing, as an element $\Omega' \in \Omega^2(V)$. Ω is non-degenerate, which is equivalent to
   $$\Omega' \wedge \cdots \wedge \Omega' \neq 0$$

6. **Corollary:** If ω a symplectic form on M, then in fact $\forall x \in M$, $\omega_x^n \neq 0$

Almost Complex Structures

(a) V an $\mathbb{R}$-vector space
(b) $J : V \to V$ a complex structure if $J^2 = -Id$
(c) Let V, Ω be symplectic. A complex structure on V, $J : V \to V$ is called
    i. Almost comptabile if J is symplectic, i.e. $\Omega(v, w) = \Omega(Jv, Jw)$
    ii. Tame if $\Omega(v, Jv) > 0$ for all $0 \neq v \in V$
    iii. Compatible if Almost compatible + Tame
(d) Claim: $V, \Omega,\ J$ is almost compatible $\iff$ $\Omega(\cdot, J\cdot)$ is a symplectic bilinear form.
    Proof: We jave
    $$\Omega(v, Jw) = \Omega(Jv, J^2w) = -\Omega(Jv, w) = \Omega(w, Jv)$$
    ending it ☐
(e) **Corollary:** (V, Ω) is compatible $\iff$ $\Omega(\cdot, J\cdot)$ is a symmetric positive definite bilinear form, i.e. an inner product
(f) **Definition:** If (M, ω) symplectic, a compatible, almost complex structure on TM, $J : TM \to TM$ and $J^2 = -Id$ such that at every $x \in M$, T_xM, ω_x, J_x is compatible
(g) Similarly, given $\langle \cdot, \cdot \rangle$ positive definite + symmetric and J an isometry and complex structure, then there exists a symplectic pairing $(\Omega, \langle J\cdot, \cdot \rangle)$. This is because
    $$\Omega(v, w) = \langle Jv, w \rangle = \langle -v, Jw \rangle = -\langle Jw, v \rangle = -\Omega(w, v)$$

(h) Claim 2: for (V, Ω) symplectic pairing, there exists a contractible (and hence non-empty) space of compatible complex structures.
Proof: Non-empty because choose basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, i.e. standard basis. Define

$$J(e_i) = f_i, \quad J(f_i) = -e_i$$

and then $\langle \cdot, \cdot \rangle := \Omega(\cdot, J\cdot)$ is the standard inner product w.r.t this basis. □

(i) This proof required a choice of basis, there is a more efficient construction that works on manifolds though

Polar Decomposition

(a) Let F be an $\mathbb{R}$-vector space with an inner product $\langle \cdot, \cdot \rangle$ and $T : V \to V$ a linear isomorphism. Then there exists a canonical $P : V \to V$ and $\mathcal{O} : V \to V$ such that:
P is symmetric, i.e. $\langle Pv, w \rangle = \langle v, Pw \rangle$ and positive definite, i.e. $\langle v, Pv \rangle > 0$ for all $v \neq 0$.
$\mathcal{O}$ is an isometry, i.e. $\langle \mathcal{O}v, \mathcal{O}w \rangle = \langle v, w \rangle$ and $T = \mathcal{O}P = P\mathcal{O}$

(b) **Proof:** TT^* is positive definite/self adjoint (because we're over reals), which implies that

$$P = (TT^*)^{1/2}$$

where the square root of this matrix exists because of the spectral theorem. Now define $\mathcal{O} = P^{-1}T$, and note that

$$\langle \mathcal{O}v, \mathcal{O}w \rangle = \langle \mathcal{O}^t\mathcal{O}v, w \rangle$$

but also

$$\mathcal{O}^t\mathcal{O} = T^t(P^2)^{-1}T = T^t(T^t)^{-1}T^{-1}T = Id$$

10/07

1. We begin with a remark about problem 4 from the last pset: A good example of non-commuting vector fields (as per Dylan's suggestions):

$$M = \mathbb{R}^2, \quad S = \{x_0\} \quad V_1 = \begin{cases} \frac{d}{dx} & |x| \leq r \\ \frac{\partial}{\partial r} & R \leq |x| \leq K \\ 0 & |x| > K \end{cases}; \qquad V_2 = \frac{\partial}{\partial \theta} \quad \forall R \leq |x| \leq K$$

For $r < R < K$. The idea is that V_1 looks like a constant vector field on the inner disk, and then a partition of unity argument is used to connect the constant vector field to the radial vector field, which is full strength on the annulus, $A = \{r \leq |x| \leq R\}$. Everything then dies outside of some radius $K = R + \epsilon$.

V_2 by contrast is just $\partial/\partial\theta$ times some smooth bump function so that the vector field is full strength on the annulus.

The idea is that the radial vector flow and the constant flow $\partial/\partial x$ agree on the positive x-axis. Thus if $x_0 = 0$, we can first flow with V_1 and then V_2, and so we're good once we get to A, the annulus. On the other hand if we first do V_2 and then V_1, we can only move x_0 along the x-axis.

Thus when the vector fields don't commute,

$$\Phi_k^{t_k}(\Phi_{k-1}^{t_{k-1}}(\cdots \Phi_1^{t_1}(x_0)))$$

is a weird set.

In the non-commuting case, the sufficiency part of the problem still holds. One needs to use that transversality is an open condition restricted to small enough $t \in \mathbb{R}$, then S_t should also satisfy the condition

2. Last time: we were trying to understand the space of compatible complex structures inside a symplectic vector space, (V, Ω)

   (a) We had seen that this space is non-empty! Choose a basis and get standard symplectic vector space

(b) **Claim 1**: There is a a compatible smooth map

$$S_1 := \left\{\text{Inner products on } V \quad\right\} \overset{\theta}{\to} \left\{\begin{array}{l}\text{Compatible complex structures on}\\ (V,\Omega)\end{array}\right\} =: S_2$$

which admits a section.
Proof: Given $\langle , \rangle$ Don't we start with $\langle , \rangle$? How do we get an Ω. It seems like we're assuming we start with an Ω we can turn Ω into a skew adjoint operator $K : V \to V$

$$\langle Kv, w\rangle := \Omega(v, w)$$

(skew adjoint because $\langle Kv, w\rangle = \Omega(v, w) = -\Omega(w, v) = -\langle Kw, v\rangle = -\langle v, Kw\rangle$). Write down the polar decomposition of K w.r.t $\langle , \rangle$
Now $K =: P(\langle , \rangle)\Theta(\langle , \rangle) = \Theta(\langle , \rangle)P(\langle , \rangle)$. We will define the map Θ such that

$$\Theta : \langle , \rangle \mapsto \Theta(\langle , \rangle)$$

we want

$$\Theta(\langle , \rangle)^2 = -Id \iff \Theta(\langle , \rangle) = -\Theta(\langle , \rangle)^t$$

because $\Theta\Theta^t = Id$ by definition of polar decomposition. From the above, the left hand side is equal to $P^{-1}K = \Theta$ while the other side is equal to $-K^t(P^t)^{-1}$, but

$$K = -K^t \qquad P = P^t$$

which readily gives the equivalence. Thus we conclude that $\Theta(\langle , \rangle) = -\Theta(\langle , \rangle)^t$ and hence $\Theta^2 = -Id$.

We also have to check that $\Omega(\Theta v, \Theta w) \overset{?}{=} \Omega(v, w)$. By definition the left side is equal to

$$\langle K\Theta v, K\Theta w\rangle = \langle \Theta^{-1}K\Theta v, w\rangle = \langle P\Theta v, w\rangle = \langle Kv, w\rangle$$

This tells us that Θ is almost compatible w.r.t. Ω, and we already know that Θ is an isometry by its definition from polar decomposition.

Now we need to show that it is tame, i.e. $\Omega(v, \Theta v) \overset{?}{>} 0$. TO see this

$$\Omega(v, \Theta v) = \langle Kv, \Theta v\rangle = \langle P\Theta v, \Theta v\rangle = \langle P(\Theta v), \Theta v\rangle > 0$$

where the last step holds because P is positive definition in the polar decomposition of K. Therefor Θ is a compatible almost complex structure for Ω. ☐

(c) **Claim 2** This map, Θ, has a **section**, i.e. there exists a map $s : S_2 \to S_1$ such that $\Theta \circ s = Id$. Note that Θ is surjective, but not injective. What is this section?
Let J be a compatible almost complex structure. We define

$$\langle \cdot, \cdot\rangle := \Omega(\cdot, J\cdot)$$

want to check that $\Theta(\langle , \rangle) = J$, so that $s : J \mapsto \Theta$ would be our section.
We check the above

$$\Omega(v, w) = \langle Kv, w\rangle = \Omega(Kv, Jw) = \Omega(JKv, -w) \implies JK = -Id \implies K = J$$

because $J^2 = -Id$. But we also know that J is an isometry w.r.t. to the inner product $\langle , \rangle$. Therefore $\Theta(\langle , \rangle) = J$ because this polar decomposition is unique. Now we have our correspondence s as desired and

$$\{\langle , \rangle\} \xleftrightarrow[s]{\Theta} \{J\}$$

but note that the left hand side is convex (i.e. the space of all inner products is convex), and hence contractible. Further note that Θ is a surjection, whereas s is simply a section, and not injective or anything. ☐

(d) **Corollary:** The space

$$\begin{Bmatrix} \text{comptabile complex} \\ \text{structures on } (V, \Omega) \end{Bmatrix}$$

is a contractible space.

Proof: The $\{\langle , \rangle\}$ space is convex. Fix a $g \in \{\langle , \rangle\}$, we can define a contraction of $\{J\}$ to $\Theta(g)$ by

$$\Theta \circ ((1-t)s(J) + t \cdot g)$$

in particular, we can map $I \mapsto g \in \{\langle , \rangle\}$. ☐

(e) A choice of J makes (V, Ω) more rigid. For $W \subseteq V$ a lagrangian subspace, JW is also a lagrangian. $JW \oplus W = V$. If you define

$$\langle , \rangle := \Omega(\cdot, J\cdot)$$

then $JW \perp W$. Just need to check that $\langle v, Jw\rangle = 0$ for all $v, w \in W$ but, this is equal to $-\Omega(v, w)$ which equals 0 because W Lagrangian.

(f) **Corollary:** (2) Every symplectic vector bundle admits a compatible complex structure. Moreover, any two compatible complex structures are homotopic to each other (which is a result of this space being a contractible space). In particular, this implies that they are isomorphic as complex vector bundles.

(g) One way to think about two bundles being homotopic is construct base space $M \times [0, 1]$ and then connect E_1 which occurs at $M \times \{0\}$ and E_1 which occurs at $M \times \{1\}$. We need a connection to do this

(h) With our work today, we can show that there exists manifolds of half dimension which are not lagrangian

10/09

1. Last time: Every symplectic vector bundle admits a compatible complex structure. Moreover the resulting isomorphism class of the resulting complex vector bundle is independent of choices

2. Ex: We know that $E \oplus E^* \to X$ (where E is a real vector bundle) is a symplectic vector bundle - so what is the associated complex vector bundle up to isomorphism?

3. **Answer**It is the complexification of E, i.e. $E \otimes_{\mathbb{R}} \mathbb{C} = E_{\mathbb{C}} \to X$ id a complex vector bundle. In particular, we have an isomorphism between $E \oplus E^* \to X$ and $E_{\mathbb{C}} \to X$ which sends the complex structure of the former to the latter

4. Note: a real vector bundle with a complex structure is not exactly a complex vector bundle if we compare the definitions

5. How do we prove that the complexification of E is the associated complex vector bundle? Fix $\langle , \rangle$ on E. Choose $J : E \oplus E^* \to E \oplus E^*$ by

   $$J : e \to \langle \cdot, e\rangle$$

   This defines J on the E component, to define it on E^*, we use the fact that $J^2 = -Id$. Another way to say this is that J is a map for the bundle $E \oplus E \to X$ where $Je = \tilde{e}$, such that $\tilde{e}$ is the copy of e in the second factor.

   Now define a map between real vector bundles $E_{\mathbb{C}} \cong E \oplus E \mapsto E \oplus E^*$,

   $$(e, \tilde{e}) \mapsto (e, \langle \tilde{e}, \cdot\rangle)$$

6. Now consider $P \to X$ a symplectic vector bundle, this induces $P_J \to X$, which is a complex vector bundle up to isomorphism. Here the J denotes the complex vector bundle, which yields an actual J as before (up to isomorphism). This in turn induces a complex line bundle $\det_{\mathbb{C}}(P_J) := \Omega^k_{\mathbb{C}}(P_{\mathbb{C}}) \to X$, which is also well defined up to isomorphism.

   If E a real vector bundle, then $\det_{\mathbb{C}}(E_{\mathbb{C}}) \cong \det(E)_{\mathbb{C}}$

7. Consequence: $\det_{\mathbb{C}}(E_{\mathbb{C}})^{\otimes_{\mathbb{C}} 2}$ is always trivial because $\det(F)^{\otimes 2}$ is always trivial, being a trivial bundle tensored with itself Don't understand. Also links online say this is not true unless the first Chern class vanishes

8. Upshot: If $P \to X$ symplectic vector bundle admits a Lagrangian subbundle, then $\det_{\mathbb{C}}(P_{\mathbb{C}})^{\otimes_{\mathbb{C}} 2}$ is trivial.

9. Now consider the symplectic vector bundle, $TS^2 \oplus \mathbb{R}^2_{st} \to S^2$ (standard symplectic structure on $\mathbb{R}^2$).
   (a) This clearly has a rank 2 Lagrangian submanifold
   (b) Also $\det_{\mathbb{C}}(T_{\mathbb{C}}\mathbb{P}^1 \oplus \mathbb{C}) = \det_{\mathbb{C}}(T_{\mathbb{C}}\mathbb{P}^1)$, by some complex analysis, we have
   $$\det_{\mathbb{C}}(T_{\mathbb{C}}\mathbb{P}^1 \oplus \mathbb{C}) = \det_{\mathbb{C}}(T_{\mathbb{C}}\mathbb{P}^1) = T_{\mathbb{C}}\mathbb{P}^1 = \mathcal{O}(2) \text{ on } \mathbb{P}^1$$
   and the square of this last thing is $\mathcal{O}(4)$ which is non-zero.
   (c) As an aside $\mathcal{O}(-1)$ is the tautological line bundle over $\mathbb{P}^1$. $\mathcal{O}(1) = \mathcal{O}(-1)^*$, and then $\mathcal{O}(2) = \mathcal{O}(1)^{\otimes_{\mathbb{C}} 2}$, i.e. the tensor of the tautological line bundle with itself.

10. Almost Complex vs. Complex Manifolds
   (a) Almost Complex Manifold: A manifold, and an endomorphism $J : TM \to TM$ such that $J^2 = -Id$
   (b) Complex Manifold: X is covered by charts which are open subsets of $\mathbb{C}^n$ with biholomorphic transition maps
   (c) It's easy to see: if X is a complex manifold, then the underlying real manifold, $X_{\mathbb{R}}$, has a canonical, almost complex structure
   (d) On charts: multiplication by i is a map, i.e. $\cdot i : \mathbb{C}^n \to \mathbb{C}^n$, and this induces a map $\cdot i : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. The differential of this map is in fact the "J" we've been talking about $J : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$. These glue together by the biholomorphism of the transition map.
   (e) Biholomorphism can be thought of as a PDE on the transition functions. When we say that J preserves the biholomorphic structure, that also yields a differential equation on J
   (f) In the Almost complex manifold case, no such differential equation exists because we don't require that our maps are holomorphic. We have trivializations, and we just want that J commutes with the transition maps, which is just a linear equation pointwise.

11. **Theorem:** An almost complex structure on M is of the form $X_{\mathbb{R}}$ for some complex manifold X if and only if a certain Nijenhuis tensor, which is in $\Gamma(T^*M \otimes T^*M \otimes T^*M)$, vanishes.

12. Think of the above tensor as a pointwise quaternion.

13. Such complex structures are called **integrable**.

14. In symplectic geometry, we don't care (95 percent of the time), whether an almost complex structure is integrable or not.

15. Symplectic structres in complex geometry
   (a) **Lemma:** Let (V, Ω, J) a symplectic vector space, with J tame. Let $W \subseteq V$ be a subspace that is closed under J. Then W is actually symplectic as a subspace, in particular it's a symplectic vector space itself via Ω.
   Proof: $\Omega(v, Jv) > 0$ for all $v \neq 0$. We need to show that $W \cap W^{\Omega} = \{0\}$. Take $v \in W \cap W^{\Omega}$, since $Jv \in W$, we have
   $$\Omega(v, Jv) = 0 \implies v = 0$$
   by J being tame. □
   (b) **Corollary:** An almost complex submanifold Y of (M, ω, J) which is tame, is symplectic itself w.r.t $\omega\big|_Y$.
   Not totally sure how this follows, is Y preserved by $\omega\big|_Y$ necessarily by nature of being almost complex?
   (c) In particular, a complex submanifold of a complex manifold whose associated complex structure is comptabile with a symplectic form ω, is symplectic
   (d) Ex: $\mathbb{C}^n$ has a symplectic structure.
   $$\omega = \sum_{i=1}^{n} d\Re(z_i) \wedge d\Im(z_i) \qquad \text{s.t. } \{z_i\} \text{ are } \mathbb{C} \text{ coordinates}$$
   If you write $z_i = p_i + \sqrt{-1}q_i$ then
   $$\omega \to \sum_i dp_i \wedge dq_i$$
   the standard almost complex structure on $\mathbb{C}^n$ is compatbile. Then any solution set of polynomial equations in $\mathbb{C}^n$ that is smooth gives you a symplectic manifold Why is this?

10/11

1. Last time: Complex submanifold of $\mathbb{C}^n$ have symplectic structures induced from being a symplectic submanifold of $(\mathbb{R}^{2n}, \omega_{st})$. Among these, the well behaved (important) ones are the properly embedded ones
2. Recall that a proper map satisfies that the preimage of a compact set is compact
3. Ex: Of a non-proper map: say we have $f : \mathbb{R} \to \mathbb{R}^2$ such that $\text{Im}(f)$ is a bounded simple curve with open endings. Then this is not proper because take a compact neighborhood of the open endings, then the preimage will be of the form (a, ∞), which is not compact
4. Ex: A proper map would be something where we send $\mathbb{R}$ to an unbounded simple curve in $\mathbb{R}^2$
5. **Definition:** A complex manifold is called stein if there exists a proper holomorphic embedding of C into $\mathbb{C}^N$, for some N
6. Ex: Subsets of $\mathbb{C}^n$ that are cut out smoothly by polynomial equations
7. Note that there may be many such embeddings, each of which can equip X with a different symplectic form
8. Fortunately, they are not really different
9. **Proposition:** (Gromov-Eliashberg) Any two symplectic structures obtained on X this way are symplecticomorphic. I.e. if we have two proper, holomorphic embeddings of X into $\mathbb{C}^N$, then both embeddings are holomorphic
10. Note that affine varieties are Stein, but there non-algebraic Stein manifolds as well
11. Aside: We don't have a good way to generally construct a symplectic manifold (compare this to the construction of 3-manifolds via knots). Umut seems upset about this, implying that it's a big problem in symplectic geometry
12. We will now show that complex submanifolds of $\mathbb{CP}^n$ are also symplectic (e.g. projective complex varieties) but these symplectic structures actually depend on the embedding
    (a) Let's start with another important definition
    (b) **Definition:** Kahler manifolds: the following are equivalent definitions
        i. (M, ω, J) symplectic manifold with complex structure such that J is **integrable**
        ii. (M, g, J) where g is a Riemannian metric and J is an almost complex structure, such that J preserves the metric at each tangent space, and J is *covariantly constant* (i.e. when you differentiate J w.r.t. to the connection), i.e.
        $$\nabla J = 0$$
        where ∇ is the Levi-Civita connection from the metric
    (c) Remark: from the first definition, we can get a Riemannian metric by composing the form with the complex structure. From the second definition, J is not assumed to be integrable as it follows from the other requirements
    After that, we get ω from J and g and we have that $\nabla J = 0 \iff d\omega = 0$. In particular
    $$\omega := g(J\cdot, \cdot) \implies \nabla\omega = \nabla g(J\cdot, \cdot)$$
    (d) Pointing out the obvious: Kahler manifolds have symplectic structures no matter which definition is used (In the second definition, is $\omega = g(\cdot, J\cdot)$?)
    (e) Ex: $\mathbb{C}^n$ is a Kahler manifold
    (f) Note: complex submanifolds of Kahler manifolds are Kahler, so a complex submanifold of $\mathbb{C}^n$ is not just symplectic, but also Kahler
    (g) **Fubini-Study form (or metric)** on $\mathbb{CP}^n$: Consider the standard Kahler structure on $\mathbb{C}^{n+1}$, i.e. ω_{st}, J_{st}, g_{st}.
    (h) We have a canonical map
    $$S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$$
    sending each point $x \in S^{2n+1}$ to the complex line it lies on.

(i) There is an $U(1)$-action (where $U(1) = S^1$ but thought of as a Lie group) on S^{2n+1} given by multiplication by $e^{i\theta}$. The map, π, is the quotient map given by this action

(j) Note that S^{2n+1} has an induced Riemannian metric from g on $\mathbb{C}^{n+1}$

(k) Define the distribution $D^{2n} \subseteq TS^{2n+1}$ defined by taking perpendiculars to the fibers of π. Some picture with the Hopf map here

Here's what's happening: Given $p \in S^{2n+1}$ we have the fiber at p which is equal to $\{pe^{i\theta} \mid \theta \in [0, 2\pi)\}$. This fiber defines a vector at p in TS^{2n+1}, and hence we can take the orthogonal complement to this vector within TS^{2n+1}, giving us a $2n$-dimensional subspace of the tangent space. To imagine this, think of great circles as being the fibers over a point in S^2 (wrong dimension, I know, but so it goes) and then we get a subspace by taking the complement of the vector which defines the great circle through p at that point.

(l) By construction of the smooth structure of $\mathbb{CP}^n$, for every $x \in S^{2n+1}$, we have

$$d\pi_x\Big|_{D_x} : D_x \to T_{[x]}\mathbb{CP}^n$$

The above is because $d\pi$ is surjective with the fiber of π as the kernel Moreover, $d\pi_x$ is invariant under the $U(1)$ action (precisely because of the fiber being the kernel remark) and

$$\begin{tikzcd} D_x \arrow[r, "U(1)action"] \arrow[dr] & D_{e^{i\theta}x} \arrow[d] \\ & T_{[x]}\mathbb{CP}^n \end{tikzcd}$$

(m) Not sure I understand the construction of D^{2n}, **ask UMUT in office hours**. What is the "fiber directions"? resolved

(n) Hence, in order to define structures on $\mathbb{CP}^n$, we need $U(1)$ invariant structures on $D = D^{2n}$

  i. Each $D_x \subseteq T_x\mathbb{C}^{n+1}$ is a complex subspace (because it is the perpendicular to a complex subspace). In particular, multiplication by $e^{i\theta}$ is a biholomorphism of $\mathbb{C}^{n+1}$, therefore, we have that the below commutes

$$\begin{tikzcd} J\Big|_{D_x} \arrow[d, "\circlearrowright"] & {: D_x} \arrow[r] & D_x \arrow[d] \\ J\Big|_{D_{e^{i\theta}x}} & {: D_{e^{i\theta}x}} \arrow[r] & D_{e^{i\theta}x} \end{tikzcd}$$

  ii. For the Riemannian metric and symplectic form on $\mathbb{CP}^n$, we follow the same strategy. Noting that $\cdot e^{i\theta}$ is an isometry and also a symplectomorphism

  iii. We need to show that

$$d\omega_{FS} = 0 \text{ or } \nabla J_{\mathbb{CP}^n} = 0$$

  where ω_{FS} denotes the Fubini-Study form. Did we show this? I think not

  iv. The point is that we've equipped each D_x with a metric, which is invariant under the $e^{i\theta}\cdot$ action and patches together well

(o) Remark: Next time we will see a conceptual explanation for why $d\omega = 0$

(p) Note the $U(n+1)$ action on $\mathbb{CP}^n$. It is a transitive action and it is an action by biholomorphic isometries. In fact g_{FS} is determined uniquely by this property (invariant under $U(n+1)$ action) up to scaling

(q) Why is $\nabla J = 0$?

∇J is a $(1, 2)$ tensor which is also invariant under the $U(n+1)$ action. For every point $p \in \mathbb{CP}^n$, there exists an element $A \in U(n+1)$ which does the following: A fixes p and $dA\Big|_p = -Id$ (automorphism of the tangent space at that point p). Note that these kind of things are called "symmetric spaces"
Trick/Proof: for a vector space V, no non-zero element of $V^{\otimes i} \otimes (V^*)^{\otimes j}$ with $i + j$ odd number can be preserved by multiplication by -1. Therefore $\nabla J = 0$

10/14 (Need to start reviewing from here)

1. Last time: Kahler manifolds (e.g. complex submanifolds of $\mathbb{C}^n$ or $\mathbb{CP}^n$, where these spaces are equipped with the fubini-study form)

2. Warning: There are closed Kahler manifolds which are not projective, i.e. they cannot be embedding in $\mathbb{CP}^n$ holomorphically. In fact, most closed Kahler manifolds are of this form (i.e. there are many of them)

3. Ex: $K3$ surfaces (e.g. 4 real dimension, 2 complex dimensions):
   (a) are closed complex surfaces
   (b) admit a Kahler structure
   (c) The first Chern class of the tangent bundle also vanishes (i.e. $c_1(T_{\mathbb{C}}M)) = 0$
   (d) and M is not equal to a 4-torus (so we assume that M is simply connected), nor an abelian variety

   Any two M satisfying the above conditions are diffeomorphic. Hence smoothly, there is one $K3$ surface. There is a 20 (complex) dimensional family of complex structures on M, and each of these support a Kahler structure. But only a 19 dimensional family admits a holomorphic embedding into $\mathbb{CP}^n$. Somehow 19 is the cut-off for a nice property for $K3$ surfaces. Note that 19 and 20 are the dimensions of the moduli space of complex structures on M, where M is a fixed $K3$ surface. Note: A moduli space means a space of objects where each object is an equivalence class of things, e.g. a moduli space of complex structures would be a space of equivalence classes of complex structures.

4. Ex: Consider $\{x^4 + y^4 + z^4 + t^4 = 0\} \subseteq \mathbb{CP}^3$

5. What is special about Stein Manifolds (i.e. properly embedded holomorphic submanifolds of $\mathbb{C}^n$ for some n) among open symplectic manifolds?
   (a) Answer 1: They have infinite volume!

   But of course there are many open symplectic manifolds with finite volume (and by volume we know that for ω a symplectic 2 form on a $2n$-dimensional manifold, then $dV = \omega \wedge \cdots \wedge \omega$)
   (b) Answer 2: If X^{2n} is Stein, then $H_k(X, \mathbb{Z}) = 0$ for $k > n$. For intuition, think of affine varieties, because they have half of the homology that they can have (i.e. half of their dimension). This is called the "Weak Lefschetz Theorem"

   As an example of this theorem, consider submanifolds of $\mathbb{C}^2$. Then affine varities will correspond to a Riemann surface, which deformation retracts onto a 1 (real) dimensional manifold, so it can have first homology and lower, but nothing above
   (c) **Remark:** There is a pure characterization of symplectic Stein manifolds (with no reference to complex structures). This says that all Stein manifolds can be obtained by "Weinstein handle attachments." This is a theorem of Eliashberg-Cielibach.

   Further note that anything obtained by Weinstein handle attachments are called Weinstein manifolds. Moreover the functor from Stein manifolds to Weinstein manifolds is denoted

$$\text{Wein} : \{\text{Stein}\} \to \{\text{Weinstein}\}$$

6. What is special about compact Kahler manifolds among closed (i.e. compact no boundary) symplectic manifolds?
   (a) Answer 1: There is a Hodge decomposition (which only depends on the complex structure of the manifold, i.e. no dependence on the symplectic form) This does not exist for arbitrary compact complex manifolds

$$H^k(M, \mathbb{C}) = \oplus_{i+j=k} H^{i,j}(M)$$

   where $H^{i,j}(\mathbb{C}) = \overline{H^{j,i}(\mathbb{C})}$ (i.e. complex conjugate under exchanging symmetry) as complex vector spaces Still not sure what it means to take the complex conjugate of a complex vector space Complexification of $H^{i,j}(\mathbb{C})$ as a (real?) vector space I think. For the dimension of these spaces, aka the betti numbers, we have the following relation

$$b_k = \sum_{i+j=k} b_{i,j}, \qquad b_{i,j} = b_{j,i}$$

These $\{b_{i,j}\}$ form the Hodge diamond.

$$\begin{array}{ccccc}
 & & \cdots & & \\
 & \cdots & & \cdots & \\
b_{2,0} & & b_{1,1} & & b_{0,2} \\
 & b_{1,0} & & b_{0,1} & \\
 & & b_{0,0} & &
\end{array}$$

For a $K3$ surface, we have the following Hodge diamond

$$\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}$$

(b) Note that if M is compact, Kahler, then b_k is even whenever k is odd

(c) Ex: Kodaira-Thurston manifold: This is a quotient of $\mathbb{R}^4$ with the standard structure, $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, and the quotient is by the following equivalence relation

$$(x_1, x_2, x_3, x_4) \sim (x_1 + 1, x_2, x_3 + x_4, x_4) \sim (x_1, x_2 + 1, x_3, x_4) \sim (x_1, x_2, x_3 + 1, x_4) \sim (x_1, x_2, x_3, x_4 + 1)$$

Note that without the first $x_3 + x_4$, we would just have the 4-torus. But this extra addition gives it an interesting structure

It's best to think about this as a torus bundle over a torus.

$$\begin{tikzcd}
T^2 \arrow[r, hook] & X \arrow[d, "{(x_1,x_2)}"] \\
 & T^2
\end{tikzcd}$$

Monodromy in the x_1 direction is a Dehn Twist. Monodromy in the x_2 direction is the identity.

Now take $T^2_{x_3,x_4} \times \mathbb{R}^2$ and then glue accordingly to make our manifold. This is not Kahler because $b_1 = 3$

(d) Answer 2: There is a Hard Lefschetz theorem for Kahler manifolds:

Theorem: If M is Kahler with symplectic class $[\omega]$, then for $0 \leq k \leq n-2$, we get a map

$$H^{n-k}(M, \mathbb{R}) \xrightarrow{\wedge[\omega]^k} H^{n+k}(M, \mathbb{R})$$

i.e. wedging with $[\omega]$ k-times gives an isomorphism between the homologies.

(e) Answer 3: Formality of rational cohomology ring.

Corollary: No Massey Products

   (f) Note: the Kodaira-Thurston manifold fails all of the above properties

7. Constructions of symplectic manifolds

   (a) Symplectic blow-up operation: Geometrically, this looks a lot more like blow up in complex manifolds then real ones. On real manifolds, blowing up a point is look removing a ball. On $\mathbb{C}^2$ though, blowing up a point amounts to removing a ball and replacing it with $\mathbb{CP}^1$

   How does this work? Given M a symplectic manifold and Z a symplectic submanifold. then

   $$(M, Z) \mapsto BL(M, Z) \text{ a closed symplectic manifold}$$

   basically "remove an open tubular neighborhood of Z and then do a partial collapsing of the boundary."

   Ex: For $M = \mathbb{R}^4$ and $Z = \{0\}$ then we go from $\mathbb{R}^4$ to S^3 by removing a ball and then we get to S^2.

   The blow up process does basically nothing if Z is codimension 2

   Not sure how the above example works. Ask Umut

   Check Macduff and Salamon

   (b) Symplectic sum along codimension 2 symplectic submanifolds

   $$V \hookrightarrow M_1, \qquad V \hookrightarrow M_2$$

   which are topologically gluable $e(N(V, M_1)) = -e(N(V, M_2))$ where e is the euler class, and then the two embeddings can be glued.

   (c) **Theorem:** (Gompf) Arbitrary finitely presented groups can arise as fundamental groups of closed symplectic 4-manifolds.

   (d) Proof uses a symplectic sum construction.

   (e) Note: This is not true at all for a Kahler manifold

10/16

1. Today we talk about Symplectic reduction

2. Linear algebra version: (V, Ω) a symplectic vector space, and $W \subseteq V$ is any subspace. Then $W/(W \cap W^\Omega)$ is naturally symplectic

3. Ex: If W is coisotropic, i.e. $W^\Omega \subseteq W$, then W/W^Ω is naturally symplectic.

4. This clearly has a symplectic bundle version: Given any symplectic vector bundle and any vector subbundle, then we apply this construction fiberwise to the subbundle. Note: we have to assume that $E_x \cap E_x^\Omega$ is constant rank, where E_x is the fiber over the point x in the subbundle E

5. If (M, ω) is a symplectic manifold and $X \subseteq M$ a submanifold, **and** (**Assumption** 1) we assume that $T_xX \cap T_xX^\Omega \subseteq T_xX$ has constant rank, then $TX \cap TX^\omega$ is a subbundle of TX.

6. Ex: For X coisotropic manifold, then $T_xX^\Omega \subseteq T_xX$ is a subbundle of codimension $X \subseteq M$ (follows by rank nullity and $\dim(W) + \dim(W^\Omega) = n$ proof)

7. Claim: the distribution $TX \cap TX^\Omega \subseteq TX$ is integrable under assumption 1

   Proof: Let ξ_1 and ξ_2 be vector fields on X which are tangent to $TX^\Omega \cap TX$.

   Let $\omega_x := \omega\Big|_X$. We need to show that for every η, a vector field on X, $\omega_x([\xi_1, \xi_2], \eta) = 0$, as this is equivalent to being integrable

   We use the Cartan formula for exterior differentiation and the fact that $d\omega = 0$:

   $$0 = d\omega(\xi_1, x_2, \eta) = \pm\xi_1\omega_x(\xi_2, \eta) \pm \xi_2\omega_x(\xi_1, \eta) \pm \eta\omega_x(\xi_1, \xi_2)$$
   $$\pm\omega_x([\xi_1, \xi_2], \eta) \pm \omega_x([\xi_1, \eta], \xi_2) \pm \omega_x([\xi_2, \eta], \xi_1)$$

note that most terms vanish because ξ_1, ξ_2 all lie in TX^Ω and hence $\omega_x(\xi_1, v) = 0$ for all v. This leaves our last term

$$0 = 0 + 0 + \omega_x([\xi_1, \xi_2], \eta) + 0 + 0$$

Thus $TX \cap TX^\Omega$ is tangent to a foliation of X. What we've done is shown that our distribution is integrable and hence by the frobenius integrability theorem, it is a folation ☐

8. Ex: If X is coisotropic, then we get what is called the characteristic foliation

9. Ex: If X is codimension 1 then $TX \cap TX^\Omega$ is rank 1, then it is automatically coisotropic and $TX \cap TX^\Omega$ is automatically integrable

10. Note: the leaves of F are isotropic

11. (**Assumption 2**): There is a smooth manifold ("the leaf space") such that the foliation F is given by the fibers of a surjective submersion, $\pi : X \to B$ with connected fibers. I.e. each leaf of our original manifold is a point in the "leaf space" manifold

12. Claim: B has a canonical symplectic structure, ω_{red} (omega reduced), such that $\pi^*\omega_{red} = \omega_X$.
Proof: By linear algebra, for each $x \in X$, $T_xX/(T_xX \cap T_xX^\Omega)$ has a symplectic structure. Let $\pi(x) = \pi(x') = b$, then

$$\begin{tikzcd}
T_xX/T_xX \cap T_xX^\Omega \arrow[rr, dashed, "\sim"] \arrow[dr, "\sim"] & & T_{x'}X/T_{x'}X \cap T_{x'}X^\Omega \arrow[dl, "\sim"] \\
& T_bB &
\end{tikzcd}$$

Need to show $(*)$: the identification $\overset{\sim}{\dashrightarrow}$ preserves symplectic structures

Lemma: If ξ is a vector field on X that is tangent on $TX \cap TX^\Omega$, then $\mathcal{L}_\xi \omega_x = 0$ where $\mathcal{L}_\xi$ is the lie derivative w.r.t. ξ.
Proof: We have by cartan magic formula

$$\mathcal{L}_\xi \omega_x = \iota_\xi d\omega_x + d(\iota_\xi \omega_x) = 0$$

where both of these terms vanish because $d\omega_x = 0$ and $\iota_\xi \omega_x$ is zero by nature of being tangent to $TX \cap TX^\Omega$. ☐

Now let's prove $(*)$ for x and x' lying in the same submersion chart, i.e. where we project the chart $U \subseteq \mathbb{R}^n$ to the first k coordinates:

Find a constant vector in U where the time 1 map takes $x \to x'$, then multiply by some smooth bump function to extend it to the entirety of the manifold, where it is supported in U. Then apply the lemma. This preserves the symplectic structure Does flowing by a vector field in general preserve the symplectic structure? This is exactly the content of the lemma and gives the identification

$$T_xX/T_xX \cap T_xX^\Omega \leftrightarrow T_{x'}X/T_{x'}X \cap T_{x'}X^\Omega$$

For the general case, choose a path $x \to x'$ and cover it with finitely many **canonical** submersion charts (finite by compactness). What is a **canonical** submersion chart?

Now we have a non-degenerate 2-form ω_{red} s.t. $\pi^*\omega_{red} = \omega_x$. This implies that $d\omega_{red} = 0$ because we can take a local section

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[d, "\pi"'] \\
U \arrow[u, bend right, "s"']
\end{tikzcd}$$

then

$$s^*\pi^*\omega_{red} = s^*\omega_x$$

but $(\pi \circ s)^* = s^* \circ \pi^* = Id$ so the left hand side is ω_{red}. Then we have

$$d\omega_{red} = d(s^*\omega_x) = s^*(d\omega_x) = s^*(0) = 0$$

☐

13. Ex: $S^{2n-1} \subseteq \mathbb{C}^n$. The characteristic foliation is given by the fibers of $S^{2n-1} \to \mathbb{CP}^{n-1}$ because the fibers are given by the S^1 action on S^{2n-1}. We want to show that the symplectic structure restricted to the tangent space of the sphere has ker $= S^1$ Did we show this

This shows that ω_{FS} (fubini-study) is closed. Because we apply the lemma in the direction of the S^1 fiber?

10/18

Symmetries of symplectic manifolds

1. Heuristically (MYTH?): If we want to find n functions (in some number of variables) which satisfy k partial differential equations, then we feel like $k \leq n$. This is a direct translation from linear algebra (PDE's in involutive form)

2. Consider the equation $d\alpha = \beta$, where β is the input and we're trying to solve for α. Here $\alpha \in \Omega^2(\mathbb{R}^4)$ and $\beta \in \Omega^3(\mathbb{R}^4)$. Here, we're trying to solve for 6 functions and we're given 3 equations. Note the following:

   If $d\beta \neq 0$, then no solution. If $\beta = 0$, then there is an ∞-dimensional family of solutions. For any $\xi \in \Omega^1(\mathbb{R}^4)$ is a solution

3. If (X^{2n}, ω) is a symplectic manifold, $\varphi : X \to X$ a symplectomorphism. If $\varphi^*(\omega) = \omega$, then this is $O(n)$ functions and $O(n^2)$ equations but it still has a solution

4. Umut says this is more proof that the Heuristic is wrong - but I'm skeptical because there are more equations encoded in a symplectic form, e.g. $\omega(u, v) = -\omega(v, u)$. However, the first example with $d\alpha = \beta$ does check out...

5. **Definition:** A vector field $V \in \Gamma(TM)$ is called symplectic if $\alpha_V \omega = 0$, i.e. the flow of V preserves ω, this is equivalent to

   $$0 = \mathcal{L}_V \omega = \iota_V d\omega + d(\iota_V \omega)$$

   This establishes a correspondence

   $$\left\{\text{symplectic vector fields}\right\} \leftrightarrow \left\{\text{closed 1-forms} \quad \right\}$$

6. **Definition:** A symplectic isotopy is a time dependent vector field that preserves ω in the same way, i.e. $\mathcal{L}_{V_t} \omega = 0$ for all t.

7. **Definition:** A symplectic vector field V is called **Hamiltonian** if $\iota_V \omega$ is exact.

8. **Definition:** Let $f : M \to \mathbb{R}$ a function ("Hamiltonian"). The **Hamiltonian vector field**, X_f, is defined by the formula

   $$\omega(X_f, \cdot) = df$$

   Two functions gives rise to the same Hamiltonian vector field if they differ by a locally constant function. The flow generated by a Hamiltonian vector field is called the **Hamiltonian flow** (might be time dependent)

9. **Proposition:** Energy Preservation: For $H : M \to \mathbb{R}$, the flow of X_H preserves the level sets of H (energy levels).
   Proof: Want to show $X_H \cdot H = 0$ but

   $$X_H \cdot H = dH(X_H) = \omega(X_H, X_H) = 0$$

10. Let c be a regular value of H, what is the characteristic foliation/line field of $H^{-1}(c)$? Intuitively, the line field is given by X_H along the curve What is the curve? Integral curve given an initial point and a Hamiltonian flow starting at that point? From here, it makes sense because the above calculation tells us that our curve is coisotropic, so we can talk about a characteristic foliation The curve is defined because X_H is defined at every point. So yes it's the integral curve, but also the tangent space is often more than a curve because the level set could be higher dimensional

    text

11. The line field generated by X_H (which is never 0 because c is a regular value) is precisely the characteristic line field. Why?

    Let V be a tangent vector to $H^{-1}(c)$ this implies that $0 = dH(v) = \omega(X_H, v)$, so X_H is ω orthogonal to v. As a result, the flow lines of X_H follow the leaves of the characteristic foliation.

    Need a picture. Is $H^{-1}(c)$ one dimensional? In this case, isn't ω-orthogonal to X_H equivalent to parallel to X_H? Nope, $H^{-1}(c)$ can be higher dimension as a level set

12. Hamilton's equations: $(\mathbb{R}^{2n}, \sum_i dp_i \wedge dq_i)$, and we also have $H : \mathbb{R}^{2n} \to \mathbb{R}$, then

$$\iota_{X_H}\omega = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i$$

Why does the above hold? I'm not sure what defining properties of the Hamiltonian vector field give us this where

$$X_H = \sum_i f_i \frac{\partial}{\partial p_i} + g_i \frac{\partial}{\partial q_i}$$

this implies that

$$\frac{\partial H}{q_i} = f_i, \quad g_i = -\frac{\partial H}{\partial p_i}$$

13. Any integral curve $(\vec{p}(t), \vec{q}(t))$ of X_H satisfies

$$\dot{q}_i(t) = -\frac{\partial H}{\partial p_i}(\vec{p}(t), \vec{q}(t)) \qquad \dot{p}_i(t) = \frac{\partial H}{\partial q_i}(\vec{p}(t), \vec{q}(t))$$

Why? Look this up myself

14. Note: this is Hamilton's equations from physics with the signs switched (many sign choices are made in symplectic geometry)

15. If

$$H = -\frac{p^2}{2m} + V(\vec{q}) \implies \dot{\vec{q}} = -\frac{p}{m}, \qquad \dot{p}_i(t) = \frac{\partial V}{\partial q_i}(\vec{q}(t))$$

these are the equations for (negated) velocity and force. The above two equations imply

$$m\ddot{q}_i(t) = -\frac{\partial V}{\partial q_i}(\vec{q}(t)), \qquad (ma(t) = F(q(t)))$$

16. **Theorem:** (Liouville): For $V \subseteq \mathbb{R}^{2n}$ a compact domain, and $H : \mathbb{R}^{2n} \to \mathbb{R}$, and let the system evolve according to $\varphi_t = \Phi_{X_H}$ (i.e. the flow associated to X_H). Then

$$\mathrm{vol}(\varphi_t(V)) = \mathrm{vol}(V)$$

at all times. This is because X_H preserves ω, which implies that X_H preserves ω^n, which is exactly the volume element.

17. **Definition:** (Poisson Brackets): For $f, g \in C^\infty(M)$, then $\{f, g\} \in C^\infty(M)$ is the Poisson Bracket

$$\{f, g\} := \omega(X_g, X_f)$$

18. Claim 1: Let ξ_1 and ξ_2 be symplectic vector fields then $\omega([\xi_1, \xi_2], \cdot) = d(\omega(\xi_1, \xi_2))$, i.e. $[\xi_1, \xi_2]$ is the Hamiltonian vector field of the function $\omega(\xi_1, \xi_2)$.

    Claim 2: $X_{\{f,g\}} = [X_f, X_g]$
    Proof: For Claim 1

$$\mathcal{L}_{\xi_1}(\omega(\xi_2, \cdot)) = (\mathcal{L}_{\xi_1}\omega)(\xi_2, \cdot) + \omega(\mathcal{L}_{\xi_1}\xi_2, \cdot) = \omega([\xi_1, \xi_2], \cdot)$$

Note that $\mathcal{L}_{\xi_1}\omega(\xi_2, \cdot) = 0$ because $\mathcal{L}_{\xi_1}\omega = 0$ by nature of being a sympelctic vector field

On the other hand, we can also expand as follows

$$\mathcal{L}_{\xi_1}(\omega(\xi_2, \cdot)) = \iota_{\xi_1}(d(\omega(\xi_2, \cdot))) + d(\omega(\xi_2, \xi_1))$$

the first term vanishes because we showed that when ξ_2 is symplectic, then $\omega(\xi_2, \cdot)$ is closed and so

$$d\omega(\xi_2, \cdot) = 0$$

and so combining the two expansions, we have

$$\omega([\xi_1, \xi_2], \cdot) = d(\omega(\xi_2, \xi_1))$$

For claim 2, we set $\xi_i = X_f$ and $\xi_2 = X_g$. Then in 1), we have

$$\omega([X_f, X_g], \cdot) = d(\{f, g\}) \implies X_{\{f,g\}} = [X_f, X_g]$$

10/21

1. Last time: symplectic vector fields, Hamiltonian vectors fields
2. **Lemma:** (M, ω) symplectic, there exists an exact sequence of Lie Algebras

$$\begin{tikzcd}
0 \arrow[r] & H^0(M,\mathbb{R}) \arrow[r, "\text{inclusion of locally constant functions}"] & C^\infty(M,\mathbb{R}) \arrow[r] \arrow[dr, "d"'] & \chi(M,\omega) \arrow[r] & H^1(M,\mathbb{R}) \arrow[r] & 0 \\
 & & & \Omega^1_{\text{closed}}(M,\mathbb{R}) \arrow[ur, "{[\,]}"'] & &
\end{tikzcd}$$

   where $H^0(M,\mathbb{R})$, $H^1(M,\mathbb{R})$ are deRham cohomology groups with trivial lie bracket, and $\chi(M,\omega)$ are symplectic vector fields with operator $[,]$, and C^∞ is equipped with the Poisson bracket
   Proof: Exactness as $\mathbb{R}$ vector spaces is fine, why is this exact as a sequence of lie algebras?

   At the first non-trivial arrow $H^0 \to C^\infty$, the map is fine I guess? Not sure how inclusion meshes with the Lie structure but sure. Also what does locally constant mean? Like on connected components

   At the second arrow, we have $X_{\{f,g\}} = [X_f, X_g]$ (a corollary of what we proved) kernel will definitely contain constant functions, but if the association is $f \mapsto X_f$, then maybe locally constant isn't enough?

   At the third arrow, we have that $\xi, \xi' \in \chi(M,\omega)$ then $[\xi, \xi']$ is Hamiltonian from last time, so exactness also follows. So here we want the map to be surjective, with kernel equal to the image of smooth functions (i.e. exact forms inside the set of closed forms). I'm not even sure what this map is considering a lie bracket requires two inputs. We proved all of these properties in previous lectures, but the point is that we want these maps to respect the lie structure, so $v, w \in L_1$ and $f : L_1 \to L_2$, then $f([v,w]) = [f(v), f(w)]$ in the appropriate spaces. ☐

3. Isotopies: Let $\Phi : [0,\epsilon] \times M \to M$ be a smooth map, define $\Phi_t = \Phi_{\{t\}\times M}$. This map is a diffeotopy if $\Phi_0 = Id$ and Φ_t is a diffeomorphism $\forall t \in [0,\epsilon]$.

$$\Big\{\Phi : [0,\epsilon] \times M \to M\Big\} \to \Big\{\text{smooth maps } : [0,\epsilon] \to \Gamma(TM)\Big\}$$

   Note this is a map in one direction, where the time dependent vector field is given by differentiation with respect to t. We can almost get a map backwards by integrating the flow **assuming some completeness of the manifold**

4. **Definition:** (Symplectic isotopy): A diffeotopy preserving ω at all times
5. **Definition:** (Hamiltonian diffeomorphism): A diffeomorphism obtained as the time $t = 1$-map of a $[0,1]$ dependent Hamiltonian vector field (at every time it's Hamiltonian)
6. **Definition:** (Hamiltonian Isotopy): A diffeotopy which is Hamiltonian at all times I think this means that $\forall t \in [0,\epsilon]$, we have $\Phi_t(x) = \varphi_{H_t}(x_0, 1)$, i.e. the flow generated by some function X_{H_t} for H_t a function (can think of $H_t(x)$ as $H(t,x)$ for $H : [0,\epsilon] \times M \to \mathbb{R}$) starting at the point x_0 up to time $t = 1$. The above is almost correct, the point is that $H_t(x)$ does not piece together in a coherent way with respect to t,i.e. $H(t,x)$ is not smooth in t! It is a big theorem to say that $H(t,x)$ is actually smooth
7. For the above definition to actually hold, we need to show that the composition of two Hamiltonian diffeomorphisms is a Hamiltonian diffeomorphism
8. We define $Ham(M,\omega)$ as the group of Hamiltonian diffeomorphisms and $Symp(M,\omega)$ as the group of symplectomorphisms.
9. We have $Ham(M,\omega) \hookrightarrow Symp_0(M,\omega) \hookrightarrow Symp(M,\omega)$ where $Symp_0(M,\omega)$ is the connected component of the identity in $Symp(M,\omega)$
10. Remark: There are many many components to $Symp(M,\omega)$! To many to count even because apparently we can do surgery with the Dehn twist at many places and get a new connected component of $Symp(M,\omega)$

11. What is the Lie Algebra (i.e. tangent space at the identity) of $Ham(M,\omega)$? Hamiltonian vector fields!

$$\chi_{Ham}(M,\omega) \cong C^\infty(M,\mathbb{R})/\{\text{locally constant functions}\} \hookrightarrow \chi(M,\omega)$$

this is because hamiltonian vector fields are in correspondence with Hamiltonian functions, up to a locally constant function, hence why we've drawn the isomorphism between $\chi_{Ham} \cong C^\infty/\{\cdots\}$

12. Geometric interpretation of Hamiltonian isotopies etc. What is?

13. Terminology: $u : \Sigma \to (M,\omega)$ a smooth map. Σ is a 2-dimension manifold

14. Symplectic area of u is

$$\int_\Sigma u^*\omega$$

15. If we take an embedding $u : \Sigma \to M$ which bounds a 3-dimensional manifold, by stokes theorem, then integration of ω along the boundary is the same as integrating $d\omega$ on the interior 3-dimensional manifold, i.e. ω closed is equivalent to u being the boundary of $N^3 \hookrightarrow M$ where N^3 is a manifold with boundary. This implies that the symplecic area of u is 0 I feel like some stronger machinery might be needed here where we can say that $\int_N d\omega = 0$ if and only if ω is closed. Yes, kind of. The actual condition is that if for every 3-dimensional manifold with a boundary of a 2-manifold, then if we have that $\int_\Sigma \omega = 0$, then we can conclude that ω is closed.

16. **Lemma:** i) $\Phi : M \to M$ is a symplectomorphism iff Φ preserves symplectic areas.

ii) $\Phi : M \times [0,\epsilon] \to M$ a diffeotopy is a symplectic isotopy iff symplectic areas are preserved at all times. This is also equivalent to the symplectic area traced by contractible loops ($\gamma : S^1 \to M$) is 0 at all times. This last statement means the symplectic area of the cylinder traced by contractible loops is zero at all times.

iii) Φ is a Hamiltonian isotopy iff symplectic areas traced by all loops are zero. Is this right? Maybe it should be "contractible loops" The original is correct

Proof: i) ($\rightarrow$) We have

$$\varphi^*\omega = \omega \qquad \{u : \Sigma \to M\} \leftrightarrow \{\varphi \circ u : \Sigma \to M\}$$

then

$$\int_\Sigma (\varphi \circ u)^*\omega = \int_\Sigma u^*\varphi^*\omega = \int_\Sigma u^*\omega$$

verifying one direction.
($\leftarrow$) Assume φ is not a symplectomorphism, i.e. there exists a point $x \in M$ and two vectors $v, v' \in T_xM$ such that

$$\omega_x(v,v') > \omega_{d\varphi(x)}(d\varphi_x v, d\varphi_x v')$$

(if it was "<", we could take $-v$ instead of v). Now we choose an embedding, u of $D(\epsilon) \subseteq R^2_{y,y'} \hookrightarrow M$ such that

$$u(0) = x, \quad \frac{du}{dy}(0) = v, \quad \frac{du}{dy'}(0) = v'$$

this implies that for sufficiently small $\epsilon' > 0$ we have that for all b, b' in $T_y u(D(\epsilon))$ for $y \in u(D(\epsilon))$ that

$$\omega_y(b,b') > \omega_{\varphi(y)}(d\varphi_y b, d\varphi_y b')$$

Still want to ask Umut about this, not totally certain about the picture/whether or not this follows by continuity.
This implies that

$$\int_{D(\epsilon')} u^*\omega > \int_{D(\epsilon')} (\varphi \circ u)^*\omega$$

done. □

Proof: ii) to see the first equivalent condition is fine by definition because preserving symplectic areas will yield diffeotopies preserving symplectic areas at all times.

For the second part: we want to show that symplectic areas preserved implies that the trace of contrabible loops have zero area. Take

$$\gamma : S^1 \to M \ \text{ s.t. } \ \exists u \in D^2 \to M \ \text{ s.t. } \partial u = \gamma$$

Define the map $f : [0, \epsilon] \times D^2 \to M$ such that $(t, z) \mapsto \Phi_t(u(z))$. Then

$$\int_{\partial([0,\epsilon]\times D^2)} (\partial f)^* \omega = \int_{[0,\epsilon]\times D^2} f^* d\omega = 0$$

then we have that

$$0 = \int_{\{\epsilon\}\times D^2} f^*\omega - \int_{\{0\}\times D^2} f^*\omega + \int_{[0,\epsilon]\times \partial D^2} f^*\omega$$

The first two terms cancel and so we get that the last term is 0. Repeat this argument for all time t.

To go the other way: first prove that symplectic areas of discs small enough are preserved, then we chop up whatever region we have into small pieces. ☐

iii) ($\to$) If we have a Hamiltonian isotopy $\gamma : S^1 \to M$ and $u : S^1 \times (0, \epsilon) \to M$, where $(\theta, t) \mapsto \Phi_t(\gamma(\theta))$

10/23

1. Recall:
   (a) **Hamiltonian Diffeomorphism**: a time-1 map of a $[0, 1]$ dependent Hamiltonian vector field
   (b) **Hamiltonian Isotopy**: A diffeotopy consisting of Hamiltonian diffeomorphisms
   (c) **Symplectic Isotopies** A diffeotopy consisting of symplectomorphisms
2. **Lemma:** For any $\Phi : [0, \epsilon] \to M$ a diffeotopy, then TFAE:
   (a) Φ is a symplectic isotopy
   (b) The corresponding $[0, \epsilon]$ dependent vector field is symplectic for every $t \in [0, \epsilon]$
   (c) For any compact oriented manifold with boundary and smooth map $u : \Sigma \to M$, then

   $$Area(u) = Area(\Phi_t \circ u) \qquad \forall t \in [0, \epsilon]$$

   here "Area" denotes symplectic area like a few classes ago

   (d) For every $\gamma : S^1 \to M$ such that there exists $u : \Sigma \to M$ such that $\partial \text{Im}(u) = \gamma$ consider the map

   $$S^1 \times [0, 1] \to M \;\text{ s.t. }\; (\theta, s) \mapsto \Phi_s(\gamma(\theta))$$

   then the image of this map has 0 symplectic area.

   Note: the above is the definition of "trace" as used in the previous class (i.e. the image of this map "traces" out a cylinder in some sense)

Proof: Last time, we showed that $(a) \iff (c)$ and $(c) \iff (d)$. We now show $(a) \implies (b)$:

The time dependent vector field of Φ (which is almost like $\partial_t \Phi$ but see Spivak for details) is a map $[0, \epsilon] \to \Gamma(TM)$ s.t. $t \mapsto X_t$.

Umut says the below is wrong $\ddot{\frown}$ I think he'll give us a correct proof later It suffices to show that X_0 is symplectic because for any other $t_0 \in [0, \epsilon]$ we can just start the isotopy at $t = t_0$.Because starting the flow from t_0, we obtain vector fields $\tilde{X}_t, t \in [t_0, \epsilon]$, $\tilde{X}_t$, $t \in [t_0, \epsilon]$ and

$$\tilde{X}_t = (\Phi_{t_0}) X_t, \quad t \in [t_0, \epsilon]$$

$\tilde{X}_{t_0}(x)$ is defined as follows. Given a starting point x and a vector v tangent to the flow line Φ_t, then we define

$$\tilde{X}_{t_0}(x) = v$$

and $X_{t_0}(x)$ is defined as follows

$$X_{t_0}(x) = (\Phi_{t_0}^{-1})_* w$$

for w the tangent vector to the flow line starting at x and defined by $\Phi_t(x)$. Correction to the above: Apparently in the last thing above, we want to define

$$X_{t_0}(x) = \frac{d\gamma_x}{dt}(t_0) = X_{t_0}(\gamma_x(t_0))$$

which means it is also equal to $\tilde{X}_{t_0}(x)$ from before, so now we change the definition of $\tilde{X}_{t_0}$. Now we have that $\tilde{\gamma}_x$ represents the flow lines of $\Phi\Big|_{[T,T+\epsilon]}$, and so we can define

$$\frac{d\tilde{\gamma}_x}{dt}(t_0) =: \tilde{X}_{t_0-T}(\tilde{\gamma}_x(t_0-T))$$

We're given $X_t(x)$ and $\frac{d}{dt}\Phi_t^*\omega = 0$, we want to show $\mathcal{L}_{X_t}\omega \overset{?}{=} 0$

Let us define a vector field on $[0,\epsilon]\times M$ which lies over M via $[0,\epsilon]\times M \xrightarrow{pr} M$.

$$\tilde{X}(t) = X_t + \frac{\partial}{t}$$

note that

$$\mathcal{L}_{\tilde{X}} pr^*\omega = 0$$

because in one component it vanishes and in the other, we get $\frac{d}{dt}\Phi_t^*\omega = 0$. We also claim that

$$\mathcal{L}_{\tilde{x}-\tilde{x}_0} pr^*\omega\Big|_{t=0} = 0 \quad \text{s.t.} \quad (X_t - X_0)$$

where $\tilde{X}_0(t) = X_0 + \frac{\partial}{\partial t}$ and so

$$\mathcal{L}_{\tilde{x}_0} pr^*\omega = 0$$

by linearity, and so we have

$$\mathcal{L}_{\tilde{x}_0} pr^*\omega = 0 \implies \mathcal{L}_{x_0} pr^*\omega = 0 \implies \mathcal{L}_{x_0}\omega = 0$$

this is what we wanted to show. But we in fact proved that

$$\frac{d}{dt}\Phi_t^*\omega\Big|_{t_0} = \mathcal{L}_{x_0}\omega$$

10/25/19

1. Last time: Symplectic Isotopies - showed that TFAE definitions of symplectic isotopies
   (a) Showed that the above are diffeotopies of symplectomorphisms
   (b) They are the flow of a time dependent symplectic vector field
   (c) Areas are preserved at all times
   (d) Areas swept by contractible (or equivalently, null-homologous) loops are zero
2. Today, we discuss **Hamiltonain Isotopies**
3. **Lemma:** Let $H,\ G : M \times [0,\epsilon] \to \mathbb{R}$ be two time dependent Hamiltonian functions. These yield flows, $\varphi_H^t,\ \varphi_G^t : M\times[0,\epsilon]\to M$. Then the composition, $\varphi_H^t \circ \varphi_G^t : M\times[0,\epsilon]\to M$ is the Hamiltonian flow of

$$H_t + G_t \circ (\varphi_H^t)^{-1} : M\times[0,\epsilon]\to\mathbb{R}$$

Note: If the Hamiltonian flows commuted, then we'd have that $G_t\circ(\varphi_H^t)^{-1} = G_t$ and our overall Hamiltonian would be $H_t + G_t$.
Proof: Let V_t, W_t be $[0,\epsilon]$ time dependent vector fields.

$$V, W \to \varphi_V^t, \varphi_W^t \to \varphi_V^t \circ \varphi_W^t$$

$$\frac{d}{dt}\varphi_V^t(\varphi_W^t(x))\Big|_{t=t_0} = V_{t_0}(\varphi_W^{t_0}(x)) + (\varphi_V^{t_0})_* W_{t_0}(\varphi_W^{t_0}(x))$$

Shouldn't there be another chain rule here? which implies that

$$V_t + (\varphi_t^V)_* W_t \quad \text{genereates} \quad \varphi_V^t \varphi_W^t$$

so what is this flow in terms of Hamiltonian vector fields? It is

$$X_{H_t} + (\varphi_t^H)_* X_{G_t}$$

Not sure why this ends the proof This is the vector field if they were hamiltonian vector fields. Now we use the below lemma to find the actual hamiltonian

Lemma: For $\varphi : M \to M$ symplectomorphism, $F : M \to \mathbb{R}$, then $\varphi_* X_F = X_{F\circ\varphi^{-1}}$

Proof: we have

$$\begin{aligned}\omega(\varphi_* X_F, V) = \varphi^*\omega(X_F, (\varphi^{-1})_*, V) = \omega(X_F, (\varphi^{-1})_* V)\\ = dF(\varphi_*^{-1} V) = (\varphi^{-1})^* dF(V) = d(F \circ \varphi^{-1})(V)\end{aligned}$$

ending the proof. ☐

Note, now that we have the lemma, we get that

$$(\varphi_t^H)_* X_{G_t} = X_{G_t \circ (\varphi_t^H)^{-1}}$$

which gives us the desired $G_t \circ (\varphi_H^t)^{-1}$ term in our Hamiltonian. ☐

4. **Corollary:** Composition of Hamiltonian diffeomorphisms is Hamiltonian. This is because a Hamiltonian diffeomorphism is a time 1 map of a Hamiltonian flow so $t = 1$ in our above composition proof

5. **Lemma:** $\Phi : M \times [0, \epsilon] \to M$ is a diffeotopy. Then TFAE
   (a) Φ is a Hamiltonian isotopy
   (b) The corresponding time dependent vector field is Hamiltonian.
   (c) For every loop, $\gamma : S^1 \to M$, the area of $u : S^1 \times [0, \epsilon] \to M$ where $(\theta, s) \mapsto \Phi_s(\gamma(\theta))$ is zero! (Not that this differs from symplectic isotopy because there is no contractible condition on the loop)
   As an example, consider translation given by $(\theta, x) \mapsto (\theta, x + t)$ on the cylinder $S^1 \times \mathbb{R}$. Then the vector field is given by $\frac{\partial}{\partial x}$ and $\omega\left(\frac{\partial}{\partial x}\right) = \pm d\theta$, but recall that $d\theta$ is not the differential of a global function (it's the same as an on the circle). Hence this is not a Hamiltonian isotopy.

Proof: $(a) \implies (b)$ is not obvious. $(b) \implies (a)$ is obvious up to the following trivial fact: the time-$t = 1$ of H is the same as the time $t = \epsilon$ map of H/ϵ, which allows us to pass from the definition of Hamiltonian isotopy as a time 1 map to a time t map for any t just via dividing.

Let's analyze (c), we have

$$u : S^1 \times [0, \epsilon] \to M \quad \text{s.t.} \quad (\theta, t) \mapsto \Phi_t(\gamma(\theta))$$

then

$$\int_{S^1 \times [0,t]} u^*\omega = \int_0^t \int_{S^1} \omega\left(\frac{du}{dt}, \frac{du}{d\theta}\right) d\theta ds$$

note that

$$\frac{du}{dt} = V_s(\Phi_s(\gamma(\theta)))$$

where V_s is the vector field generating Φ. Now we define

$$\alpha_s = \omega(V_s, \cdot)$$

and hence from the above

$$\int_{S^1 \times [0,t]} u^*\omega = \int_0^t \int_0^1 \alpha_s \left(\frac{du}{d\theta}\right) d\theta ds = \int_0^t \left(\int_{S^1} \left(u\Big|_{S^1 \times \{s\}}\right)^* \alpha_s ds\right)$$

Now we know that V_s is Hamiltonian if and only if α_s is exact. Yes because if V_s is Hamiltonian, then by definition $\omega(V_s, \cdot) = dH$ for some H function

Lemma: (basic) A closed 1-form is exact iff it integrates to zero on every loop.

With this, we can prove $(b) \implies (c)$ because if V_S Hamiltonian then α_S is exact and we have $\left(u\Big|_{S^1 \times \{s\}}\right)^* \alpha_s$ is exact and so integrating it along S^1 will yield 0, meaning that the original area integral is 0.

For $(c) \implies (b)$, we get that α_s integrates to 0 on every loop, so somehow we can cook up a Hamiltonian which will yield the vector field of Φ

For $(a) \implies (c)$: we feel like this should follow by homotoping our trace to the trace of a time dependent hamiltonian vector field, but we will do this on homework. ☐

6. **Remark**: If $\Phi : [0,\epsilon] \times M \to M$ a symplectic isotopy, then you canonically obtain an element of $H^1(M,\mathbb{R})$. This is called the flux of Φ.

   If the flux is zero at all times, we have a Hamiltonian isotopy

7. Infinitesimal group actions

   (a) Let $\mathfrak{g}$ be a Lie algebra (something with a lie bracket that is antisymmetric, bilinear, and satisfies the Jacobi equation)

   (b) An infinitesimal (symplectic) action on M (for (M,ω)) is a map $\mathfrak{g} \mapsto \Gamma(TM)$, a lie algebra homomorphism, which is symplectic and in particular, $\mathfrak{g} \mapsto \chi(M,\omega)$, i.e. we map into the collection of *symplectic* vector fields. Recall (see 10/18) that a symplectic vector field is a vector field whose flow preserves ω, i.e. $\Phi_V^*\omega = 0$, which is equivalent to $\mathcal{L}_V\omega = i_V d\omega + d(i_V\omega) = 0$.

   (c) A Hamiltonian action is a map (lie algebra homomorphism) $\mathfrak{g} \to C^\infty(M,\mathbb{R})$, where we equip the space of smooth functions with Poisson bracket.

   (d) Assume that the image of $\mathfrak{g} \mapsto \chi(M,\omega)$ lies inside Hamiltonian vector fields. Does this imply that we have a Hamiltonian action? Not necessarily!

   (e) The obstruction comes from the fact that

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[ld, dashed] \arrow[d] \\
C^\infty(M,\mathbb{R}) \arrow[r] & \chi^{Hom}(M,\omega)
\end{tikzcd}$$

   lies in the second lie algebra cohomology of $\mathfrak{g}$ with coefficients in $H^0(M,\mathbb{R})$ what lies?. Concretely: choose a basis $g_1,\ldots,g_k$ of $\mathfrak{g}$ and lift each g_i to a function. But now look at $[g_1,g_2]$, this is a linear combination, $\sum_i a_i g_i$, and so it has two possible values under the lift because of the lie algebra homomorphism, i.e. $\{f_{g_1}, f_{g_2}\}$ but also $f_{\sum_i a_i g_i}$ and we want these two to be equal.

10/28

1. Lie Group Actions on Symplectic Manifolds

   (a) Lie groups: G is a smooth manifold and also a group which satisfies

      i. $G \times G \xrightarrow{m} G$ the multiplication operator is smooth

      ii. $G \xrightarrow{x^{-1}} G$ s.t. $x \mapsto x^{-1}$ is also smooth

   (b) Each Lie group produces a lie algebra, $\mathfrak{g} = T_{Id}G$

   (c) To define the lie algebra structure, we discuss the left-invariant vector field

   (d) We can naturally identify the vector fields at $T_{Id}G$ with

$$T_{Id}G = \{V \in \Gamma(TG) \mid (m_g)_*V = V\} = \{\text{left invariant vector fields}\}$$

   so for $v \in T_{Id}G$, define $\xi_v(g) = (dm_g)_*v$

   (e) The Lie bracket of left invariant vector fields make $\mathfrak{g}$ into a Lie Algebra

   (f) In fact, every Lie Algebra occurs via this construction and this is a one-to-one correspondence when we restrict our Lie Group to be simply connected. Pictorially

$$\left\{\begin{matrix}\text{connected} \quad + \quad \text{simply}\\ \text{connected Lie Groups}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{finite dimensional Lie}\\ \text{Algebras}\end{matrix}\right\}$$

   (g) The above is also a functorial assignment, i.e. given two lie groups and a lie group homomorphism, then we induce a homomorphism of lie algebras under the above correspondence

(h) **Lemma:** If $f : G \to H$ is a Lie group homomorphism. Then $df_0 : \mathfrak{g} \to \mathfrak{h}$ is a lie algebra homomorphism.
Proof: (sketch) If $v \in T_{Id}G$, then we get our left-invariant vector field ξ_v^G, and similar for $f_*v \in T_{Id}H$, then we get $\xi_{f_*v}^H$.
Claim: These vector fields are f-related, i.e.

$$g \in G,\ f_*(m_g)_*v \overset{?}{=} (m_{f(g)})_*f_*v$$

but the above actually holds because f commutes with multiplication by g by nature of being a Lie Group homomorphism. Not sure how the commutation works. What's the second element we are multiplying by? To see this we first start with

$$g, h \in G \implies f(g \cdot h) = f(g)f(h)$$

now consider the map $f(g \cdot something)$ and take the differential on both sides and use the chain rule. ☐
Claim: f-relatedness is preserved under Lie brackets.
Proof: (sketch) To see this, use the differential operator description of vector fields. ☐
The above two claims imply that $[\xi_v^G, \xi_{v'}^G]$ and $[\xi_{f_*v}^H, \xi_{f_*v'}^H]$ are f-related by looking at the identities of each Lie group. ☐

2. Actions of Lie Groups on Manifolds

   (a) The ation of G is a smooth map $G \times M \to M$ which includes the axioms of being an action. In particular

$$\Big\{G\text{-action on } M\ \Big\} \leftrightarrow \Big\{f : \mathfrak{g} \to \Gamma(TM) \text{ lie algebra hom}\Big\}$$

$$\Big\{\alpha : G \times M \to M\Big\} \to \forall v \in \mathfrak{g},\ x \in M, \quad \frac{d}{dt}\exp(tv) \cdot x = \frac{d}{dt}\alpha(\exp(tv), x)\Big|_{t=0}$$

   How does α manifest in the lie algebra homomorphism here? Maybe implicitly through the exponential map resolved

   (b) For $f : \mathfrak{g} \to \Gamma(TM)$ a lie algebra homomorphism $T_0G = \mathfrak{g}$ then if we have

      i. G simply connected and connected
      ii. There exists a basis of $\mathfrak{g}$ whose vector fields are complete

   THEN, we get that G acts on M inducing f by looking at the infinitesimal generator of the action

3. Adjoint action:

   (a) G acts on $\mathfrak{g}$ by lie algebra homomorphisms

   (b) Namely: $\forall g \in G$, then $c_g : G \to G$ such that $c_g(x) = gxg^{-1}$

   (c) As such, c_g is a lie group homomorphism with

$$(dc_g)_{Id} : \mathfrak{g} \to \mathfrak{g}$$

   a lie algebra homomorphism

   (d) Moreover, we have

$$Ad(\cdot) : G \times \mathfrak{g} \to \mathfrak{g}$$

   defined as follows: $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ I don't think I caught this definition in time In book and I also get this

   (e) Make sure to ask Umut why for a matrix lie group, we have that $Ad_g(Y) = gYg^{-1}$ - also why doesn't this generalize for all lie groups? What goes wrong in the differentiation argument. And how this tells us that locally, this is always the case for any Lie Group? Because locally, we have an isomorphism between a lie group and a matrix group in $\mathbb{R}^{n^2}$ or something This isn't really a welldefined statement to say that $Ad_g(Y) = gYg^{-1}$ always, because g doesn't really act on Y in the lie algebra unless we define the action to be by the pushforward of multiplication, i.e. $(m_g)_*$

   (f) The infinitesimal action of the adjoint Does this mean infinitesimal action of $Ad(g)$? See board picture sends $\mathfrak{g} \to \Gamma(T\mathfrak{g})$ where $v \mapsto V_w = \pm[v, w]$ for $w \in \mathfrak{g}$ thought of as the base point on $T\mathfrak{g}$.

   (g) The above can also be thought of as a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ where $(v, w) \mapsto [v, w]$, and we call this map $ad(\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

   (h) Coadjoint action of G on $\mathfrak{g}^*$ is given by g acting by $(Ad(g^{-1}))^*$
   (i) From Da Silva, ask umut: when $G = \mathbb{R}$ or $G = S^1$, the coadjoint action is trivial, why? see middle bottom of p. 134 Both the adjoint and coadjoint action are trivial actually

4. Symplectic structures on coadjoint orbits
   (a) Aside: Poisson Structures:
      i. A poisson structure is assigned on the entire dual of the lie algebra
      ii. For M a closed manifold, a lie algebra structure on $C^\infty(M, \mathbb{R})$, denoted by $\{\}$, is called a **Poisson Structure** if it satisfies the Leibniz rule:

$$\{f, gh\} = g\{f, h\} + h\{f, g\}$$

      iii. Ex: The poisson bracket for ω a symplectic form
      iv. Umut says that when you turn a classical mechanical system, you take a symplectic manifold with a poisson bracket and turn it into something with a parameter, which is a quantization? This is how you go from classical mechanics to quantum mechanics apparently I'd like more info on this
      v. A poisson structure is equivalent to a bivector field π (an element of $T(\wedge^2 TM)$) which satisfies $[\pi, \pi] = 0$ (here we're using the Schouten bracket, not the lie bracket)
      vi. Further note that $[\pi, \pi]_s = 0$ is equivalent to the jacobi identity
      vii. Consider $\tilde{\pi} : T^*M \to TM$ which is defined by contraction with π. The image of this map is a "distribution." Using Frobenius, we get a *singular* foliation whose leaves are symplectic
   (b) Poisson Structure on the lie algebra, $\mathfrak{g}^*$:
      i. Think of $\wedge^2 V$ as an antisymmetric pair on V^*
      ii. This implies that $T_a^* \mathfrak{g}^* = \mathfrak{g}$
      iii. Then we define π by

$$\forall a \in \mathfrak{g}^*, \quad \pi_a(g, g') = a([g, g']) \qquad \forall g, g' \in \mathfrak{g}$$

      iv. The Symplectic leaves of the induced foliation are the coadjoint orbites of G
      v. We can directly describe the symplectic structure of an orbit, θ:
      For every $X, Y \in T_0 G$, can define $a([X, Y]) := \omega(X^\#, Y^\#)$. Here's what I think is happening: X, Y lie in $T_0 G = \mathfrak{g}^*$ so their lie bracket also lies there. We draw an isomorphism between $\mathfrak{g} \cong T_a^* \mathfrak{g}^*$ via ii (not sure how though) so that.... actually not sure how that relates So $X^\#$ is the same vector field we were talking about before, i.e. with this map from $\mathfrak{g} \to TG$ and this vector field also lies in the tangent space of the coadjoint orbit. But we need that if X or Y is in $T_{Id} Stab(a)$ then the number above is equal to 0.
      vi. Here $X^\#$, $Y^\#$ are some lifts given that π is a surjective map. We then want to show that $\omega(X^\#, Y^\#)$ is independent of the choice of lift
      vii. Ex: $G = SL(2, \mathbb{R}) = 2$ by 2 matrices with determinant 1, then $\mathfrak{g} = sl(2, \mathbb{R}) = 2$ by 2 matrices with trace $= 0$. Can identify $g \leftrightarrow g^*$ in a G-equivariant manner. Consider $Adj(\cdot)$ (conjugation matrices)
      viii. There's some nice picture for this, but we ran out of time

10/30

1. Last time: Given G, a Lie group, then G acts on $\mathfrak{g}^*$ (dual of lie algebra) by the coadjoint action and the orbits of this action have natural symplectic structures. Formally, we have the coadjoint action is given by $K : G \to Aut(\mathfrak{g}^*)$ such that

$$\forall g \in G, \ \forall Y \in \mathfrak{g}, \ F \in \mathfrak{g}^* \qquad \langle K(g)F, Y\rangle = \langle F, Ad(g^{-1}Y)\rangle$$

   so we can think of K, the coadjoint action, as the formal adjoint of the adjoint representation

2. Note: If G smoothly acts on any smooth manifold, then its orbits are images of of injective immersions
   Proof: Let $p \in M$, want to show the orbit, $\mathcal{O}(p)$ is the image of an injective immersion. Consider

$$Stab(p) = \{g \in G \mid gp = p\} \subseteq G \quad \text{is a closed subset}$$

by the closed subgroup theorem. Moreover, $Stab(p)$ is a Lie group itself. Then we consider a map $\iota : G/Stab(p) \to M$ with $[g] \mapsto g \cdot p$. Then the image of this map gives the orbit of p, and it is an injective map from the quotient space to M.

Using that $Stab(p)$ is a closed subset, we can show that $G/Stab(p)$ is a manifold (quotient manifold theorem). Moreover, ι has constant rank, and so it is an immersion. ☐

3. Aside: If G is compact, then ι is a proper map and the orbits are submanifolds

4. Examples of coadjoint action 1)

$$G = SU(2), \quad \mathfrak{g} = su(2) = \left\{ \begin{pmatrix} ai & z \\ -\bar{z} & -ai \end{pmatrix} \,\middle|\, z \in \mathbb{C},\ a \in \mathbb{R} \right\}$$

   we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$, G-equivariantly How is this identification happening? (something to do with the killing form). We can just consider G acting on $\mathfrak{g}$ by conjugation. Note that elements of $\mathfrak{g}$ are all diagonalizable with imaginary eigenvalues. The orbits of our G action are just round spheres (of varying radii) centered at the origin. The same result occurs for $SO(3)$

5. Example of coadjoint action 2)

$$G = SL(2,\mathbb{R}) \quad \mathfrak{g} = sl(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \right\}$$

   moreover, we can identify $\mathfrak{g} \leftrightarrow \mathfrak{g}^*$. Here the identification between the lie algebra and its dual has to do with semisimplicity of the lie group.

   We look at the conjugation action of G on $\mathfrak{g}$ and we get 6 conjugacy classes

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \right\} \quad \lambda > 0$$

   the point is that for some $\lambda > 0$ fixed, the last three matrices give us 3 distinct conjugacy classes. Moreover, such a collection of conjugacy classes exist for each $\lambda > 0$. The change in λ can be thought of as the different leaves in a foliation.

6. **Theorem:** (Kirilov-Kostant-Souriau) Every symplectic homogenous manifold of a Lie Group G is, up to a possible covering, a coadjoint orbit of some central extension of G.

   Here a (symplectic) homogenous manifold is a (symplectic) manifold which admits transitive G-action (by symplectomorphisms). Central extensions are those such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \tilde{G} \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

   for some A, where $\mathrm{Im}(A) \subseteq Z(G) = center(G)$

7. Hamiltonian Lie Group Actions

   (a) For $G \times M \to M$ a symplectic action is called Hamiltonian if the map $\mathfrak{g} \to \chi(M,\omega)$ lifts to $\mathfrak{g} \to C^\infty(M,\mathbb{R})$ as a Lie algebra homomorphism

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, dashed] \arrow[d] \\ C^\infty(M,\mathbb{R}) \arrow[r, "X"] & \chi(M,\omega) \end{tikzcd}$$

   The lifted map, $\alpha : \mathfrak{g} \to C^\infty(M,\mathbb{R})$ is called the comoment map. What is X here? Also in order to have a lift, I assume we need X to be surjective? Or at least having imagine contained in the image of $\mathfrak{g} \to \chi(M,\omega)$) We're not saying this lift always exists, so X may not be surjective. Also X is the map which sends a Hamiltonian to its hamiltonian vector field

   (b) Claim: α is G-equivariant.
   Proof: We have for any G-action that

$$\mathfrak{g} \xrightarrow{\#} \chi(M,\omega) \quad \text{s.t.} \quad v \mapsto v^{\#}$$

   where

$$(Ad(g)\cdot v)^{\#} = (a_g)_* v^{\#}$$

   What is a_g? Maybe this is supposed to be ad_g? Instead of m_g, which we used before because our lie group action on itself was just multiplciation, we replace the multiplication action with a more general action, a, and we call this a_g. suppose $v^{\#}$ generates the flow φ^t. Then we have $(Ad(g)v)^{\#}$ generates the flow $a_g\varphi^t a_g^{-1}$. This implies that $(Ad(g)v)^{\#} = (a_g)_* v^{\#}$. Moreover, the Hamiltonian vector field of $H \circ \varphi^{-1}$ is $\varphi_* X_H$. ☐

   (c) There is also a moment map defined by

$$\mu : M \to \mathfrak{g}^* \ \text{ s.t. } \ x \mapsto [g \in \mathfrak{g} \mapsto \alpha(g)(x)]$$

   where $\alpha(g) \in C^\infty(M,\mathbb{R})$ Not sure what's happening in this map. Maybe we take the left invariant vector field associated to $g \in \mathfrak{g}$ and then apply some α, our comoment map, to g all at the point x. Here, we implicitly have a G action on M. Note that each point $x \in M$, yields a map on $\mathfrak{g}$ through the above defined map, which is the image of x

   (d) It's easy to check that M is G-equivariant using the coadjiont action on $\mathfrak{g}^*$.

   (e) Ex: If $\mathcal{O} \subseteq \mathfrak{g}^*$. The action of G on $\mathcal{O}$ is Hamiltonian, via the inclusion map $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ maybe? Not sure about this last sentence. We need to check this, in particular that the action of G is Hamiltonian

   (f) Ex: Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$ with the area form on the symplectic structure. The action we consider is an S^1 action given by rotation along the z-axis. Intuitively, this action is Hamiltonian because of the geometric criterion from a few classes ago.

11/1

1. Last time: Hamiltonian G-actions
   (a) We have the following diagram

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "\text{comoment map}"'] \arrow[d, "\text{Infinitesimal action}"] \\ C^\infty(M,\mathbb{R}) \arrow[r] & \chi(M,\omega) \end{tikzcd}$$

   (b) $\mu : M \to \mathfrak{g}^*$ (moment map) which satisfies $\forall X \in \mathfrak{g}$, the Hamiltonian associated to X is equal to $X \cdot \mu : M \to \mathfrak{g}^* \xrightarrow{\cdot X} \mathbb{R}$

2. Ex: $S^2 \subseteq \mathbb{R}^3$ with area form. S^1 action given by rotation along z-direction. We introduce cylindrical coordinates in $\mathbb{R}^3$

$$\text{area form} = \sin\phi d\phi d\theta = -d(\cos\phi)d\theta = -dzd\theta$$

$$\text{generator of the action on } S^1 = \frac{\partial}{\partial\theta}$$

$$\text{Hamiltonian of } S^1 \text{ action} = z = \text{moment map}$$

   So the moment map is the projection from $S^2 \to \mathbb{R}$ of the z-coordinate. The "generator of S^1 action" is the image of 1 under the map from $\mathfrak{g} \to \chi(M,\omega)$ where $1 \in \mathbb{R} = \mathfrak{g}$. This is because $\mathbb{R}$ is the lie algebra for S^1.

3. In fact, $SO(3)$ acts on S^2 with a Hamiltonian action

4. This is a coadjoint orbit of $SO(3)$, with the moment map: $S^2 \to \mathfrak{g}^* = \mathbb{R}^3$ the inclusion. This follows from general story but can also be directly shown from our computations from a moment ago.

5. Remark: Comoment maps are not unique, e.g. in the rotation example we could take $\mathbb{R} \mapsto C^\infty(M,\mathbb{R})$ via $1 \mapsto z+1$ instead of $1 \to z$

6. Note that $0 \in \mathfrak{g}^*$ is a special point, because $\{0\}$ is always preserved and hence its preimage is a coadjoint action. This corresponds to Noether's principle of conserved quantities somehow!
7. If $\mathfrak{g}$ is semi-simple, then the moment map, μ, is unique
8. If $\mathfrak{g}$ is abelian then we have $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$ and so if μ is a moment map, then μ + translation by a vector in $\mathfrak{g}^*$ also is
9. Exercise: $\mathbb{R}^3 \circlearrowright \mathbb{R}^3$ via translations. This can be lifted to $\mathbb{R}^3 \circlearrowright T\mathbb{R}^3$. This is a Hamiltonian group action, so it has a moment map $T^*\mathbb{R}^3 \to \mathbb{R}^3$, given by projection to the p coordinates, where $T^*\mathbb{R}^3$ parameterized by (q, p)
10. Ex: $SO(3)$ acts on $\mathbb{R}^3$ which lifts to an action $SO(3) \circlearrowright T^*\mathbb{R}^3$. The Hamiltonian group action gives a moment map $\mu : T^*\mathbb{R}^3 \to \mathbb{R}^3$ which sends $(p, q) \to p \times q$, i.e. it maps the point in phase space to the angular momentum.
11. Symplectic Reduction
    (a) We have $G \times M \to M$ a Hamiltonian action with $\mu : M \to \mathfrak{g}^*$
    (b) Claim:
    $$d\mu_p : T_pM \to T_{\mu(p)}\mathfrak{g}^* \cong \mathfrak{g}^*$$
    $$\ker d\mu_p = (T_p\mathcal{O}(p))^\omega$$
    $$\text{Im}(d\mu_p) = \text{annihilator of LieAlg(Stab(p)) inside of } \mathfrak{g}^*$$
    where the right hand side of the second line is the symplectic orthogonal complement of the tangent space (at p) of the orbit of p.
    Proof: We have $\omega_p(X_p^\#, v) = \langle d\mu_p(v), X\rangle$ for every $X \in \mathfrak{g}$, and $v \in T_pM$. This is because for every $X \in \mathfrak{g}$,
    $$d(X \circ \mu) = \omega(X^\#, \cdot)$$
    this is the definition of the comoment map if you stare at it for a while. But
    $$d(X \circ \mu)(v) = dX \circ d\mu(v)$$
    we then have
    $$W \in \ker d\mu_p \iff W \in (T_p\mathcal{O}_p)^\omega$$
    This is because of the first equality
    $$\omega_p(X_p^\#, v) = \langle d\mu_p(v), X\rangle$$
    For the image statement, if $X_p^\# = 0$, then $d\mu_p(W)$ annihilates Im$d\mu_p \subseteq$ annihilator. □
    (c) What's the Upshot of all of this?
    (d) **Theorem:** (Marsden-Weinstein-Meyer) For G compact, $G : M \to M$ with Hamiltonian $\mu : M \to \mathfrak{g}$. Assume compact G acts freely on $\mu^{-1}(0)$.
    Then $\text{Im}(d\mu_p) = \mathfrak{g}^*$ for all $p \in \mu^{-1}(0)$, which implies that 0 is a regular value of μ, which implies that $\mu^{-1}(0)$ is a manifold.
    Moreover, $\ker d\mu_p = T_p\mu^{-1}(0)$ and $\mathcal{O}(p) \subseteq \mu^{-1}(0)$. If we take the symplectic orthogonal complements of these spaces, we get
    $$(T_p\mu^{-1}(0))^\omega = T_p\mathcal{O}_p$$
    which tells us that $\mu^{-1}(0)$ is a coisotropic manifold, with the characteristic folation given by the orbits of G because the tangent directions are given by $T_p\mathcal{O}_p$.
    Formally, $\mu^{-1}(0)$ is coisotropic with characteristic foliation biven by the G-orbits, which implies that $\mu^{-1}(0)/G$ admits a symplectic structure. This is called the symplectic reduction, $M//G := \mu^{-1}(0)/G$

11/4/19

1. **Theorem:** Let (M, ω) be a connected symplectic manifold. Then for any $p, q \in M$, there exists a Hamiltonian isotopy $\Phi : M \times I \to M$ such that $\Phi_1(p) = q$, i.e. "any two points in a symplectic manifold look locally the same"
   Proof: Heuristically: If we were trying to just find a diffeotopy, we could just find a path from $p \to q$ and then create a vector field along this path which integrates to the path. The vector field could then be extended

to a compact neighborhood and we'd get a diffeomorphism which sends $p \to q$ and equals the identity outside some compact set containing the path. Note also that if M is connected, then it is path connected by nature of being a manifold

Actual proof:
Step 1: Choose a smooth path (embedded) $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and let $v_t = \gamma'(t) \in T_{\gamma(t)}M$

Step 2: Find a $[0,1]$-dependent vector field X_t such that $X_t(\gamma(t)) = v_t$ for all $t \in [0,1]$ and there exists a neighborhood of $\text{Im}(\gamma)$ such that $X_t = 0$ outside of N for all $t \in [0,1]$. For v_0, choose $H_0 : M \to \mathbb{R}$ such that $X_{H_0}(\gamma(0) = p) = v_0$. This is equivalent to $dH_0(p) = \omega(v_0, \cdot)$. By multiplying H_0 with cutoff functions, we can assume that its support is the closure of an arbitrarily small open subset

Choose a neighborhood N of $\text{Im}(\gamma)$. Find a diffeotopy $\psi_t : M \times I \to M$ such that ψ_t is just the identity outside of N and $\psi_t(\gamma(0)) = \gamma(t)$ and $\partial\psi_t(v_0) = v_t$.

A naive thing to do (which Umut thought was the correct answer) is to consider $H_t := H_0 \circ \psi_t^{-1}$. But this doesn't work because $X_{H_t}(v_t) \neq v_t$, as

$$dH_t(v_t) = (\psi_t)^* dH_0(v_0)$$

we do have that $\partial_t H_t = v_t$ though, but the Hamiltonian vector field is often not equal to $\partial_t H_t$.

Try 2: Fix an embedding of $B_\epsilon(0) \hookrightarrow M$ which sends $0 \to \gamma(0)$. Call this embedding E. Then we get a series of embeddings, $\psi_t \circ E : B(\epsilon) \hookrightarrow M$ which sends $0 \mapsto \gamma(t)$. Find a linear hamiltonian (i.e. linear in B_ϵ and standard $\mathbb{R}^n$ coordinates), $\tilde{H}_t$, at each time t such that $X_{\tilde{H}_t}(\gamma(t)) = v_t$ (solve this in B_ϵ). Note that $\tilde{H}_t$ will be smooth as a function of t because there's only one way to solve the equation $X_{\tilde{H}_t} = v_t$ when $\tilde{H}_t$ is linear. Now to finish the proof, we cutoff $X_{\tilde{H}_t}$ with a smooth function $\rho : B_\epsilon \to \mathbb{R}$. Step 3 is to flow along $X_{\tilde{H}_t}$ and we finish the theorem. □

2. **Theorem:** (Darboux): Let (M, ω) symplectic and $x \in M$, then there exists a coordinate system with coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ such that $\omega = dp_1 \wedge dq_1 + \dots + dp_n \wedge dq_n$ in the domain of these coordinates. This is called the normal form about x. Note that via symplectic linear algebra, this is easy to achieve at a point, but not on a chart, i.e. in all of the neighborhood of a point.
Proof: It suffices to prove: Let $y \neq x$, then there exists another symplectic structure on M, ω^{new} on M, s.t. $\omega^{new} = \omega$ near x and $\omega^{new} = dp_i \wedge d_i$ near y in some coordinate system. Then use the previous lemma to map the symplectic structure near y to the symplectic structure near x.

Choose some coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ at y such that $dpdq\Big|_y := dp_i \wedge dq_i\Big|_y = \omega\Big|_y$. Now we want to interpolate between $dpdq$ and ω in a neighborhood of y. Think of it as the two forms agree at y and then immediately stop agreeing outside of the point of y. Now we want to make them equal in a small neighborhood.

Remark: If $\rho_1 + \rho_2 = 1$ partitions of unity and ω_1, ω_2 are symplectic forms, then $\rho_1\omega_1 + \rho_2\omega_2$ is not symplectic in general, because this form may not be closed (i.e. $d\rho_i \neq 0$).

One can patch together exact two forms, i.e. if we have $d\lambda_1$, $d\lambda_2$, then $d(\rho_1\lambda_1 + \rho_2\lambda_2)$ is exact and therefore closed. However, non-degeneracy becomes harder to achieve as we also need bounds on $d\rho_i$.

Patching together $dpdq$ and ω now becomes a pain. We need a 1-form, θ, which is compactly supported near y such that

(a) $\omega + d\theta = dpdq$ in a neighborhood of y.

(b) $d\theta$ is small.

This is possible because $\omega - dpdq\Big|_y = 0$. We could get away with much less.

Here's a more local argument near x. We have ω and $dpdq$ at x such that $\omega_x = dpdq_x$ by choice of coordinate chart. We can try to find a diffeomorphism defined near x, taking $\omega \to dpdq$ but also fixing x.

Now we use **Moser's trick**: construct a diffeomorphism as a time-1 map of a vector field of a $[0,1]$-dependent vector field that vanishes on x for all $t \in [0,1]$. We do this by rigidifying the problem, i.e. making the problem more constrained and hence easier to solve. Let $\omega_t = t\omega + (1-t)dpdq$. Note that this is still closed because its a convex combination of *constant functions* what? These forms are constant? How What Umut means is that for a fixed t, we have t and $(1-t)$ are constant, and so $d\omega_t = td\omega + (1-t)d(dpdq) = 0$. Also $\omega_t\Big|_x = \omega = dpdq$.

Note that for some neighborhood $U \ni x$, ω_t is symplectic for all t. Nondegeneracy is thus satisfied because at $t = 0$, ω_t is non-degenerate at x and non-degneracy is an open condition.

Now we want to find V_t with flow φ_t such that $\varphi_t^* \omega_t = \omega$ for all $t \in [0, 1]$

11/6

1. Last time: We were priving the Darboux theorem and were just setting up the Moser Argument:

   We have (M, ω) symplectic with $x \in M$ and $\omega_x = \left(\sum_i dp_i \wedge dq_i\right)_x$ where the coordinates are $p_1, \dots, p_n, q_1, \dots, q_n$

   Consider $t\omega + (1-t)dpdq$ is a symplectic form in some neighborhood $U_2 \ni x$ when we fix any $t \in [0, 1]$. We want to find $V_t : [0, 1] \to \Gamma(TU_2)$ such that:

   (a) $V_t(x) = 0$ for all $t \in [0, 1]$ which implies that there exists a U_3 such that the flow $I \times U_3 \to U_2$ is defined, i.e. the flow never leaves U_2. This comes from an ODE argument

   (b) If the flow is $\varphi_t : U_3 \to U_2$, then $\varphi_t^* \omega_t = \omega$ for every t

   Now note that
   $$\varphi_t^* \omega_t = \omega \iff \frac{d}{dt} \varphi_t^* \omega_t = 0 = \varphi_t^* \mathcal{L}_{v_t} \omega_t + \varphi_t^* \left(\frac{d}{dt}\omega_t\right)$$
   The above expansion can be thought of as the chain rule because we have a function
   $$\mathbb{R} \mapsto \mathbb{R}^2 \to \Omega^k(M) \text{ s.t. } t \mapsto (t, t) \mapsto \varphi_t^* \omega_t$$
   now because φ_t^* is a diffeomorphism, we get that
   $$\mathcal{L}_{v_t} \omega_t + \frac{d}{dt}\omega_t = 0 \implies d(\iota_{v_t}\omega_t) = \mathcal{L}_{v_t}\omega_t = -\frac{d}{dt}\omega_t = dpdq - \omega$$
   Now note that $dpdq - \omega$ is an exact 2-form which vanishes at x by this inequality. Choose a random primitive for this, i.e. $dpdq - \omega = d\theta$. If we choose the appropriate constant so that $\theta = \iota_{v_t}\omega_t$, then this uniquely defines v_t.

   Now for our first statement (a), we need $\theta(x) = 0$. We can do this by either subtracting a constant (i.e. differential of a linear one form) or use the proof of the Poincare lemma to produce a specific θ. ☐

2. The Moset argument at it's best: (M, ω) a closed sympelctic manifold, ω_t is $[0, 1]$ family of symplectic forms $\omega_0 = \omega$ and $[\omega_t] = [\omega]$ for every t (here we formulate the equivalence classes in $H^2_{dR}(M)$)

3. **Lemma:** There exists a diffeotopy, φ_t, such that $\varphi_t^*(\omega_t) = \omega$ and in particular, (M, ω_0) is symplectomorphic to (M, ω_1)

   Proof: Note that now every vector field is complete. Run the Moser argument and get
   $$0 = \frac{d}{dt}\varphi_t^* \omega_t = \cdots = d(\iota_{v_t}\omega_t) = -\frac{d}{dt}\omega_t$$
   Note first that $\left[\frac{d}{dt}\omega_t\right] = \frac{d}{dt}[\omega_t] = 0$. Now we need to choose some $\xi_t \in \Omega^1(M)$ such that $\frac{d}{dt}\omega_t = d\xi_t$ and ξ_t depends smoothly on t. Umut says there are many ways to go about this, but they're all kind of a hassle. One involves choosing a Riemannian metric, which gives a primitive of our closed 2-form, $\frac{d}{dt}\omega_t$. We then use the Hodge theorem via the hodge star operator $*$, which gives an L^2 metric on $\Omega^*(M)$, which yields the adjoint operator d^*. This gives us a canonical choice of primitives.

   We can also try to choose a covering of our manifold and patch together local primitives by using the Mayer-Vietoris theorem.

   We can also try to make a bundle over $[0, 1]$ with all of the choices we have, which will give a contractible, infinite dimensional vector space. ☐

4. Warning: If $[\omega_0] = [\omega_1]$ and they're both symplectic, then that does not mean that they can both be connected by a ω_t such that $[\omega_t] = [\omega_0]$ for all t. I.e. Two symplectic forms with the same cohomology class are not necessarily connected by a one-parameter family of symplectic forms with the same cohomology. Note: the naive approach $t\omega_0 + (1-t)\omega_1$ might be degenerate at a point!

5. **Theorem:** Let Σ be a closed surface, ω_0, ω_1 two area forms, i.e. symplectic forms, then

$$(\Sigma, \omega_0) \overset{\textbf{symplectomorphic}}{\sim} (\Sigma, \omega_1) \iff \int_\Sigma \omega_1 = \int_\Sigma \omega_0$$

   This tells us that up to scaling and symplectomorphism, there is a canonical symplectic form on a surface.
   Proof: Key point, for every $t \in [0,1]$, if ω_0 and ω_1 have the same sign (makes sense because $\Lambda^2(\Sigma)$ is one dimensional because the manifold is dimension 2), then $t\omega_0 + (1-t)\omega_1$ is a symplectic form and we can use the Moser argument to get a primitive. $\square$

6. We can prove more stuff with Moser:

7. **Theorem:** (Extension): Let P a manifold, $N \subseteq P$ a closed submanifold,
   (a) Let Ω be a skew-symmetric bilinear form on $T_N P := TP\Big|_N$ whose restriction to TN is a closed two form. Then Ω extends to a closed two form on in a neighborhood of N. If Ω is symplectic, then the extension can be made to be symplectic (this is because non-degeneracy is an open condition so if Ω non-degenerate, then surely it is non-degerate in a neighborhood).
   (b) Let Ω_0, Ω_1 be symplectic structures on P whose restriction to $T_N P$ are equal (e.g. when we were proving Darboux ω and $dpdq$ satisfied this condition where $N = \{x\}$). Then there are neighborhoods U and V of $N \subseteq P$ and a sympletomorphism, F, from $(U, \Omega_0) \to (V, \Omega_1)$ such that $F\Big|_N = Id$ and $dF\Big|_{T_N P} = Id$.

11/8 (Notes Checkover starting here)

1. Extension theorem: P a manifold, $N \subseteq P$ a closed submanifold
   (a) Let Ω be a skew symmetric bilinear form on $T_N P := TP\Big|_N$ whose restriction to TN is closed. Then Ω can be extended to a closed form in a neighborhood
   (b) Let Ω_0, Ω_1 symplectic structures on P, $\Omega_0\Big|_{T_N P} = \Omega_1\Big|_{T_N P}$, then there exists neighborhoods, U, V of N which are symplectomorphic (extending the identity on N and $T_N P$)

2. Homotopy formula (is deRham theory): $f, g : X \to Y$ and $F : X \times I \to Y$ a homotopy between f and g.

3. Let $f^*, g^* : \Omega^*(Y) \to \Omega^*(X)$ be chain homotopic

4. Define $h : \Omega^*(Y) \to \Omega^{*-1}(X)$ such that

$$\omega \mapsto \int_0^1 (\iota_t^* \iota_{\partial/\partial t} F^*) \omega dt$$

   where

$$\iota_t : X \to X \times I \;\text{ s.t. }\; x \mapsto (x, t)$$

   i.e. first pullback ω by F, then contract by $\partial/\partial t$, then pull back by ι_t.

5. Note that as defined above,

$$f^* - g^* = dh + hd$$

   as a consequence of Cartan's magic formula

6. **Corollary:** Let $E \to M$ be a vector bundle, with

$$\begin{tikzcd} D \subseteq E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "z"'] \end{tikzcd}$$

   a disk bundle. We have a homotopy between $z \circ \pi$ and $id : D \to D$ such that

$$F : D \times I \to D \;\text{ s.t. }\; ((v, m), t) \mapsto (tv, m)$$

thinking of D as the disk bundle. Is this for the disk bundle??? yes works out because $0 \leq t \leq 1$ If ω is a form on D, then we have

$$\omega - (z \circ \pi)^* \omega = dh\omega + hd\omega$$

why tho

7. For any $\alpha \in \Omega^1(D)$, then

$$h\alpha\Big|_{T_{z(M)}D} = 0$$

Why?

$$i_{\partial/\partial t} F\Big|_{z(M) \times I} \alpha = 0$$

$$i_{\partial/\partial t} F^* \alpha(v_1, \dots, v_k) = \alpha(F_* \frac{\partial}{\partial t}, F_* v_1, \dots, F_* v_k)$$

I don't understand this computation

8. Also note that

$$dh\alpha\Big|_{T_{z(M)}D} = 0 \quad \text{if} \quad \alpha\Big|_{T_{z(M)}D} = 0$$

as then we have

$$\alpha - \pi^* z^* \alpha = hd\alpha + dh\alpha$$

but the first term is 0 by assumption, the second term vanishes for some reason what reason?, and $d\alpha = 0$ by the first thing we proved, so $dh\alpha\Big|_{T_{z(M)}} = 0$ when restricted

9. Proof of theorem:

   (a) Given Ω on $T_N P$, we need this one form in a neighborhood. Can assume P is a tubular neighborhood of N.

   (b) Step 1: Extend Ω to so some form

   (c) Step 2: Consider P as a disk bundle of the normal bundle of N. Choose a connection, which tells us that every point $(x, v) \in P$, we have

   $$T_{(x,v)} P \cong (T_N P)_x$$

   Note that $(T_N P)_x \ncong T_x N \subseteq TP$.

   (d) With the above isomorphism, we get out extension α and write the homotopy formula for the contraction:

   $$\alpha - \pi^* z^* \alpha = dh\alpha - hd\alpha$$

   Also don't get this, but I assume it's similar to the first iteration of this. Note that because α restricted to the zero section is closed, then $\pi^* z^* \alpha$ is closed. We also have $dh\alpha := d(h\alpha)$ is closed because it is exact, and so

   $$\alpha - hd\alpha = \text{ closed 2-form}$$

   and so the left hand side still extends Ω on $T_N P$. This proves part a) of the extension theorem ☐

   (e) For part b), we have Ω_0 and Ω_1, and $\Omega_0\Big|_{T_N P} = \Omega_1\Big|_{T_N P}$.

   (f) We can choose a nice primitive of $\Omega_0 - \Omega_1$. Notice that $\omega_t = t\Omega_0 + (1-t)\Omega_1$ is again symplectic in a neighborhood of V. Now we proceed exactly as in the Darboux theorem:

   $$i_{v_t} \omega_t = \xi$$

   (g) For $df\Big|_{T_N P} = id$, $v_t\Big|_N = 0$ is not enough. Here f is the time 1 flow of v_t. We need

   $$d\varphi_t^{v_t} : T_n P \to T_n P, \qquad n \in N$$

   to be the identity map. We check that this follows from $d\xi = 0$. "The derivation of v_t along any direction $v \in T_p N$ is zero"

11/11

1. Wedn: No class (to be rescheduled). Friday we'll have a sub
2. Last time: Extension theorem: For $N \subseteq P$
   (a) Ω on T_NP and closed on TN implies there exists a closed extension to a neighborhood
   (b) $\Omega_0 = \Omega_1$ on T_NP, then we can find a symplectomorphism
3. How do you do this?
4. Ex 1: Darboux theorem: Tubuluar neighborhood theorem: Let E be the normal bundle of $N \subseteq P$, Fix an isomorphism $TN \supseteq T_NE \xrightarrow{\sim} T_NP \subseteq TN$. Then there exist open neighborhoods, $U \supseteq N \subseteq E$ and $V \supseteq N \subseteq P$ and diffeomorphisms $U \to V$ which induce the isomorphism $T_NE \xrightarrow{\sim} T_NP$
5. Ex 2: Let $L \subseteq M$ be a lagrangian, then the normal bundle is isomorphic to T^*L and so we have

   $$\begin{tikzcd} N = T_LM/TL \arrow[r, leftrightarrow, "\cong", "\omega"'] \arrow[d] & T^*L \arrow[d] \\ L & L \end{tikzcd}$$

   Choose a lagrangian complement subbundle to $TL \subseteq T_LM$ so that $T_LM \cong T^*L \oplus TL \cong T_L(T^*L)$, where $T_L(T^*L)$ means the tangent bundle of T^*L when restricted to the zero section, which is L. T^*L already has a symplectic structure extending the natural skew-symmetric pairing on $T_L(T^*L)$
6. **Theorem:** (Weinstein Neighborhood Theorem) For $L \subseteq M$ a lagrangian, there exists neighborhoods $U \supseteq L \subseteq T^*L$ and $V \supseteq L \subseteq M$ such that there exists a symplectomorphism

   $$\begin{tikzcd} U \arrow[r, "\sim"] \arrow[d, "\supseteq"] & V \arrow[d, "\supseteq"] \\ L \arrow[r, "\sim"] & L \end{tikzcd}$$

7. Ex 3: $C \subseteq M$ coisotropic. Remark: Here the normal bundle and (T_CM, Ω) doesn't just depend on C, it can depend on the embedding. Compare this to the lagrangian case where the normal bundle is T^*L and hence has a canonical symplectic form, hence symplectic structure.

   Linear algebra: V symplectic vector space, $W \subseteq V$ isotropic (or $W \subseteq W^\Omega$ coisotropic)
   (a) Choose any complement to W in W^Ω, call it S
   (b) Then W^Ω is also symplectic
   (c) $W \subseteq S^\Omega$ is lagrangian
   (d) Choose lagrangian complement to W in S^Ω

   $$\implies V \overset{\text{sympl. isom.}}{\cong} W \oplus W^* \oplus S$$

8. From the above linear algebra, we get $T_CM \cong S(C) \oplus TC \oplus T^*C$ where $S(C) \cong TC^\Omega/TC$. So the choice for the symplectic structure for arbitrary C not lagrangian, comes from how many choices of a normal bundle $S(C)$ that we can make
9. The normal bundle satisfies
   $$T_CM/TC \cong S(C) \oplus T^*C$$
   where the right hand side lies over C as a vector bundle. We can now use previous results to get a symplectic form extending the symplectic form on the normal bundle, which is $S(C) \oplus T^*C$, to a neighborhood of it.
10. Hence, neighborhoods equipped with a symplectic structure of C correspond to rank $\dim M - 2\dim C$ symplectic vector bundles over C
11. $S(C)$ corresponds to the part of the normal bundle sensitive to its embedding. In the lagrangian case, all choices are contractible and hence there's a canonical choice, but the same is not true for isotropic and coisotropic

12. Gromov non-squeezing:
    (a) $B_r \subseteq \mathbb{R}^4$ is a ball of radius r. Let $Z(R) = D(R) \times \mathbb{R}^2$ where $D(R)$ is the disk of radius R, this is called the cylinder.
    (b) **Theorem:** If $B(r)$ symplectically embeds into $Z(R)$ then $R \geq r$
    (c) The reason this is a non-trivial result is because if we just require a volume preserving diffeomorphism, then we can squeeze the ball into a thin, elongated ellipsoid so that the volume is preserved and we can fit the ball in the cylinder
    (d) **Theorem:** (Guth, Polterovizh) If $\Sigma = T^2 \backslash \{\text{open disk}\}$ and $area(\Sigma) = 1$, then $\forall r > 0$, $B(r)$ symplectically embeds into $\Sigma \times \mathbb{R}^2$. Here, we replaced $S^2 - \{\text{open disk}\}$ with $T^2 - \{\text{open disk}\}$ and there was no obstruction to the embedding.
    (e) Remark: the idea is that in the Gromov theorem, we consider $D(R) \times \mathbb{R}^2 \cong [S^2 - \{\text{open disk}\}] \times \mathbb{R}^2$ because a sphere minus an open disk is diffeomorphic to an open disk
    (f) **Remark**: For every R, there are symplectic embeddings $B^4(R) \subseteq \mathbb{R}^4$, with arbitrarily large percentage (volumewise) of $B^4(R)$ maps into $Z(1) = D(1) \times \mathbb{R}^2$. This is called a Katok Embedding
    (g) The proof of Gromov non-squeezing and Guth requires the knowledge of J-holomorphic curves
    (h) Sketch: Consider (Σ, j) a Riemann Surface for j an almost complex structure which is actually complex. Let (M, J) be an almost complex manifold and finally a map

    $$u : (\Sigma, j) \to (M, J) \text{ s.t. } du \circ j = J \circ du$$

    then u is called a (j, J)-holomorphic curve. If one chooses holomorphic coordinates, $z = s + it$ locally on Σ, then our above equation becomes

    $$du \circ j \left(\frac{\partial}{\partial s} \right) = J \circ du \left(\frac{\partial}{\partial s} \right)$$
    $$du \left(\frac{\partial}{\partial t} \right) = \frac{\partial u}{\partial t} = J \circ \frac{\partial u}{\partial s}$$

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1. Gromov Nonsqueezing (1985): For $B(r) \subseteq \mathbb{C}^n$ the open ball of radius r and $Z(R) = D^2(R) \times \mathbb{C}^{n-1} \subseteq \mathbb{C}^n$, $R > 0$ (a cylinder).
   Theorem: There exists a symplectic embedding $\varphi : B(r) \hookrightarrow Z(R)$ iff $r \leq R$ and $\varphi^* \omega_0 = \omega_0$.
   Note: $\omega_0 = \sum_i dx_i \wedge dy_i$ where $\mathbb{C}^n$ is thought of as a $2n$-real dimensional manifold parameterized by $(x_1, y_1, \ldots, x_n, y_n)$.

2. Morally, whenever a symplectic structure is more rigid than differential structure, it's because there's a holomorphic curve obstructing the equivalence of the two

3. Proof of theorem: Assume that φ exists. Let $\epsilon > 0$, then

   $$\varphi(B(r - \epsilon)) \subseteq D^2 \times [-A, A]^{2n-2}$$

   because the image of the closed ball $\varphi(\overline{B(r - \epsilon)})$ will be compact (compactness preserved by symplcetic embeddings) and hence the image of the interior will be contained in a compact set.

   Moreover, we can embed $D(R) \hookrightarrow S(\pi R^2 + \epsilon)$ where $S^2(\pi R^2 + \epsilon)$ means a two-sphere of area $\pi R^2 + \epsilon$. This means we have the following embedding

   $$\tilde{\varphi} : B(r - \epsilon) \hookrightarrow (S^2(\pi R^2 + \epsilon) \times T^{2n-2}, \sigma \oplus \tau)$$

4. **Theorem:** For (M, τ) symplectic dimension $2n - 2$. If there exists a symplectic embedding

   $$\varphi : B(r) \hookrightarrow S^2(a) \times M$$

   where the image has symplectic form $\sigma \oplus \tau$. Then $\pi r^2 \leq a$. Here $S^2(a)$ again means a sphere of area a.

5. **Corollary:** (non-squeezing) If we have the above theorem, then

$$\pi(r-\epsilon)^2 \leq \pi R^2 + \epsilon \qquad \forall \epsilon > 0 \implies r \leq R$$

6. This concludes the proof of the non-squeezing theorem, once we prove our stronger theorem

7. **Definition:** For (Σ, j) a riemann surface and (X, J) an almost complex manifold, with $J : TX \to TX$ and $J^2 = -Id$, then $u : \Sigma \to M$ such that $du \circ j = J \circ du$ is called pseudoholomorphic, (j, J)-holomorphic, or J-holomorphic.

8. In the above, the point is that in local charts the equation $du \circ j = J \circ du$ amounts to the cauchy Riemann equations

9. **Lemma:** (Main Lemma) For $\pi_2(M) = 0$, there exists an ω-compatible almost complex structure J on $S^2 \times M$ such that $\varphi^* J = i$ on $B(r)$ and a J-holomorphic curve

$$u : \mathbb{C}P^1 \to S^2 \times M$$

   s.t.

   (a) $[u] = [S^2 \times \{\text{pt}\}] \in H_2(S^2 \times M)$
   (b) $\varphi(0) \in \text{Im}(u)$

10. **Lemma:** (Monotonicity Lemma) (from minimal surface theory) For $\tilde{u} : (\Sigma, j) \to (B(r), i)$ nonconstant proper pseudoholomorphic curve and $0 \in \text{Im}(\tilde{u})$ then $Area(\tilde{u}) = \int_\Sigma \tilde{u}^* \omega_0 \geq \pi r^2$, where πr^2 is the area of the intersection of any complex line and $B(r)$

11. With the monotonicity lemma (which we won't prove or verify the conditions needed to apply), we have $\pi r^2 \leq Area(\tilde{u}) \leq Area(u)$, where the second inequality follows from φ being a symplectic embedding.

    This secretly comes from the fact that u is J-holomorphic with $\omega(\cdot, J(\cdot)) = g(\cdot, \cdot)$ and

$$\int u^* \omega = \int |du|^2$$

    and so when we pull back by $\tilde{u}$ instead of u, we get the same integrand but over a smaller domain. From here, we have

$$a = \sigma([S^2]) = \omega([S^2 \times \text{pt}]) = \int u^* \omega$$

    and so we get out desired bound of $a \leq \pi r^2$.

12. Finally, to prove the main lemma:

$$\zeta = \{\omega - \text{compatible almost complex structures on } S^2 \times M\}(W^{k,2})$$

    For $J \in \zeta$, let

$$\tilde{M}_J = \left\{u : \mathbb{C}P^1 \to S^2 \times M \;\middle|\; u \text{ is J-holo}, \quad u(\infty) = \varphi(0), \quad [u] = [S^2 \times \text{pt}]\right\}$$

    Moreover

$$\tilde{M}_J = \overline{\partial}_J^{-1}(0) \subseteq B_J \xleftrightarrow{\overline{\partial}_J} \epsilon_J = \bigcup_u W^{k-1}(\Gamma(\overline{\text{Hom}}_{\mathbb{C}}(T\mathbb{C}P^1, u^* T(S^2 \times M))))$$

    Here B_J is a Banach manifold and is the same as $\overline{M}_J$ but we don't require that they're J-holomorphic and u is in $W^{k,p}$. Also

$$\overline{\partial}_J = J \circ du \circ j + du$$

$$\overline{\partial}_J^{(u)} = 0 \iff u \text{ is J-holo}$$

$$D_u \overline{\partial_J} : T_u B_J = \Gamma u^* T(S^2 \times M) \to (E_J)_u$$

    Now we use the implicit function theorem for Banach manifolds:

    Theorem: If $D_u \overline{\partial_J}$ is surjective for all $u \in \overline{\partial}_J^{-1}(0)$ then $\overline{M}_J$ is smooth manifold of dimension 4 via Riemann Roch

    Now define

$$M_J := \tilde{M}_J / Aut(\mathbb{C}P^1, \infty) \quad \text{which is of smooth dimension } = 4 - 4 = 0$$

    Since $[u] = [S^2 \times \text{pt}]$, M_J is compact.

13. Outline of the remainder of the proof
    (a) $J_0 = \iota_{\mathbb{C}P^1} \times \iota_{T^{2n-2}}$ is Fredholm regular (so the implicit function theorem applies). Can check that $|M_{J_0}| = 1$
    (b) There exists a J_1 Fredholm regular such that $\varphi^* J_1 = i$ (warning: it's very luck that J_1 exists due to choice of the homology class $[S^2 \times \text{pt}]$)
    (c) For a generic path J_t, $t \in [0,1]$ from $J_0 \to J_1$, $\overline{M} = \bigcup_{t\in[0,1]} \{t\} \times M_{J_t}$, this has the structure of a compact 1-dimensional manifold, with $\partial M = M_{J_0} \sqcup M_{J_1}$
    (d) Now M_{J_0} is a single point, and M_{J_1} may have many points, M is a 1-dimensional manifold with M_{J_0}, M_{J_1} as a boundary, and hence there is a cobordism between the two. Because M_{J_0} has one point, then M_{J_1} is nonempty. So we get **some** holomorphic curve in M_{J_1}, which is what we wanted

11/18

1. Recap of Gromov Non-squeezing (one last summary):
    (a) $B(r) \subseteq \mathbb{R}^4$ and $Z(R) = D(R) \times \mathbb{R}^2 \subseteq \mathbb{R}^4$
    (b) **Theorem:** $B(r)$ symplectically embeds into $Z(R)$ iff $r \leq R$

    Proof: (sketch) Assume that $E : B(1) \hookrightarrow Z(R)$ then for any $\epsilon > 0$, there exists a symplectic embedding $\tilde{E} : B(1) \hookrightarrow (S^2(\pi R^2 + \epsilon)) \times T^2(A)$ for $A \gg 1$. Umut says this step isn't too bad and amounts to decomposing $B(1)$ into $D^2(1)$ times another part.

    Now we want to show that if $B(1) \hookrightarrow S^2(\alpha) \times T^2(A)$ when $A \gg a$, then area of a linear slice of $B(1)$ is less than a. This implies the claim because then we use

    $$\forall \epsilon > 0, \quad \pi < \pi R^2 + \epsilon \implies R \geq 1$$

    From here, the key points are:
    i. If $A > a$, then for any generic, tame, compatible J, for any point in $S^2(a) \times T^2(A)$, there exists a J-holomorphic curve $\mathbb{C}P^1 \to T^2(a)$ (here we think of $\mathbb{C}P^1$ as S^2 with the complex holomorphic structure), which is in the homology class of $S^2 \times \{\text{pt}\}$
    ii. Monotonicity: $B(1) \subseteq \mathbb{R}^4 = \mathbb{C}^2$ in a metric sense so consider a J-holomorphic curve, $u : \Sigma \to B(1)$ with $\partial\Sigma$ mapping to $\partial B(1)$ and passing through the origin. Here, Σ is a compact surface with boundary.

       Then, $area(u) = \int u^*\omega \geq \pi$. This is not true in general, but is true for J-holomorphic curves. The examples of holomorphic curves are given by graphs of polynomials $\mathbb{C} \to \mathbb{C}$

    From these two facts, assume that we have $F : B(1) \hookrightarrow S^2(a) \times T^2(A)$. Then the standard symplectic structure, J on $B(1)$ pushes forward to the image, and this is compatible because E is a symplectic embedding. Note: our monotonicity theorem is only true when we're considering $\mathbb{R}^4$ with the standard structure.

    Now we want to extend our compatible structure to a tame J on the whole space of $S^2 \times T^2$. As a result, $\omega(v, Jv) > 0$ for $v \neq 0$. Moreover

    $$a = \int u^*\omega > \int \left(u\Big|_{\text{the part mapping inside B}} \right)^* \omega$$

    where strict inequality comes from the fact that some portion of the curve leaves B. But we know that the right hand side integral is greater than π by monotonicity, and so $a > \pi$. ☐
2. Existence of J-holomorphic curves: through every point in $S^2(a) \times T^2(A)$ for generic J
    (a) Show that $S^2 \times T^2$ with the almost complex structure obtained by $S^2 = \mathbb{C}P^1$ and $T^2 = \mathbb{C}/\Omega$ (and taking the product of the two complex structures) counts as generic
    (b) Then show that the foliation persists under generic deformation of J
    (c) Warning: Notice that $\Sigma_g \times T^2$ (for $g \geq 1$) also is foliated by J-holomorphic curves in the split, almost complex structure. Yet, almost all of these curves just disappear under generic deformation of J
3. Symplectic reduction: (Lagrangian)

    (a) Recall: basic linear algebra, V is symplectic $Q \subseteq V \mapsto Q/Q \cap Q^c$ symplectic. Q is coisotropic and maps to Q/Q^c
    (b) Linear Algebra: V a symplectic vector space, $W \subseteq V$ coisotropic so that W/W^Ω is symplectic.
    (c) **Lemma:** : $L \subseteq V$ lagrangian. Then, the image of $L \cap W \mapsto W/W^\Omega$ is a lagrangian (no assumptions on $\dim L \cap W$).
    Proof: Let's denote the image $L_W \subseteq W/W^\Omega$. L_W is isotropic (obvious). We want $L_W^\Omega \subseteq L_W$ as well to show that L_W is lagrangian.
$$v \in W, \qquad [v] \in L_W^\Omega \iff \omega(v, \varphi) = 0, \quad \forall \varphi \in L \cap W \iff v \in (L \cap W)^\Omega$$
    We want to show $v \in (L \cap W) + W^\Omega$. Note that $(L \cap W)^\Omega = L^\Omega + W^\Omega = L + W^\Omega$. Now we take
$$v \in (L \cap W)^\Omega, \ \text{ s.t. } \ v = v_1 + v_2, \qquad v_1 \in L, \ v_2 \in W^\Omega$$
    Note $W^\Omega \subseteq W$ and $v_2 \in W$ so $v_1 = v - v_2 \in W$. ☐
    (d) If $L \pitchfork W$ (this is what I use for transverse, $\pitchfork$), i.e. $L + W = V$, W is $n + k$ dimension (so $n - k$ codimension), then $\dim L \cap W = k$ and $\dim W/W^\Omega = 2k$.
    The image of $L \cap W \to W/W^\Omega$ is isomorphic to $L \cap W/L \cap W^\Omega$, which has dimension k. This implies that $L \cap W^\Omega = 0$ by dimension counting.

11/20

1. Last time: V a symplectic vector space, $W \subseteq V$ coisotropic, $L \subseteq V$ Lagrangian
    (a) Then $L_W \subseteq W/W^\Omega$, where L_W is the image of $L \cap W \to W/W_\Omega$
    (b) If L transverse to W, then $L \cap W^\Omega = 0$, and $L \cap W \hookrightarrow W/W_\Omega$ and injection
2. Today: Symplectic vector bundles
    (a) Let $E \downarrow M$ be a symplcetic vector bundle and $Q \subseteq E$ a coisotropic subbundle
    (b) This implies we get a symplectic vector bundle
$$\begin{tikzcd} Q/Q^\Omega \arrow[d] \\ M \end{tikzcd}$$
    Let $\mathcal{L} \subseteq E$ a lagrangian subbundle
    (c) Assume: $\mathcal{L} \cap Q$ is constant rank (hence a subbundle). Then the image $\mathcal{L} \cap Q \to Q/Q^\Omega$ is also a subbundle
    (d) To see this, note that the image at a point is isomorphic to $L_x \cap Q_x/(L_x \cap Q_x^\Omega)$ where
$$\mathcal{L}_x \cap Q_x^\Omega = \mathcal{L}_x^\Omega \cap Q_x^\Omega = (\mathcal{L}_x + Q_x)^\Omega$$
    and the fibers of the last object forms a bundle if $\mathcal{L}_x \cap Q_x$ does. This implies that $\mathcal{L} \cap Q/\mathcal{L} \cap Q^\Omega$ forms a bundle.
    (e) Therefore, we obtain the reduced Lagrangian subbundle in Q/Q^Ω:
$$\mathcal{L} \xrightarrow{reduced} \mathcal{L}_Q$$
3. Manifold case
    (a) M a symplectic manifold, $C \subseteq M$ coisotropic, and $L \subseteq M$ Lagrangian
    (b) Assumption 1: C is foliated by the isotropic leaves. The leaf space, X, is a manifold, e.g. the leaves are the fibers of a submersion $C \to X$ = reduction of C. Umut says that this came from the construction of our foliation in Frobenius integrability. This means X has a natural symplectic structure

(c) Assumption 2: L should intersect C cleanly, i.e. $L\cap C$ is a submanifold and for any $a\in L\cap C$, $T_aL\cap C = T_aL\cap T_aC$. In particular, any $P\subseteq M$ intersects $P\subseteq M$ cleanly. Note this is slightly weaker than transverse

(d) We want to obtain a Lagrangian inside X. We have a smooth map, $g : L\cap C \hookrightarrow C\to X$. The differential $T_aL\cap T_aC \to T_aC/(T_aC)^{\Omega}$, has a lagrangian image. In particular, g has constant rank. Now by the constant rank theorem, the image is an immersed Lagrangian. But note, that g is not necessarily an immersion map itself, just that the image of g is the image of another immersion.

(e) This image is the reduced Lagrangian, L_X, where $L \pitchfork C$ implies $L\cap C\to X$ is an immersion itself.

(f) Warning: This is not an embedding because L can intersect a leaf more than once

4. Now we'll talk about **Weinstein's Symplectic Category**

   (a) **Definition:** A lagrangian correspondence (canonical relation) from (M,ω_M) to (N,ω_N) is just a lagrangian in the twisted product, $(M\times N, \pi_M^*\omega_M - \pi_N^*\omega_N) = M\tilde{\times}N$.

   (b) Ex: If $M\xrightarrow{\varphi}N$ is a symplectomorphism, then $graph(\varphi)\subseteq M\tilde{\times}N$ is a Lagrangian correspondence.

   (c) If G a Hamiltonian action on M with μ a moment map $M\to\mathfrak{g}^*$. Assume the fact $G \circlearrowright \mu^{-1}(0)$ is free. This implies that
   $$\mu^{-1}(0)\hookrightarrow M\tilde{\times}(M//G)(=\mu^{-1}(0)/G)$$
   here the map is $\mu^{-1}(0)\hookrightarrow M$, i.e. inject into the first component, and then $a\in\mu^{-1}(0)\mapsto [a]\in\mu^{-1}(0)/G$ in the second component. Note that $\mu^{-1}(0)$ is a coisotropic with leaves given by the orbits of G

   (d) Ex: $S^{2n-1}\to\mathbb{C}^n\times\mathbb{CP}^{n-1}$

   (e) If $P_1=\{\text{pt}\}$, then a Lagrangian correspondence $P_1\to P_2$ is just a lagrangian in P_2.

   (f) **Proposition:** : Lagrangian correspondences can be composed, $L_1\subseteq P_1\tilde{\times}P_2$, $L_2\subseteq P_2\tilde{\times}P_3$ gives rise to $L_1\circ L_2\subseteq P_1\tilde{\times}P_3$
   Proof: $L_1\times L_2\subseteq (P_1\tilde{\times}P_2)\times(P_2\tilde{\times}P_3)$, this is a Lagrangian submanifold. $\Delta_{P_2}\subseteq -(P_2\tilde{\times}P_2)$ is the diagonal, and
   $$P_1\times\Delta_{P_2}\tilde{\times}P_3\subseteq(P_1\tilde{\times}P_2)\times(P_2\tilde{\times}P_3)$$
   is coisotropic.
   The reduction of $P_1\times\Delta_{P_2}\tilde{\times}P_3$ is just $P_1\tilde{\times}P_3$, if the intersection is clean, then we're good.

11/22

1. Let $(T^*M, d\lambda_{taut})$. Then $L\subseteq T^*M$ is called exact if $\lambda\Big|_L$ is exact

2. Ex: zero section of T^*M (in fact, $\lambda\Big|_{T_{Z_M}(T^*M)} = 0$ for $\lambda = pdq$)

3. $\varphi : T^*M\to T^*M$ Hamiltonian diffeomorphism, then $\varphi(Z_M)$ is also exact

4. Why? You can show this using a flux argument. Pick a Hamiltonian isotopy and compute integrals of λ over 1-cycles on $\varphi(Z_M)$ via stokes theorem

5. Nearby Lagrangian conjecture: Any compact exact Lagrangian in T^*M is Hamiltonian isotopic to the zero section

6. Generating functions (families):

   (a) Let $\pi : E\to M$ be a submersion (e.g. $M\times\mathbb{R}\xrightarrow{N}M$, OR a vector bundle over M), and $S: E\to\mathbb{R}$ a function

   (b) Inside T^*E, we have a coisotropic,
   $$N := \{(e,\xi)\mid \xi\Big|_{\ker d\pi_e} = 0\}$$
   Note that $\ker d\pi_e$ represents the collection of vertical vectors

(c) Note that we have

$$N \hookrightarrow T^*E \xrightarrow{d\pi} T^*M$$

it is easy to see that $N \subseteq T^*E$ is a coisotropic with isotropic leaves being the fibers of the composite map (i.e. the one from $N \to T^*M$)

(d) If $graph(dS) \subseteq T^*E$, then we can consider the reduced lagrangian:

$$\iota_S : N \cap graph(dS) \to T^*M$$

is a lagrangian immersion where M is the reduction of N. Denote $N \cap graph(dS) =: \Sigma_S$

(e) This is called "the lagrangian generated by S

(f) We further have

$$\Sigma_S \subseteq T^*E \to E \text{ s.t. } \Sigma_S \xrightarrow{inject} E$$

with this, it's best to think of $\Sigma_S \subseteq E$, and it corresponds to "vertical critical points of S." Then ι_S is defined by recording the horizontal part of dS at a point of $\Sigma_S \subseteq E$.

(g) In coordinates,

$$\begin{tikzcd}\mathbb{R}^{n+k} \arrow[d] \\ \mathbb{R}^n\end{tikzcd}$$

such that $(\xi, \eta) \mapsto \eta$ such that

$$\Sigma_S := \{(\xi, \eta) \mid \frac{\partial S}{\partial \xi}(\xi, \eta) = 0\}$$

$$\iota_S : \Sigma_S \to T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$$

$$(\xi, \eta) \mapsto (\frac{\partial S}{\partial \eta}(\xi, \eta), \eta)$$

this is always an immersion because lagrangian reduction always yields an immersion. It need not be an embedding

(h) Ex: $S : \mathbb{R}_a \times \mathbb{R}_x \to \mathbb{R}$ such that $\mathbb{R}_x = M$. We have the map

$$(a, x) \mapsto \frac{a^3}{3} + (x^2 - 1)a$$

Then

$$\Sigma_S = \{(a, x) \mid a^2 + (x^2 - 1) = 0\} \subseteq \mathbb{R}^2_{a,x}$$

and

$$\iota_s : \Sigma_S \to \mathbb{R}^2 \text{ s.t. } (a, x) \mapsto (2xa, x)$$

this is a mapping of the unit circle into $\mathbb{R}^2$ such that the image looks like a figure 8. Furthere more, in this case, our ambient $E = \mathbb{R}_a \times \mathbb{R}_x$ so

$$N = \{(e, \xi) \mid \xi|_{\ker d\pi} = 0\} = \{(a, x, p_a, p_x) \mid p_a = 0\}$$

where $T^*E = \{(a, x, p_x, p_a)\}$

(i) **Lemma:** $\iota_s^* \lambda_{taut} = d(S\Big|_{\Sigma_S})$ so ι_S is an exact lagrangian immersion.
Proof: this is trivial to check in coordinates

(j) **Theorem:** Every Lagrangian $L \subseteq T^*M$ (for L and M compact) that is Hamiltonian isotopic to the zero section admits a generating function. In fact, this generating function can be chosen so that $S : \mathbb{R}^N \times M \to \mathbb{R}$ and S is quadratic at infinity in the $\mathbb{R}^n$ direction.
Proof: The following construction does not really work, but gives the right idea:
Let $\gamma : [0, 1] \to T^*M$ and $\gamma(t) = (p(t), q(t))$, fix a time-dependent hamiltonian $H : T^*M \times [0, 1] \to \mathbb{R}$. Then the action of γ is defined as

$$A(\gamma) := \int_{[0,1]} \gamma^* \lambda_{taut} + \int_0^1 H_t(\gamma(t))dt = \int_0^1 p(t)\dot{q}(t) + H_t(p(t), q(t))$$

Claim: consider $E = \{\gamma : [0,1] \to M \mid \gamma(0) \in Z_M\}$, i.e. all paths which start at the zero section. Moreover, the projection

$$p : E \to M \;\text{ s.t. }\; \gamma \mapsto \pi(\gamma(1))$$

for $\pi : T^*M \to M$. Then A "generates" the time 1 image of Z_M under the flows of H_t. This is almost our generating function, but E is infinite dimensional, so we look at critical points of the action. ☐

(k) Exercise: $\gamma(t) = (p(t), q(t)) \mapsto (p(t) + \delta p(t), q(t) + \delta q(t))$. Then

$$A(\gamma) \to A(\gamma) + \delta A$$

where

$$\delta A = \int \delta p(t) \left(\dot{q}(t) + \frac{\partial H_t}{\partial p}(p(t), q(t)) \right)$$
$$- \int \delta q(t)(\dot{p}(t) - \frac{\partial H_t}{\partial q}(p(t), q(t))) + p(1)\delta q(1) - p(0)\delta q(0)$$

note that when we look at vertical variation, we get $\delta q(1) = 0$ and we know $p(0) = 0$, so the last two terms dies. This vanishes for all small variation when γ satisfies Hamilton's equations, i.e. $\partial_p H_t = -\dot{q}$ and $\partial_q H_t = \dot{p}$.

(l) $\gamma \in \Sigma_S$ iff γ satisfies Hamilton's equations. Moreover, if so, then

$$\Sigma_S \to T^*M \;\text{ s.t. }\; \gamma \mapsto (p(1), q(1))$$

the image if precisle the image of Z_M under the Hamiltonian flow under H_t.

(m) In order to turn this into a real proof, we need to discretize H_t