Symplectic Geometry Seminar Talk: Sobolev Embedding Theorem

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Oct. 11th, 2019

Things to talk about

1. Weak Derivatives

   (a) Preliminary definitions: for $U \subseteq \mathbb{R}^n$ open, we define

   $$D(U) := C_c^\infty(U, \mathbb{C}) = \{\text{smooth functions with compact support lying in } U\}$$

   (b) $D(U)$ has a standard convergence topology, i.e. $\{\varphi_k\} \overset{D(U)}{\to} \varphi \iff \lim_{k\to\infty} ||\partial^\alpha \varphi_k - \partial^\alpha \varphi||_\infty = 0$.

   (c) $D'(U) := (D(U))^*$ is the space of distributions such that $T \in D'(U)$ and $\varphi \in D(U)$ then we denote $T(\varphi)$ as $\langle T, \varphi\rangle$ or (T, φ).

   (d) **Remark:** the above notation arises because if we have any $f \in L^1_{\text{loc}}(U)$ then we have a natural distribution

   $$T_f \text{ s.t. } \forall \varphi \in D(U), \qquad T_f(\varphi) = \int_U f\varphi$$

   which is well defined because remember that φ has compact support.

   (e) **Remark:** In some sense, integrating φ against an integrable function is the normal case. This is because $\{T_\varphi\}_{\varphi \in D(U)}$ is dense in $D'(U)$, i.e. for any $T \in D(U)$, there exists a sequence $\{T_n\} = \{T_{f_n}\}$ such that $f_n \in D(U)$ and

   $$\forall \varphi \in D(U) \qquad \lim_{n\to\infty} (T_n, \varphi) = (T, \varphi)$$

   One direct way to show this is $T_n := T * \rho_n$ where ρ_n is a function which approaches δ_0 in the distributional sense

   (f) Ex: Consider

   $$\delta_0 \in D'(U) \text{ s.t. } \delta_0(\varphi) = \varphi(0)$$

   This is clearly linear. Boundedness follows, but we'd have to delve into the topology of $C^\infty(U)$ which is some locally convex topology defined by the collection of semi-norms

   $$\rho_{K,m} : C^\infty(U) \to \mathbb{R} \text{ s.t. } \rho_{K,m} \sup_{\substack{x\in K \\ |\alpha|\le m}} |\partial^\alpha \varphi(x)|$$

   for K a compact subset of U and $m \geq 0$.

   (g) For now, I just note that T is bounded if there exists an $m \geq 0$ such that

   $$T : D(U) \to \mathbb{C} \quad \text{a linear operator is bounded} \iff \forall K \subseteq U \text{ compact}$$
   $$\exists C_K \text{ s.t. } \forall \varphi \in D(U) \text{ s.t. } supp(\varphi) \subseteq K, \qquad |T(\varphi)| \leq C_K \sup_{\substack{x\in K \\ |\alpha|\le m}} |\partial^\alpha \varphi(x)|$$

   (h) We can differentiate distributions to get $\partial_j T$ by

   $$(\partial_j T, \varphi) = -(T, \partial_j \varphi)$$

   If we have that

   $$(\partial_j T, \varphi) = -(T, \partial_j \varphi) = (g, \varphi)$$

   for some g locally integrable, then we say that g **is the weak derivative of** T

(i) Here is a fun example which makes high school me happy. Let $f(x) = \ln|x|$, then f is locally integrable because its bounded outside of $(-\epsilon, \epsilon)$ but also you can integrate it on $(-\epsilon, \epsilon)$. The weak derivative is then

$$(\frac{d}{dx}f, \varphi) := -(\varphi', f) = -\int_{\mathbb{R}} \varphi' \ln(|x|)dx$$

φ has compact support so this is well defined. In particular, we can write

$$\int_{\mathbb{R}} \varphi' \ln(|x|)dx = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \varphi' \ln(|x|)dx$$

Now we perform integration by parts

$$\int_{x>\epsilon} \varphi' \ln(|x|)dx = \varphi(x)\ln(|x|)\Big|_{\epsilon}^{\infty} - \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}dx$$

and

$$\int_{x<-\epsilon} \varphi' \ln(|x|)dx = \varphi(x)\ln(|x|)\Big|_{-\infty}^{-\epsilon} - \int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x}dx$$

Now note that φ has compact support so

$$\int_{|x|>\epsilon} \varphi' \ln(|x|)dx = \ln(\epsilon)[\varphi(\epsilon) - \varphi(-\epsilon)] - \int_{|x|>\epsilon} \frac{\varphi(x)}{x}dx$$

But now we write

$$\varphi(x) = \varphi(0) + xg(x) \implies \varphi(\epsilon) - \varphi(-\epsilon) = \epsilon[g(\epsilon) - g(-\epsilon)]$$

for g smooth. But note that $\lim_{\epsilon\to 0}\epsilon\ln(\epsilon) = 0$. And thus we have

$$(f', \varphi) = \text{P.I.}\int_{\mathbb{R}} \frac{\varphi(x)}{x} = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{\varphi(x)}{x}dx$$

(j) I think this is cool because when I was younger I saw stuff like

$$\int_{\mathbb{R}} \frac{\cos(x)}{x}dx$$

and I was like "EZ" it's zero because its an odd function. But my teacher said its not defined because it's not absolutely integrable. However, note that

$$\lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{\varphi(x)}{x}dx = \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{\varphi(x) - \varphi(0)}{x}dx$$

and so this is all nice and well defined.

(k) **Remark:** We can define weak derivatives in the context of higher order partial differential operators: suppose we have

$$L\Big|_x = \sum_{|\alpha|\le m} c_\alpha(x)\partial^\alpha$$

for $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index, then we say that $Lf = g$ **weakly** if

$$\forall\varphi \in C_c^\infty(U), \qquad \int_U g\varphi dx = \int_U fL^*\varphi dx$$

where L^* is the formal adjoint given by

$$L^*\varphi = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha(c_\alpha(x)\phi)$$

2. Sobolev space $\to$ What is this?

(a) Staying with $U \subseteq \mathbb{R}^n$, we define

$$W^{0,p}(U) = L^p(U) \qquad W^{k,p} = \{f \mid \partial_i f \text{ exists and } \partial_i f \in W^{k-1,p}(U)\} = \{f \mid f \in L^p(U) \text{ and } \forall |\alpha| \leq k\}$$

with the norm

$$||f||_{W^{k,p}} = \sum_{|\alpha| \leq m} ||\partial^\alpha f||_p$$

(b) Some nice properties of Sobolev spaces:

i. Sobolev spaces are Banach spaces

ii. They are reflexive for all k and $1 < p < \infty$

iii. For $p = 2$, they form a Hilbert space with inner product

$$\langle u, v \rangle_k = \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \leq k} \partial^\alpha u \cdot \partial^\alpha v \right) dx$$

iv. $C_C^\infty(U)$ is dense in $W^{k,p}(U)$ for all k and p

(c) For example, define

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < R \end{cases} \quad \Longrightarrow \in W^{1,p}(\mathbb{R})$$

(d) Why do we care?

i. **Definition:** Suppose (Σ, j) is a Riemann surface (complex manifold of dimension one) with almost complex structure j (hey look I know what that means) and (M, J) is an almost complex manifold with almost complex structure J. A smooth map $u : \Sigma \to M$ is **J-Holomorphic** if its differential at every point is complex-linear, i.e.

$$Tu \circ j = J \circ Tu$$

The above only makes sense if $u \in C^1$, or more generally when $u \in W^{1,p}$ (apparently)

ii. Often in symplectic geometry, we have these "J-holomorphic maps" $u : \Sigma \to M$ which might be smooth or might not. When we look at integrable maps, i.e. the complex structure arises from multiplication by i after pushing forward through the coordinate charts, we can choose a coordinate basis and get that $J = i$ so that it is of course smooth.

iii. In the non-integrable case, no such nice expression exists, and thus we have fewer smoothness assumptions on J. In this case, the a priori lack of regularity says that we need to work in Sobolev spaces to extract information about u

iv. Sobolev spaces also show up everywhere, and in particular, we can convert weak derivatives into actual regularity. The most notable result is the Sobolev embedding theorem

3. Sobolev Embedding Theorem

(a) **Definition:** A compact operator $T : X \to Y$ sends bounded sets to precompact sets, i.e. for $B \subseteq X$ s.t. $\sup_{b \in B} ||b|| < R$ then $\overline{T(B)}$ is compact. Equivalently we have that T sends bounded sequences to sequences with a weakly convergent subsequence.

(b) **Definition:** The **STRENGTH** of a sobolev space, $W^{k,p}(\mathbb{R}^n)$, is defined to be

$$\sigma_N(k, p) = k - N/p$$

this value is a measure of how big a sobolev space is: the larger the **STRENGTH**, the more regular the functions are, and thus the space has fewer total functions.

(c) Strength is derived from a scaling argument, i.e. suppose that we want to bound

$$||\partial^\alpha u(\lambda \cdot)||_p \leq A ||\partial^\alpha u||_p$$

for some scalar λ, then note that

$$||\partial^\alpha u(\lambda\cdot)||_p = \left(\int_{\mathbb{R}^N} |\partial^\alpha u|^p\right)^{1/p} \implies$$

$$||\partial^\alpha u(\lambda\cdot)||_p = \left(\int_{\mathbb{R}^N} |\partial^\alpha u(\lambda\cdot)|^p dx\right)^{1/p} = \left(\frac{1}{|\lambda|^N}\int_{\mathbb{R}^N} |\lambda^{|\alpha|}(\partial^\alpha u)(\lambda\cdot)|^p |\lambda|^N dx\right)^{1/p}$$
$$= \left(|\lambda|^{|\alpha|p-N}||\partial^\alpha u||_p^p\right)^{1/p} = |\lambda|^{|\alpha|-N/p}||\partial^\alpha u||_p$$

For $|\alpha| = k$, this is exactly the strength. For some reason or another, mathematicians care about this top term in the sobolev norm, which is the quantity above.

(d) **Theorem:** (Sobolev Embedding) For $W^{k,p}(\mathbb{R}^N)$ and $W^{m,q}(\mathbb{R}^N)$ such that

$$\sigma_N(k,p) = \sigma_N(m,q) < 0 \qquad \text{and } k > m$$

then

$$\iota : W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{m,q}(\mathbb{R}^n)$$

is continuous, i.e.

$$||f||_{W^{m,q}} \leq C(N,k,m,p,q)||f||_{W^{k,p}}$$

Proof: (sketch) We prove a lemma

Lemma 0.1. *For $N \geq 2$ and $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. Then define*

$$\xi^i = (x_1, \ldots, \hat{x}_i, \ldots, x_n) \in \mathbb{R}^{N-1}$$

and

$$f(x) = \prod_{i=1}^N f_i(\xi^i)$$

then $f \in L^1(\mathbb{R}^N)$ with

$$||f||_1 \leq \prod_{i=1}^N ||f_i||_{N-1}$$

With the above lemma, we prove an inequality of the form

$$||u||_{N/(N-1)} \leq C||du||_1 \text{ s.t. } |du|^2 = |\partial_1 u|^2 + \cdots + |\partial_N u|^2$$

for smooth functions of compact support. Then using density of smooth functions in $W^{1,1}(\mathbb{R}^N)$, we get our desired result for $k = 1$, $p = 1$, $m = 0$, and $q = N/(N-1)$.

To go from here to the general case of (k,p,m,q) is not entirely clear to me but I believe it uses an interpolation inequality of the form

(e) **Remark:** that most of these results can be extended $W^{k,p}(U)$ for $U \subseteq \mathbb{R}^N$, when U has a nice boundary, i.e. either C^1 or lipschitz.

(f) **Corollary:** When $k = 1$ and $\ell = 0$, we have that

$$W^{1,p}(U) \hookrightarrow L^{p^*}(U) \text{ s.t. } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

(g) **Theorem:** (Rellich-Kondrachov) For $W^{k,p}(\mathbb{R}^n)$ and $W^{m,q}(\mathbb{R}^n)$, if $k > m$ and $0 > \sigma_N(k,p) > \sigma_N(m,q)$, then we have

$$\iota : W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{m,q}(\mathbb{R}^n) \text{ is compact}$$

The inclusion $W^{k,p}(U) \hookrightarrow W^{k-1,p}(U)$ is also compact. **which Dylan asked me to say**

Morally this should make sense because we have an extra derivative to peel off and get some equicontinuity bound so that we can Arzela-Ascoli this bad boy and get that our embedding is compact.

Proof: (sketch) We'll sketch for $k = 1$ and hence $m = 0$, which forces

$$q < \left(p^{-1} - N^{-1}\right)^{-1}$$

We first show that for

$$\tau_h : L^q(\mathbb{R}^N) \to L^q(\mathbb{R}^n) \text{ s.t. } \tau_h(f)(x) = f(x+h)$$

we get the following bound

$$||\tau_h u - u||_{q,B_r} \leq C|h|^\alpha ||u||_{W^{1,p}(\mathbb{R}^N)}$$

remember, for $k = 1$. Then we use an interpolation equality for $p^* = Np/(N-p)$ and α such that

$$\frac{1}{q} = \alpha + \frac{1-\alpha}{p}$$

the inequality is of the form

$$||f||_{L^q(B_R)} \leq ||f||^\alpha_{L^1(B_R)} ||f||^{1-\alpha}_{L^{p^*}(B_R)}$$
$$\implies ||\tau_h u - u||_q \leq ||\tau_h u - u||_1^\alpha ||f||^{1-\alpha}_{L^{p^*}} \leq C|h|^\alpha ||u||_{W^{1,p}}$$

having used our first bound and one of the consequences of the Sobolev embedding theorem.
Now consider $S = \{u_n\} \subseteq W^{1,P}(\mathbb{R}^n)$ a bounded sequence of smooth functions with support lying in B_R. We want to prove that it has a convergent subsequence in L^q.

The idea from here is that we form

$$S_\delta = \{u_\delta := \rho_\delta * u \mid u \in S\}$$

i.e. we convolve our functions by ρ_δ where ρ is a bump function on $B_1(0)$ and $\rho_\delta = \frac{1}{\delta^N}\rho(x/\delta)$, and then we show that for each δ, we get a convergent subsequence, and then we take an appropriate diagonalizing subsubsequence.

From the sobolev embedding theorem and properties of convolutions, we can show that S_δ is equicontinuous and pointwise bounded. By Arzela-Ascoli, S_δ is precompact in C^0 (i.e. has compact closure), and hence, we get

$$\forall \delta > 0 \qquad \exists \{u_{n_k}\}_{k=1}^\infty \text{ s.t. } \lim_{k,\ell\to\infty} ||u_{n_k} * \rho_\delta - u_{n_\ell} * \rho_\delta|| = 0$$

Now let $\epsilon_k = 1/k$. For ϵ_i find a
It takes a bit of work to show, but for S bounded in $W^{k,p}$, the uniform bound tells us that

$$\forall \epsilon > 0, \qquad \exists \delta \text{ s.t. } \forall u \in S, \qquad ||u * \rho_r - u||_{q,B_R} < \epsilon \qquad \forall 0 \leq r \leq \delta$$

i.e. we get δ uniform in S. From here, let $\epsilon_m = 1/m$. For ϵ_m find a δ_m such that the above bound holds. Let $\{u_{n_i,1}\}$ be the sequence of functions such that $\{u_{n_i,1} * \rho_{\delta_1}\}$ is the convergent subsequence given by Arzela Ascoli. Set $f_1 = u_{n_1,1}$. Now find a subsequence of this, call it $\{u_{n_i,2}\}$ such that $\{u_{n_i,2} * \rho_{\delta_2}\}$ is convergent by Arzela Ascoli. Define $f_2 = u_{n_2,2}$, and repeat this to define f_m.
Now note that for $j > m \geq N$, we have

$$||f_m - f_j|| \leq ||f_m - f_m * \rho_{\delta_N}||_q + ||f_j - f_j * \rho_{\delta_N}||_q + ||f_m * \rho_{\delta_N} - f_j * \rho_{\delta_N}||_q$$

By our choice of indices, the first two terms are bounded above by $1/N$ independent of j and m. And if we let m, j large while leaving N fixed, we know that the last term goes to 0 as $f_m * \rho_{\delta_N}, f_j * \rho_{\delta_N} \to g_N$ for g_N continuous when N is fixed. Thus $\{f_m\}$ is our cauchy sequence of functions in L^q. □

(h) **Theorem:** (Also Rellich) If $U \subseteq \mathbb{R}^n$ is a bounded open domain with smooth boundary and $kp > n$, then there are natural continuous inclusion $W^{k+d,p}(U) \hookrightarrow C^d(U)$ for each integer $d \geq 0$. Moreover, these inclusions are compact!

(i) Motivation: Let's prove this on $(0,1)$. Consider $\{\varphi_n\}$ bounded in $W^{1,p}((0,1))$, then we want to show that some subsequence $\{\varphi_{n_k}\}$ converges in L^p. To see this, first replace each $\{\varphi_n\}$ with its continuous representative, i.e. we have

$$\varphi_n(x) = \tilde{\varphi}_n(x) \quad \text{a.e.} \quad \text{s.t. } \tilde{\varphi}_n(x) = \int_0^x \varphi'(x)dx$$

To see that this is really equal to φ a.e., note that φ and $\tilde{\varphi}$ have the same weak derivative (see p. 204-206 in Brezis). Then from the above, we have that

$$|\tilde{\varphi}_n(x) - \tilde{\varphi}_n(y)| \leq \int_x^y |\varphi_n'(x)|dx \leq ||\varphi_n'||_p |x-y|^{1/q} \leq K|x-y|^{1/q}$$

where $K \geq \sup_n ||\varphi_n||_{W^{1,p}}$. Now by arzela ascoli, there exists a uniformily convergent subsequence and so $\{\tilde{\varphi_{n_k}}\} \to \varphi_0(x)$ continuous.

4. Elliptic Bootstrapping

   (a) Idea: Suppose we have $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^k operator, and we have a C^1 solution to the ODE

   $$\dot{x} = F(x)$$

   we see that for $k \geq 1$, we have $\dot{x} = F(x)$ is at least C^1 as it is the composition of C^1 functions. This implies that $x = \int \dot{x}$ is actually C^2. If $k \geq 2$ as well then x will be C^3.

   (b) In general, we can upgrade our initial regularity of x being C^1 up to C^{k+1} regularity. This process of upgrading is known as "bootstrapping," which comes from the expression "lifting yourself up by your own bootstraps"

   (c) **Definition:** a linear differential operator L of order m on a domain $\Omega \subseteq \mathbb{R}^n$ to be of the form

   $$Lu = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u$$

   This operator is **elliptic** if we have that

   $$\forall x \in \Omega, \qquad \forall 0 \neq \xi \in \mathbb{R}^n, \quad \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0$$

   where we have $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}$.

   (d) We denote the principal symbol, σ_m^L is given by all the order m terms in the operator and operates on n-tuples

   $$\sigma_m^L(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$$

   Ellipticity then amounts to the principle symbol, $\sigma_m^L(x,\cdot)$ being non-zero away from 0

   (e) **Theorem:** (Powerful due to Morrey) Let L be an elliptic operator of order m of the above form. Then for every $k \geq 0$, there exists a constant c such that

   $$||u||_{W^{m+k,p}(B_r)} \leq K(r)\left(||Lu||_{W^{k,p}(r)} + ||u||_{L^p(B_r)}\right)$$

   where $K(r)$ is a constant depending on r and our elliptic operator

   (f) Ex: Consider the Laplacian

   $$\Delta = -\sum_{i=1}^n \partial_i^2 \implies \sigma_2^\Delta(\xi) = -\sum_{i=1}^n \xi_i^2 = -|\xi|^2$$

   so Δ is elliptic. In particular, let's say we have a harmonic function, i.e. one that satisfies $\Delta u = 0$. Then if $u \in L^p$, we have that u is smooth, as

   $$||u||_{W^{2+k,p}(B_r)} \leq K(r)\left(||Lu||_{W^{k,p}(r)} + ||u||_{L^p(B_r)}\right) \leq K(r)||u||_{L^p(B_r)}$$

   Taking k arbitrarily large and using the sobolev embedding theorem, we can get infinite regularity of u, i.e. u is smooth!

   (g) Cor: For $\Delta : W^{3,p} \to W^{1,p}$ then $\ker(\Delta)$ is finite dimensional. From the above, we have that

   $$\Delta u = 0 \implies ||u||_{W^{3,p}} \leq C||u||_p$$

   so if we consider a bounded sequence $\{u_n\} \subseteq \ker(\Delta)$, then we know that $B_{W^{3,p}} \hookrightarrow W^{2,p}$ compactly, and so there exists a convergent subsequence in $W^{2,p}$. In particular, we get that $||u_{n_k} - u_{n_\ell}||_p \to 0$, and by the above bound, this tells us that $||u_{n_k} - u_{n_\ell}||_{W^{3,p}} \to 0$. Thus the unit ball in $\ker(\Delta)$ is compact, so it must be finite dimensional