BANFF 2024: Determining the Metric from Minimal Surfaces in Asymptotically Hyperbolic Spaces

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Inverse Problems

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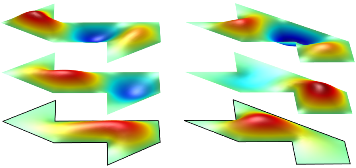
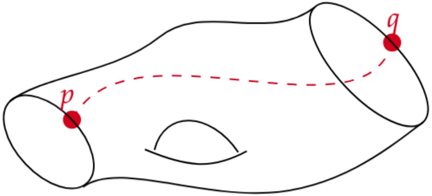


Inverse Problems: distances

Suppose M has boundary: does distance between any two points on the boundary determine g?



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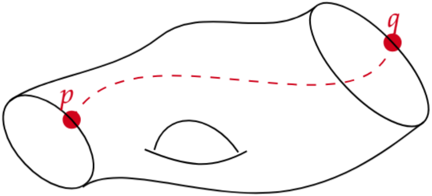

(yes, locally near the boundary) (Stefanov-Uhlmann-Vasy)

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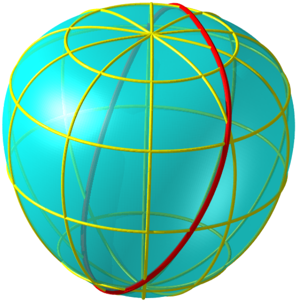

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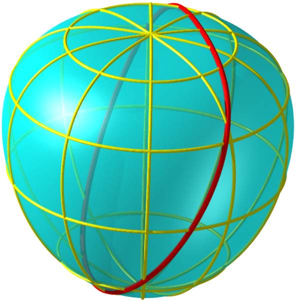
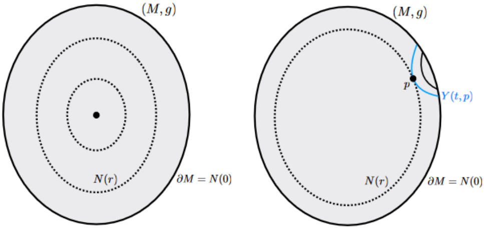


- Similar question for the "p-widths", $\omega_p(M, g)$ (MK '24, Ambrozio-Marques-Neves '24)

Inverse Problem with Minimal Surfaces: Alexakis-Balehowsky-Nachman



Figure

- (B^3, g) compact
- Suppose $\{Y_t\}$, foliation of minimal surfaces near ∂B.

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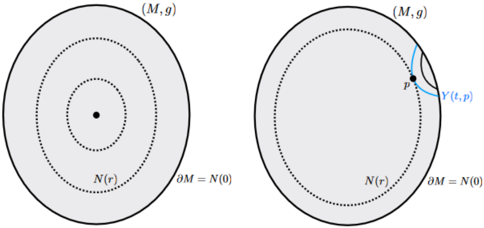

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- (B^3, g) compact
- Suppose $\{Y_t\}$, foliation of minimal surfaces near ∂B.
- Knowledge of $A(Y_t)$ for all t does determine g globally

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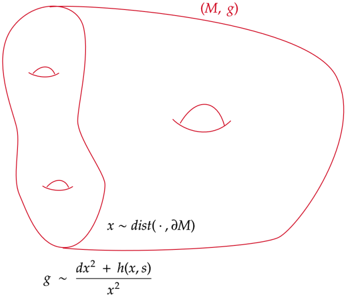

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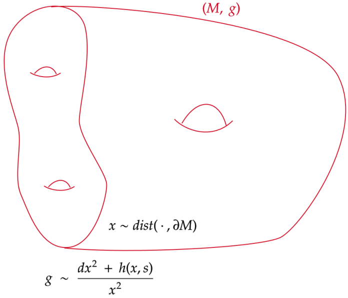

- Equivalent: "The D-to-N map for Plateau problem determines the metric" (Asymptotically hyperbolic Calderon problem?)

Conformally Compact Asymptotically Hyperbolic Metrics

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A **complete** manifold with **topological boundary**. Near boundary, sectional curvature tends to -1 (**asymptotically hyperbolic**)

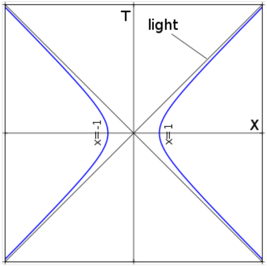

Figure: Path of particle with constant acceleration

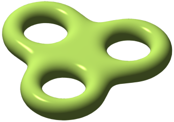

Figure: Noncompact Hyperbolic 3-manifold

"**CC AH**" = Conformally Compact Asymptotically Hyperbolic

CC AH Motivation: Poincaré-Einstein manifolds

- Given closed (N, h), conformal invariants of $[h]$?

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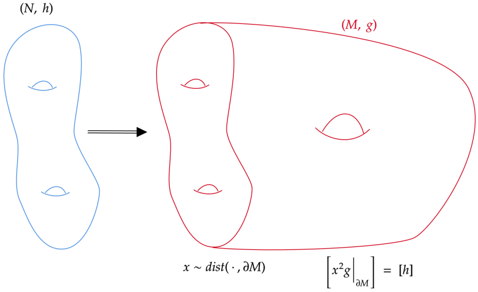

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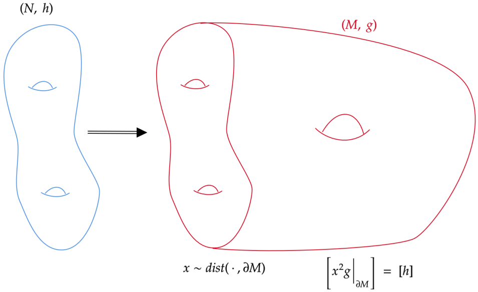

M is a "**Poincaré–Einstein**" manifold. (Given (N, h), $\exists (M, g)$?)

Poincaré-Einstein Manifolds

- Studied by Witten, Anderson, Qing, Chang, Yang, Gursky, Ozuch, Mazzeo, Pacard, Case, Tyrrell, Wang, Blitz, Waldron, McKeown, Wei, Saez, Curry, Hirachi, Lin, Ratzkin, Takeuchi, Yan, Matsumoto, Case, Han, Ge, Kuo, Wang, Kopinski among others

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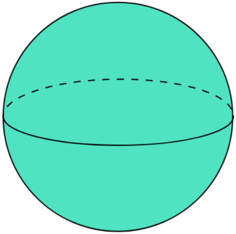
- Example of **Poincarè-Einstein** (PE) metric: AdS Schwarzschild, Hyperbolic Space

Formal Definitions CC AH Metrics



$$(M, g) = \left(B^3, \frac{4g_{euc}}{(1 - r^2)^2}\right)$$

$$x = \frac{(1 - r)}{(1 + r)}$$

Figure

- (M, g) complete with boundary

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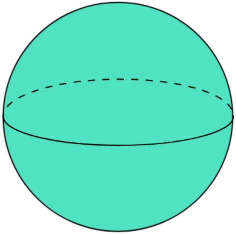

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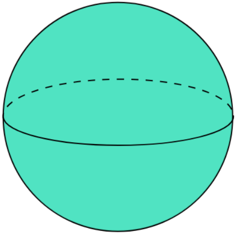

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- g is **conformally compact** (CC) if $\exists x : \overline{M} \to \mathbb{R}^{\geq 0}$ with
    - $x\Big|_{\partial M} = 0$
    - $\overline{g} = x^2 g$ metric on $\overline{M}$
    - $||dx||^2_{\overline{g}} = 1$ on ngbd of ∂M (special)

x - "(special) boundary defining function (bdf)"

Conformally Compact Asymptotically Hyperbolic Metrics

- g is **Asymptotically Hyperbolic** (AH) if

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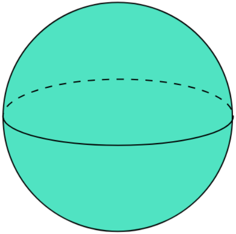
- g is CC, define the **conformal infinity**:

$$c(g) = \left[x^2 g\Big|_{\partial M}\right] = [\omega_0]$$



$$g = \frac{4g_{euc}}{(1-r^2)^2} = \frac{4dr^2 + 4r^2 g_{S^2}}{(1-r^2)^2}$$

$$x = \frac{(1-r)}{(1+r)} \qquad g = \frac{dx^2 + \frac{(1-x^2)^2}{4} g_{S^2}}{x^2}$$

$$\left[x^2 g\Big|_{x=0}\right] = [g_{S^2}]$$

Figure

Renormalized Volume

- Suppose (M^{n+1}, g) CC AH with g **even** up to order m (even)

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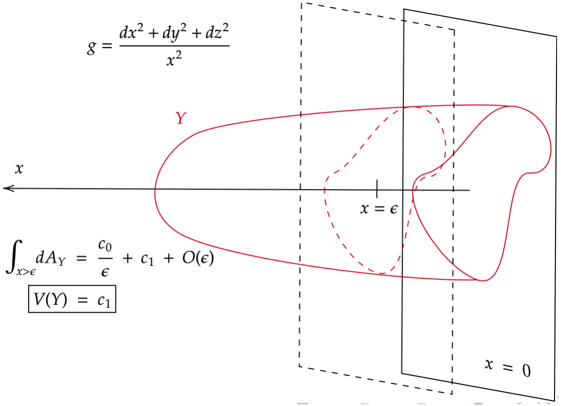

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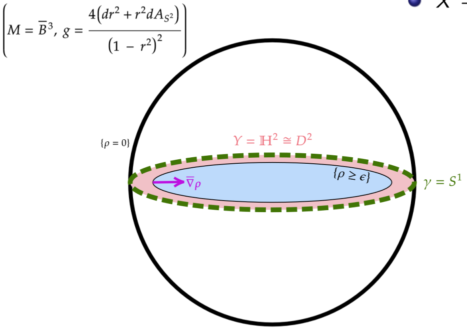

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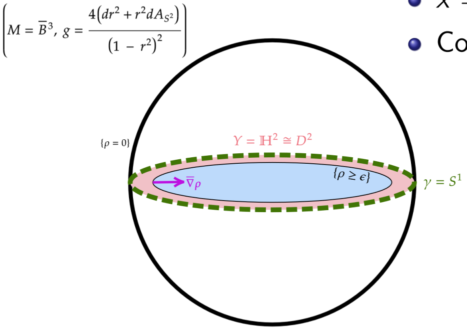

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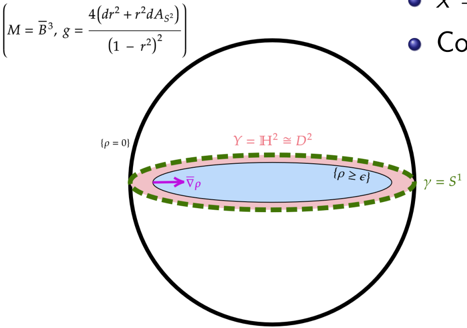

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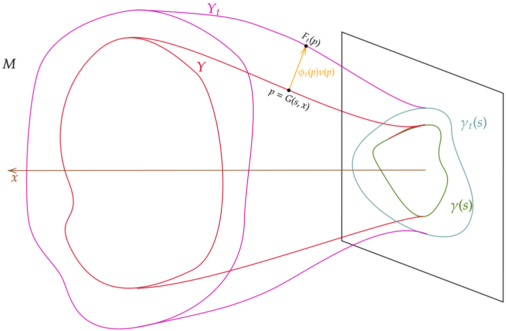

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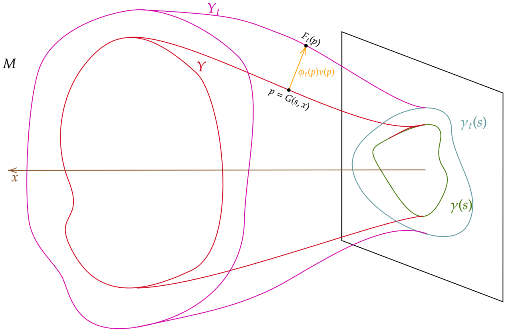

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- In 2021, MK computed variations of renormalized volume and proved regularity of minimal submanifolds (high codim)
- Previous work on existence/uniqueness in codim 1: Lin (89), Guan, Spruck, Szapiel (09), Tonegawa (96)

Significance of Renormalized Volume

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- Bernstein (2020) introduced an analogy of entropy for surfaces in hyperbolic space, $\lambda_{\mathbb{H}}(Y)$.

Theorem (Bernstein 2021)

Let Y^2 a minimal surface in $\mathbb{H}^n$ with asymptotic boundary $\partial Y = Y \cap \partial \mathbb{H}^n$. Then $-2\pi\lambda_{\mathbb{H}}(Y) \geq \mathcal{V}(Y)$

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Theorem (Anderson 82', Bernstein 21', Qing 99')

Let $Y^m \subseteq \mathbb{H}^n$ minimal with asymptotic boundary. For any $p \in \mathbb{H}^n$,

$$\psi_p(r) = \frac{Vol^m(Y \cap B_p(r))}{Vol^m(B_p^m(r))}$$

is monotone non-decreasing. When $m = 2$

$$\lim_{r \to \infty} \psi_p(r) \leq -\mathcal{V}(Y)$$

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Physics: Type IIB String theory (Witten). Renormalized Area of Y represents the entanglement entropy of the region it bounds (Ryu–Takayanagi)

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Proposition (Alexakis, Mazzeo 2008)

Suppose (M^{n+1}, g) *Poincare Einstein (PE). For* $Y^2 \subset M^{n+1}$ *with* $\partial Y = \gamma \subseteq \partial M^{n+1}$ *and* Y *intersecting the boundary orthogonally,*

$$\mathcal{V}(Y) = -2\pi\chi(Y) + \frac{1}{2}\int_Y 2|H|^2 - |\hat{k}|^2 dA_Y + \int_Y tr_Y(W_M) dA_Y$$

Rigidity of $\mathcal{V}$

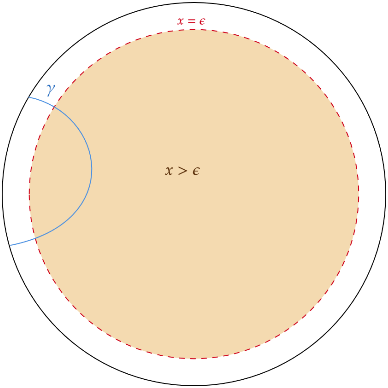
- When Y^2 minimal in $\mathbb{H}^{n+1}$, have

$$\begin{aligned}\mathcal{V}(Y) &= -2\pi\chi(Y) - \frac{1}{2}\int_Y |\hat{k}|^2 dA \\ \mathcal{V}(Y) &\leq -2\pi\end{aligned}$$

with equality if and only if $\chi(Y) = 1$ and $\hat{k} = 0$, i.e. $Y = HS^2$ and $\gamma = S^1$ (Non-trivial, due to Bernstein).

- **Question**: How can we use **rigidity** and **variations** of renormalized area to determine the metric?

Prior work: Graham-Guillarmou-Stefanov-Uhlmann



Figure

- **Renormalized Length functional** on geodesics in AH spaces

$$\mathrm{RL}(\gamma) = \lim_{\epsilon \to 0} \left[\ell_g(\gamma \cap \{x > \epsilon\}) + 2 \log(\epsilon) \right]$$

- Knowledge of $\mathrm{RL}(\gamma)$ determines expansion of metric near boundary

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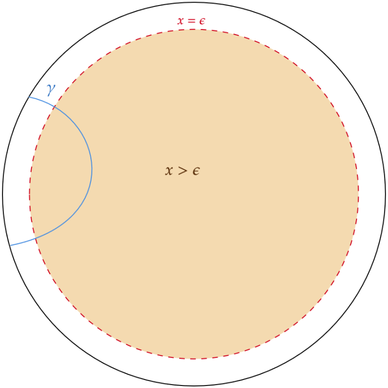

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- (**Remark**: RL(γ) not conformally invariant)

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We extend metric determination results to Renormalized Area of minimal submanifolds

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Theorem (MK, 2024)

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Theorem (MK, 2024)

Suppose (M^{n+1}, g) is CCAH, even to order 2, and $c(g)$ is known. Fix $\omega_0 \in c(g)$. Then knowledge of $\mathcal{V}(Y^m)$ for any m even, $0 < m < n+1$, determines the asymptotic expansion of the metric.

(Recall)

$$g = \frac{dx^2 + \omega_0 + \omega_2 x^2 + (\text{even}) + \omega_m x^m + O(x^{m+1})}{x^2}$$

Applications: Rigidity

Corollary

Suppose (M, g), (M, g') *CC AH even to order* 2. *For* $\gamma^1 \subseteq \partial M$, *let* $Y^2_{\gamma,g}$ *and* $Y^2_{\gamma,g'}$ *be minimal with* $\partial Y^2_{\gamma,g} = \partial Y^2_{\gamma,g'} = \gamma$.

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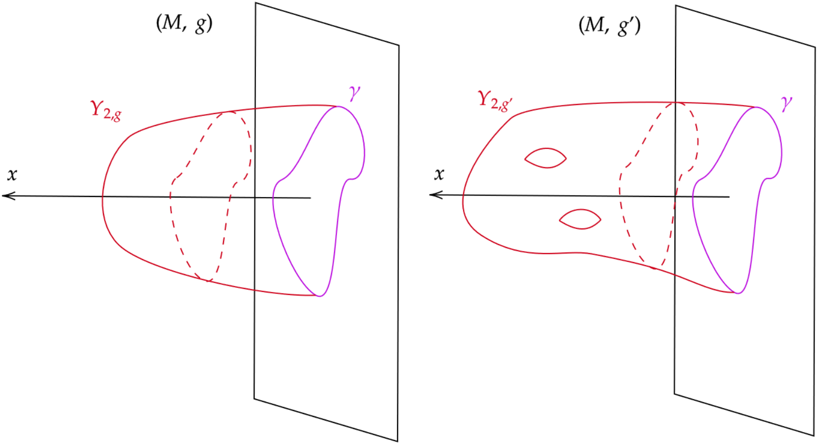
Suppose (M, g), (M, g') CC AH even to order 2. For $\gamma^1 \subseteq \partial M$, let $Y^2_{\gamma,g}$ and $Y^2_{\gamma,g'}$ be minimal with $\partial Y^2_{\gamma,g} = \partial Y^2_{\gamma,g'} = \gamma$.

If $\mathcal{V}_g(Y^2_{\gamma,g}) = \mathcal{V}_{g'}(Y^2_{\gamma,g'})$ for all γ, then $\exists \psi : \overline{M} \to \overline{M}$ diffeomorphism

$$\psi\Big|_{\partial M} = Id$$
$$\psi^*(g') - g = O(x^\infty)$$

If g, g' are real-analytic, then $\psi^*(g') = g$

Rigidity Illustrated



Figure

$$\mathcal{V}(Y_{\gamma,g}) = \mathcal{V}(Y_{\gamma,g'}) \quad \forall\gamma \implies g \text{``}=\text{''} g'$$

Applications: Poincaré–Einstein Metrics

- For n odd and (M^{n+1}, g) Poincaré–Einstein:

$$g = \frac{dx^2 + \omega_0 + x^2\omega_2 + \text{even} + x^{n-1}\omega_{n-1} + x^n \boxed{\omega_n} + O(x^{n+1}\log(x)^k)}{x^2}$$

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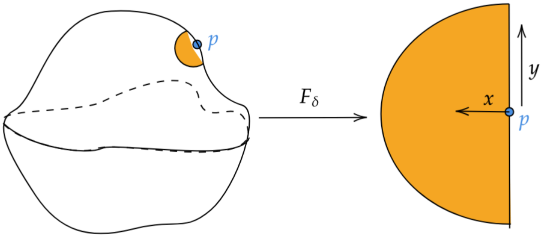
Corollary

Let (M^{n+1}, g) PE. If $\mathcal{V}(Y^2)$ is known on all minimal surfaces, or $c(g)$ and $\mathcal{V}(Y^m)$ is known for all m even with $2 < m < n+1$, then the non-local term, ω_n, is determined.

Proof of theorem 1: Determining $c(g)$

Preliminary: Blow up



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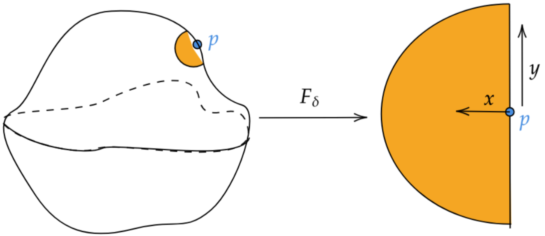

$$g = \frac{dx^2 + [(\omega_0)_{ij} + x^2(\omega_2)_{ij} + O(x^3)]dy^i dy^j}{x^2}$$

$$F_\delta(\tilde{x}, \tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^n) = (x = \delta\tilde{x}, y^1 = \delta\tilde{y}^1, \ldots, y^n = \delta\tilde{y}^n)$$

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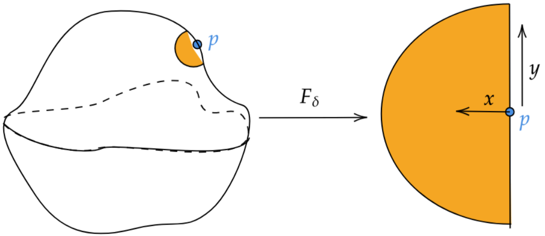

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F_\delta^*(g) &= \frac{d\tilde{x}^2 + (\omega_0(p))_{ij} d\tilde{y}^i d\tilde{y}^j + O(\delta^2)}{x^2}
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Determining ω_0

Determine $(\omega_0(p))_{ij}$ up to scalar factor $\rightarrow$ determine $c(g) = [\omega_0]$

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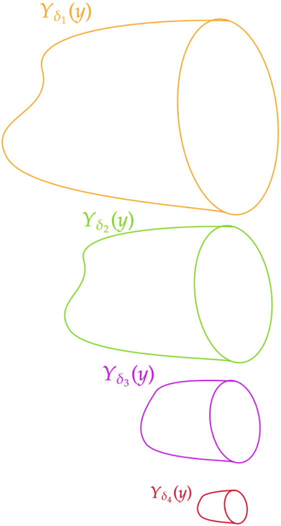

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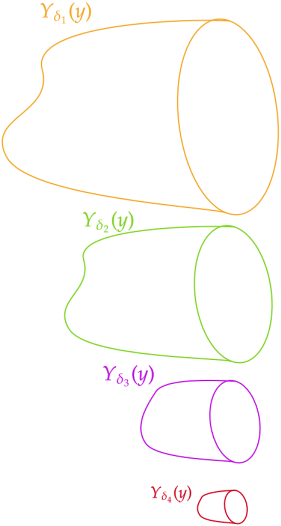

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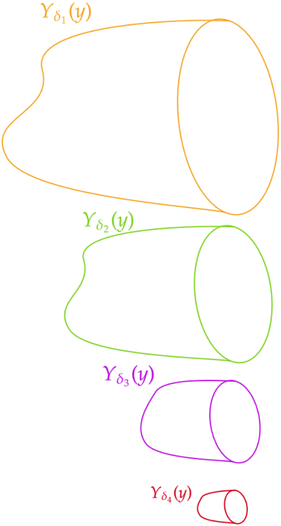

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$$\mathcal{V}(Y) = -2\pi\chi(Y) + \int_Y |\hat{k}|^2 dA_Y$$

$$\mathcal{V}(Y) = -2\pi \overset{\text{Bernstein}}{\Longleftrightarrow} Y = \mathbb{H}^2$$

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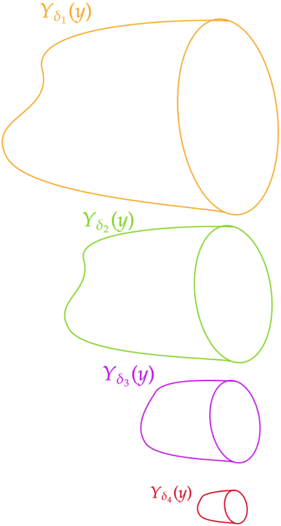

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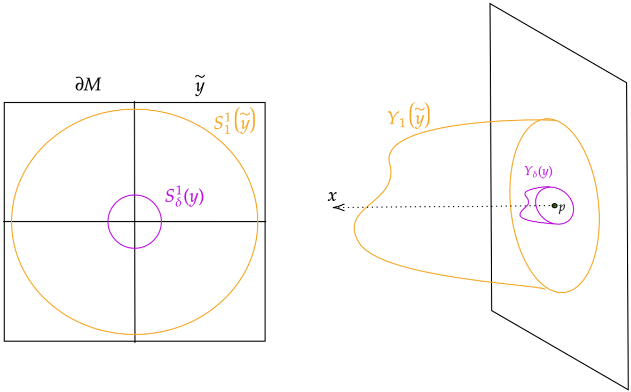
$$\iff \text{“Good choice” of } \{y^i\}$$

$$\iff (\omega_0(p))_{ij} = \lambda\delta_{ij}$$

What are Y_δ?



$$\begin{aligned} S^1_\delta(y) &= \{(y^1)^2 + (y^2)^2 = \delta^2\} = \{(\tilde{y}^1)^2 + (\tilde{y}^2)^2 = 1\} \\ \lim_{\delta \to 0} \mathcal{V}(Y_{S^1_\delta(y)}, F^*_\delta(g)) &= \mathcal{V}(Y_{\tilde{y}}, g_0) \end{aligned}$$

g_0 standard hyperbolic metric, $Y_{\tilde{y}}$ minimal w.r.t g_0 (Apply rigidity here)

Determining The Conformal Infinity

- Recall

$$\begin{aligned} \mathcal{V}(Y) &= -2\pi - \frac{1}{2}\int_Y |\hat{k}|^2 dA \\ \mathcal{V}(Y_{\tilde{y}}) &= -2\pi \iff Y_{\tilde{y}} \cong HS^2 \\ (\partial Y_{\tilde{y}} &= \{(\tilde{y}^1)^2 + (\tilde{y}^2)^2 = 1\}) \end{aligned}$$

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- Up to an orthonormal change of basis

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g_0 = \frac{dx^2 + \omega_0}{x^2} = \frac{dx^2 + A(d\tilde{y}^1)^2 + B(d\tilde{y}^2)^2}{x^2}
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Determining The Conformal Infinity (Cont.)

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- **Strategy**: Vary $\tilde{y}^2 \mapsto \tilde{y}^2 = c_1\tilde{y}^1 + c_2\tilde{y}^2$, recompute

$$\mathcal{V}(Y_\gamma), \qquad \gamma = \{(\tilde{y}^1)^2 + (\tilde{y}^2)^2 = 1\}$$

until it equals -2π.

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Tells us we have good coordinates, $\tilde{y}$, for which $\omega_0(p) = \lambda\sum_i (d\tilde{y}^i)^2$. This gives $[\omega_0]$, conformal class.

Determining ω_2: δ derivatives

Recall

$$g = \frac{dx^2 + \omega_0 + x^2\omega_2 + \dots}{x^2}$$

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If we know ω_0, $\partial_\delta^2 F_\delta^*(g)$ gives ω_2!

Determining ω_2: Perturbations of Minimal Surfaces

Idea: vary curve when computing $\partial_\delta^k \mathcal{V}(Y_\delta)$

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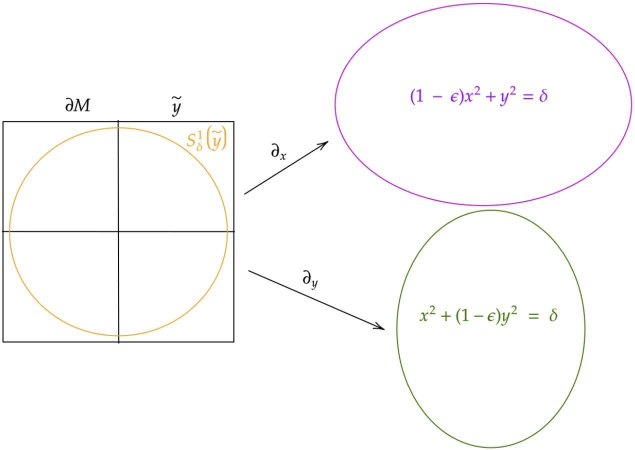
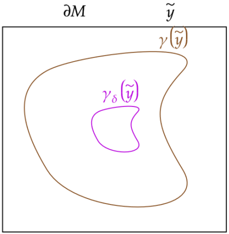
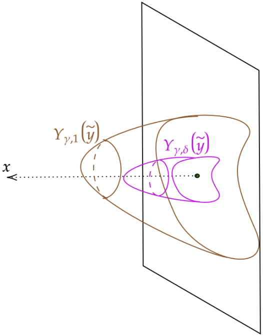
$$\frac{d}{d\gamma}\frac{d^k}{d\delta^k}\mathcal{V}(Y_{\gamma,\delta}) \leftrightarrow \frac{d^k}{d\delta^k}F_\delta^*(g) \leftrightarrow \omega_k(0)$$



Determining ω_2: Perturbations of Minimal Surfaces

- For each δ, exists $Y_{\gamma,\delta}$ minimal w.r.t. $g_\delta = F_\delta^*(g)$.





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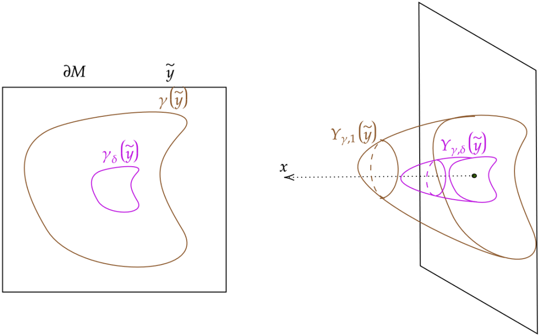

- Assume we know ω_0, compute

$$\frac{d^2}{d\delta^2}\mathcal{V}(Y_{\gamma,\delta}, g)\Big|_{\delta=0} = \mathrm{FP}_{z=0}\int_{Y_{\gamma,0}} x^z \boxed{\mathrm{Tr}_{Y_{\gamma,0}}(\omega_2)}\, dA_{Y_{\gamma,0}} + K(\omega_0)$$

$$\left(\text{Recall } g_\delta = \frac{\omega_0 + \delta^2\tilde{x}^2\omega_2 + O(\delta^3)}{\tilde{x}^2}\right)$$

Determining ω_2

- Compute variation of $\mathcal{V}$ w.r.t. boundary curve $\gamma \to \gamma_t$

$$\frac{d}{dt}\frac{d^2}{d\delta^2}\mathcal{V}(Y_{\gamma_t,\delta}, g)\Big|_{\delta=0}\Big|_{t=0} = \mathrm{FP}_{z=0}\int_{Y_{\gamma,0}} x\boxed{\dot{\phi}\mathrm{Tr}_{Y_{\gamma,0}}(\omega_2)}dA_{Y_{\gamma,0}} + K(\omega_0)$$

$\dot{\phi}$ a *jacobi field* on $Y_{\gamma,0}$.

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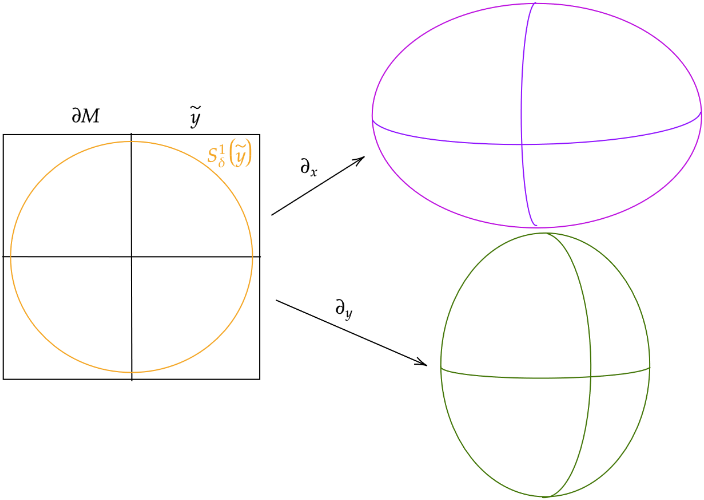
- Compute Jacobi fields explicitly about $Y = HS^2 \subseteq \mathbb{H}^{n+1}$

$$\dot{\phi}_1(\theta, x) = \frac{1}{x}$$
$$\dot{\phi}_2(\theta, x) = \cos(2\theta)\frac{(1-x)(2x+1)}{x(1+x)}$$

Determining ω_2

Up to linear combination:



Figure

Determining ω_2

- Explicitly compute

$$\frac{d}{dt}\frac{d^2}{d\delta^2}\mathcal{V}(Y_{\gamma_t,\delta},g)\Big|_{\delta=0}\Big|_{t=0}(\dot{\phi}_1) = \frac{3\pi}{8}\left[\omega_{2,11}+\omega_{2,22}\right]$$
$$\frac{d}{dt}\frac{d^2}{d\delta^2}\mathcal{V}(Y_{\gamma_t,\delta},g)\Big|_{\delta=0}\Big|_{t=0}(\dot{\phi}_2) = \frac{3\pi}{40}\left[\omega_{2,11}-\omega_{2,22}\right]$$

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- Now can recover $\omega_{2,11}$, $\omega_{2,22}$ individually

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- Now can recover $\omega_{2,11}$, $\omega_{2,22}$ individually
- Proof essentially generalizes to higher dimensions: some indicial root analysis + hypergeometric functions

Future Work

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- Rigidity/Upper bounds for renormalized volume of higher dimensional minimal surfaces $Y^m \subseteq \mathbb{H}^{n+1}$
- Determining conformal infinity using Y^m for $m \geq 4$?

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- Determining conformal infinity using Y^m for $m \geq 4$?
- Geometric interpretations/classification of "stable minimal surfaces" w.r.t renormalized area? How about Stable PE manifolds? Stable Q-curvature?
- Radon transform for minimal surfaces in AH spaces?

Thank You!