Geometric Variations of an Allen–Cahn Energy on Hypersurfaces

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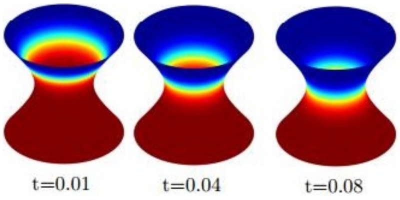
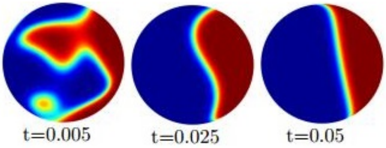
Phase Transitions and Allen–Cahn Equation



Figure



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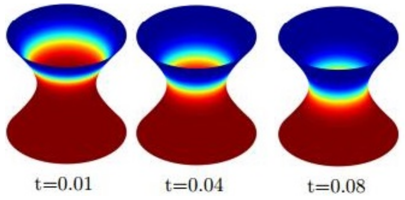
- Phase transitions: two different materials mixed together - how do they equilibriate?

Phase Transitions and Allen–Cahn Equation



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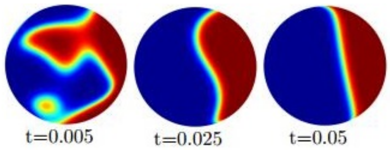

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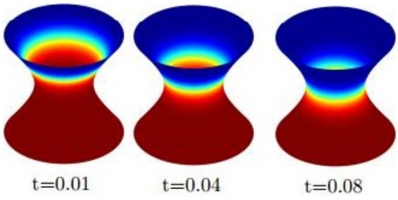
- Minimize configuration energy $\leftrightarrow$ transition region is small

Phase Transitions and Allen–Cahn Equation



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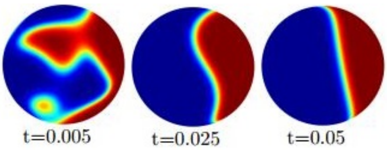

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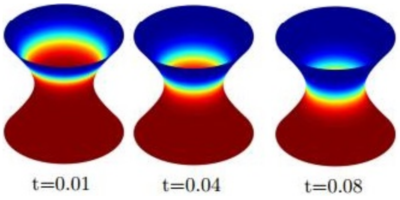

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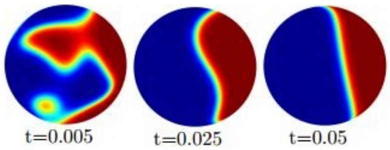

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- Build a model for phase transitions, $u_\epsilon : M \to \mathbb{R}$ such that

$$\lim_{\epsilon \to 0} u_\epsilon^{-1}(0) \text{``} = \text{''}\, Y \text{ minimal}$$

$$\lim_{\epsilon \to 0} E_\epsilon(u_\epsilon) = \sigma_0 A(Y)$$

Allen–Cahn Background

Let (M, g) closed manifold. The Allen–Cahn equation is a model for phase transitions given by

$$\epsilon^2 \Delta_g u = u(u^2 - 1) \tag{1}$$

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Solutions are critical points of

$$E_\epsilon(u) = \int_M \epsilon \frac{|\nabla^g u|^2}{2} + \frac{W(u)}{\epsilon} \tag{2}$$

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- Small energy (or critical point) $\rightarrow$ balancing between $u \approx \pm 1$ and small dirichlet energy

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$$E_\epsilon(u_\epsilon) \xrightarrow{\epsilon \to 0} P(\{u_\epsilon = 0\})$$

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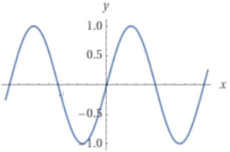

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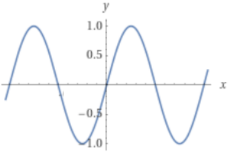

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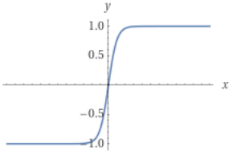
or $u = \tanh\left(\frac{t}{\epsilon\sqrt{2}}\right) =: g_\epsilon(t)$, the "heteroclinic" solution



- Gluing (**Pacard-Ritore**, '03): Near a minimal surface, one can find a solution to (1)

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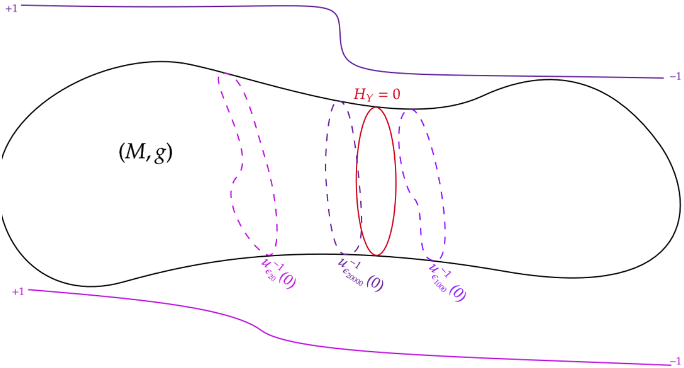

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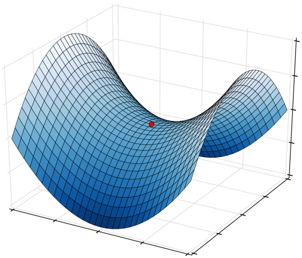
- Index and Nullity bounds:



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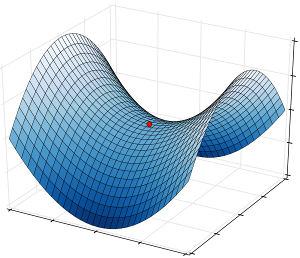

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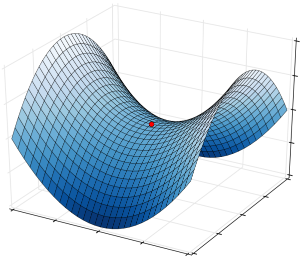

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(*Gaspar, Hiesmayr, Le*)

$$\mathrm{Ind}_{\mathrm{AC},\epsilon}(u_\epsilon) \geq \mathrm{Ind}(Y)$$

(*Chodosh–Mantoulidis*)

$$\mathrm{Ind}_{\mathrm{AC},\epsilon}(u_\epsilon) + \mathrm{Null}_{\mathrm{AC},\epsilon}(u_\epsilon) \leq \mathrm{Ind}(Y) + \mathrm{Null}(Y)$$

Theorem (Chodosh-Mantoulidis, 2018)

Let (M^3, g) a closed manifold with bumpy metric. Then there is $C > 0$ and smooth embedded minimal surfaces Σ_p for all $p > 0$ so that each component of Σ_p is two-sided and

$$\begin{aligned} C^{-1}p^{1/3} \leq Area_g(\Sigma_p) &\leq Cp^{1/3} \\ Ind(\Sigma_p) &= p \\ genus(\Sigma_p) &\geq \frac{p}{6} + 1 - Cp^{1/3} \end{aligned}$$

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Theorem (Chodosh-Mantoulidis, 2021)

Let (S^2, g) *the round sphere. Then for every* $p \in \mathbb{Z}^+$

$$\omega_p(S^2, g_0) = 2\pi\lfloor\sqrt{p}\rfloor$$

where ω_p *is the "p-width" of the length functional.*

Theorem (Caselli, Florit-Simon, Serra (2023))

Let (M^n, g) closed. There exists $C > 0$ such that for every $p \geq 1$ and $s \in (0,1)$, there exists an s-minimal surface $\Sigma^p = \partial E^p$ with morse index at most p and

$$C^{-1} p^{s/n} \leq (1-s) Per_s(E^p) \leq C p^{s/n}$$

In particular, M has infinitely many s-minimal surfaces

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- Only interested in Allen–Cahn in connection to minimal surfaces
- Why not look at $u \in H^1$ vanishing on hypersurfaces?

BE Set up

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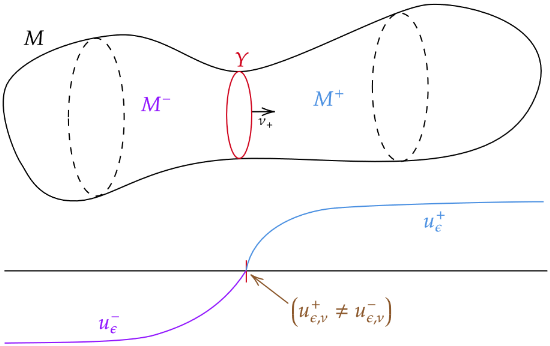
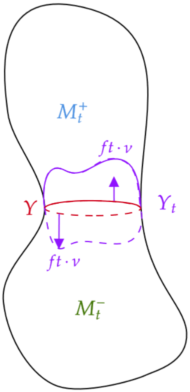


Figure

1st Variation



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Theorem (MK, Silva, 2023)

The first variation is given by

$$\frac{d}{dt} BE_\epsilon(Y_t)\Big|_{t=0} = \frac{\epsilon}{2}\int_Y f[(u_{\epsilon,\nu}^+)^2 - (u_{\epsilon,\nu}^-)^2]$$

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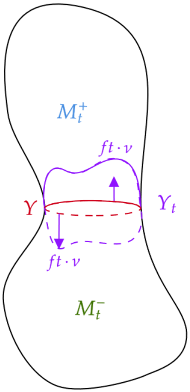

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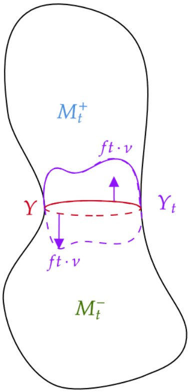

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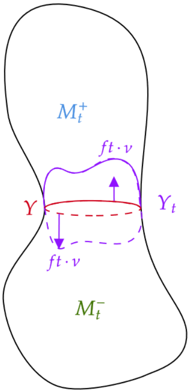

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- Critical points $\implies u^+_{\epsilon,\nu} = u^-_{\epsilon,\nu}$
- Finding critical points $\leftrightarrow$ computing dirichlet-to-Neumann map for Allen–Cahn equation on manifold with boundary

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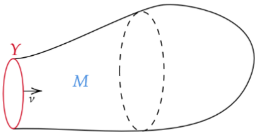
For Y a $C^{4,\alpha}$ hypersurface,

$$\nu^+(u_\epsilon^+) = \frac{1}{\epsilon\sqrt{2}} + \sigma_0 H_0 + \sigma_1 \epsilon[Ric_Y(\nu,\nu) + |A_Y|^2] + O(\epsilon^{2-\alpha})$$



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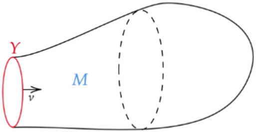

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- Above implies

$$\implies \frac{d}{dt}\mathrm{BE}_\epsilon(Y_t)\Big|_{t=0} = \sigma_0 \epsilon \int_Y f H_0 + O(\epsilon^2)$$

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If Y satisfies mild geometric assumptions, then

$$\frac{d^2}{dt^2} BE_\epsilon(Y_t)\Big|_{t=0} = D^2 A|_Y(f) + E(f)$$

$$|E(f)| \leq K\epsilon^{1/2} ||f||^2_{H^1}$$

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- $\dot{u}^\pm_\epsilon$ satisfies linearized Allen–Cahn system on $M^\pm$

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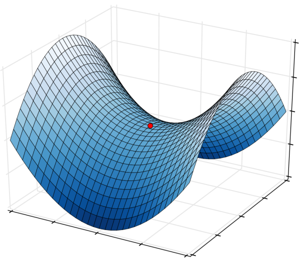
  - $(3g(t)^2 - 1)^{-1}(0) = \pm 0.93123$

Application: Index computation



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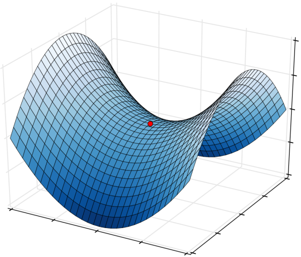

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- Let $Q(u_\epsilon)(v) = \frac{d^2}{dt^2} E_\epsilon(u + tv)\Big|_{t=0}$. Recall

$$\mathrm{Ind}_{AC}(u) := \max\{\dim V \mid V \subseteq H^1(M),\ Q(u)\Big|_{(V,V)} < 0\}$$

$$\mathrm{Null}_{AC}(u) := \dim \ker(\epsilon^2 \Delta_g - W''(u))$$

Application: Index computation

Theorem

Let $Y \leftrightarrow u_\epsilon$ a critical point for BE_ϵ. Then

$$Ind_{AC}(u_\epsilon) = Ind_{BE_\epsilon}(Y)$$
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Theorem says we can compute index/nullity on *smaller space* of

$$W = \{\dot{w}(f) \in H^1(M) \mid f \in H^1(Y), \epsilon^2 \Delta_g \dot{w} = W''(u)\dot{w}, \dot{w}\Big|_Y = -f u_\nu\}$$

Proof Sketch of $\mathrm{Ind}_{AC}(u_\epsilon) \leq \mathrm{Ind}_{BE}(Y)$

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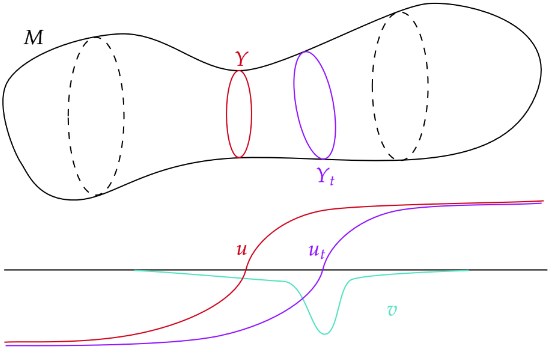
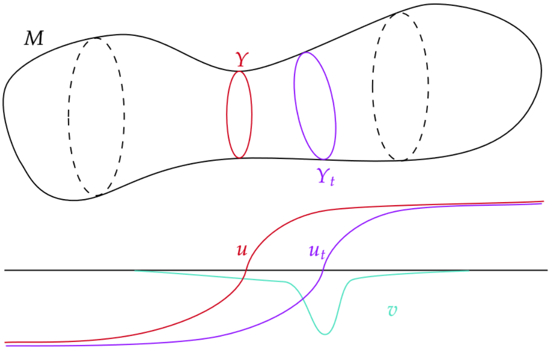


Proof Sketch (continued)



- Rewrite $u + tv = u_t + \psi_t$, u_t is a minimizer on $M_t^{\pm}$

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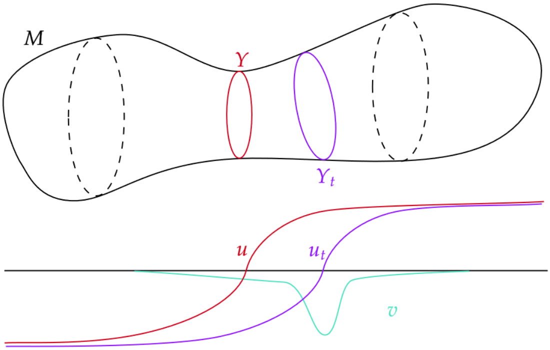

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$$\frac{d^2}{dt^2} E_\epsilon(u + tv) \stackrel{!}{=} \frac{d^2}{dt^2} \mathrm{BE}_\epsilon(Y_t)\Big|_{t=0} + Q(u)(\dot{\psi}, \dot{\psi})$$

for $\dot{\psi} = \frac{d}{dt}\psi_t\Big|_{t=0}$

Proof Sketch (continued)

- $\dot{\psi}\Big|_Y = 0$ and u_ϵ is a minimizer gives:

$$
\begin{aligned}
Q(\dot{\psi}, \dot{\psi}) &\geq 0 \\
\implies \frac{d^2}{dt^2} E_\epsilon(u + tv)\Big|_{t=0} - \frac{d^2}{dt^2} \mathrm{BE}_\epsilon(Y_t)\Big|_{t=0} &\geq 0 \\
\implies \mathrm{Ind}_{AC}(u_\epsilon) - \mathrm{Ind}_{\mathrm{BE}_\epsilon}(Y) &\leq 0
\end{aligned}
$$

Applications of 2nd Variation: Solutions on S^1

Let $u_{\epsilon,2p} : S^1 \to \mathbb{R}$ be the unique Allen–Cahn solution on S^1 vanishing on D_{2p}-symmetric points:

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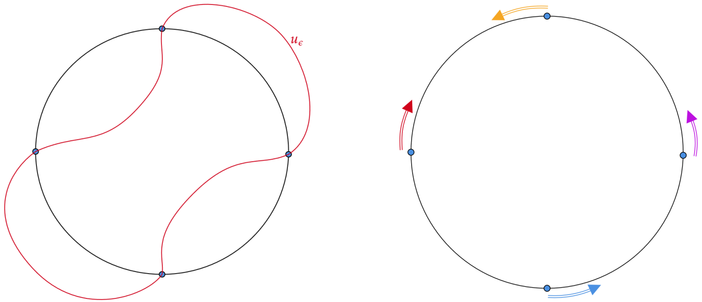
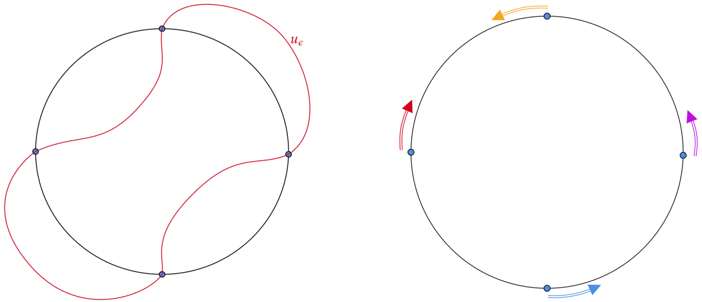


Applications of 2nd Variation: Solutions on S^1

Let $u_{\epsilon,2p} : S^1 \to \mathbb{R}$ be the unique Allen–Cahn solution on S^1 vanishing on D_{2p}-symmetric points:



Theorem

Fix $p > 0$. There exists ϵ_p such that for all $\epsilon < \epsilon_p$, $u_{\epsilon,2p}$ has Allen–Cahn Morse index $2p - 1$ and nullity 1. The nullity is realized by rotations and every other variation produces a strictly negative variations.

Motivation for solutions on S^1

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Theorem (Mantoulidis, 2022)

Let u_{ϵ_i} Allen–Cahn solutions on $M \times S^1$ such that

$$\lim_{i \to \infty} u_{\epsilon_i}^{-1}(0) = \{\theta_1, \ldots, \theta_m\} \times M$$

then m is even and $\theta_i - \theta_{i+1} = 2\pi/m$

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  - Gives example of minimal surfaces which *can not* be approximated by Allen–Cahn solutions
  - Shows that the set of Allen–Cahn min-max varifolds is a strict subset of Almgren–Pitts min-max varifolds

Proof Sketch

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$$\frac{d^2}{dt^2}\mathrm{BE}_\epsilon(Y+tf) \quad = \quad \sum_{i=0}^{2p-1} f\left(\frac{i}{2p}\right) u_\nu\left(\frac{i}{2p}\right)\left[\dot{u}_{i,x}^{+} - \dot{u}_{i,x}^{-}\right]\left(\frac{i}{2p}\right)$$

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$$\text{(rearrangement)} = \epsilon c \sum_{i=0}^{2p-1} f\left(\frac{i}{2p}\right)\dot{u}_{i,x}\left(\frac{i}{2p}\right) + f\left(\frac{i+1}{2p}\right)\dot{u}_{i,x}\left(\frac{i+1}{2p}\right)$$

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where $v(\epsilon) < 0$ - relies on explicit computation of $\dot{u}_{i,x}$

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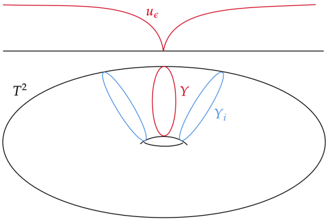

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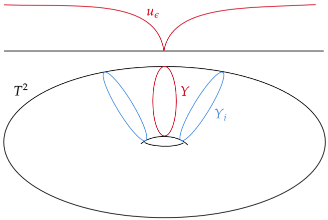

- Development of BE_ϵ-surface flow

$$\partial_t x = [u_\nu^+(x)]^2 - [u_\nu^-(x)]^2$$