University of Tennessee Knoxville Geometry Seminar: On the Existence of Infinitely Many Surfaces with Prescribed Mean Curvature

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Minimal Surfaces

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(M^{n+1}, g) manifold, we are interested in *minimal surfaces*, $Y^n \subseteq M^{n+1}$, which appear as

- Critical Points for Area
- Stationary points of Mean Curvature Flow
- Horizons of Black Holes
- Soap bubbles

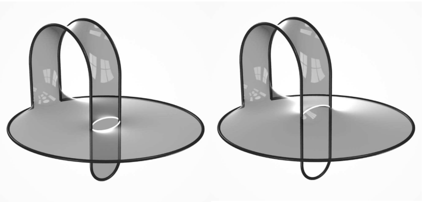

Figure: Plateau's problem

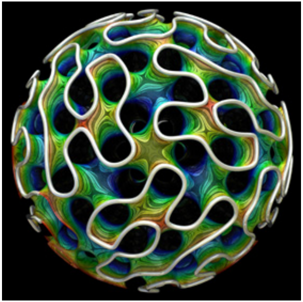

Figure: Gyroid

Yau's conjecture

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Motivation: geodesics

Theorem (Birkhoof, Bangert)

On any closed surface, (M^2, g), there exist infinitely many geodesics.

Resolution of Yau's conjecture

Theorem (Marques–Neves, Irie–Marques–Neves, Chodosh–Mantoulidis, Song)

For any closed manifold (M^{n+1}, g), $3 \leq n+1 \leq 7$, *there exist infinitely many embedded minimal surfaces.*

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    - for $\text{Ric}_g > 0$ (Marques–Neves)
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Key tools: Min-Max constructions, p-widths, $\{\omega_p\}$ (Gromov, Guth, Liokumovich–Marques–Neves)

One-parameter Min-max

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"Sweepout" whole manifold, and take the longest hypersurface.

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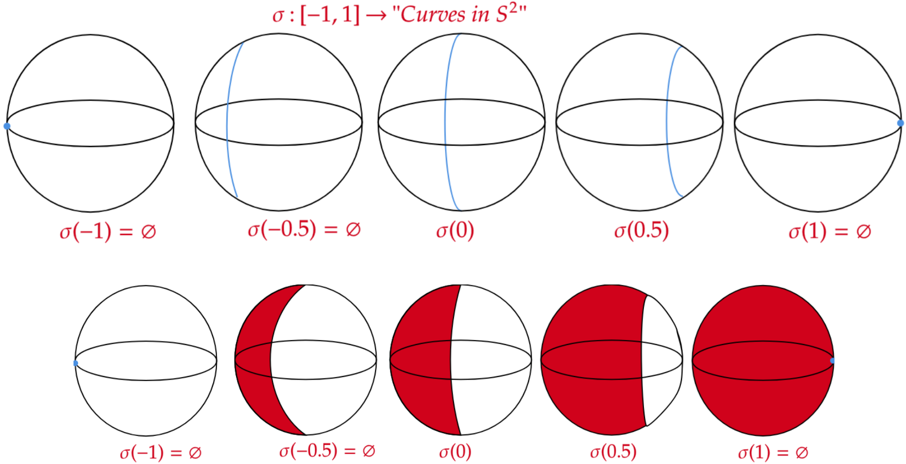
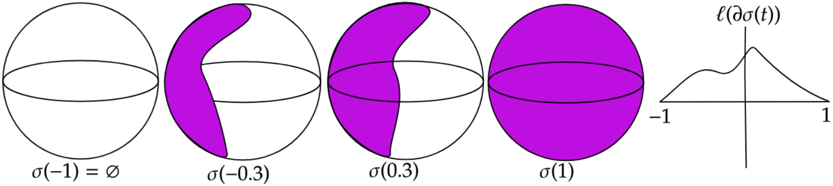


Figure: Map at the level of sets, whose boundaries are the curves.

Min-Max continued

Given $\sigma : [0,1] \to \{\text{Sets}\}$, there is always a t_0 such that Area$(\partial\sigma(t_0))$ is maximized.



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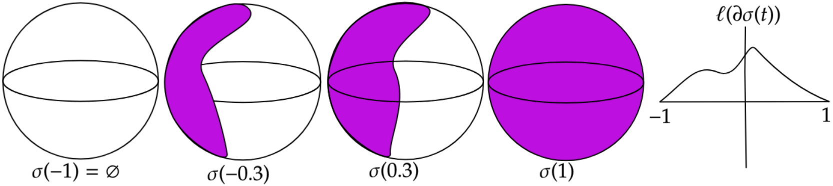

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Minimize over all paths

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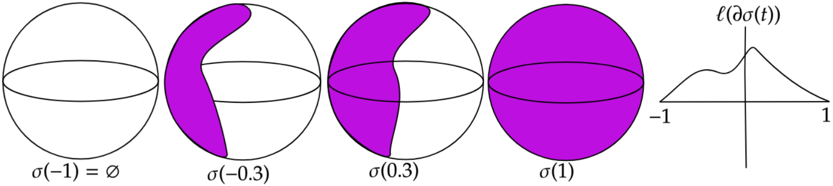

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$\Sigma_{t_0} = \partial\sigma(t_0)$, may not be minimal - how do we find a *minimal* surface? Minimize over all paths

$$\mathcal{P} := \{\sigma : [0,1] \to \{\text{Sets in } M^n\} \,\big|\, \sigma(0) = \emptyset, \sigma(1) = M\}$$

$$\omega_1 := \inf_{\sigma\in\mathcal{P}} \sup_{t\in[0,1]} \mathcal{H}^{n-1}(\partial\sigma(t))$$

$$\omega_1 = \text{Area}(\Sigma) \qquad \text{for some } \Sigma \textbf{ a minimal surface}$$

$p > 1$

- Higher parameter analogue

$$\mathcal{P}_p := \{\Phi : X \to \{\text{Hypersurfaces in } M^n\} \;\Big|\; \Phi \text{ ``p-sweep out of'' } M\}$$

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- Morally, for each p, find a new minimal surface

$$\omega_p = \mathcal{H}^{n-1}(\Sigma_p)$$

"P-widths", $\{\omega_p\}$, Formal Background

- Gromov (1980s): Introduced p-widths, $\{\omega_p\}$, as non-linear analogue of spectrum of laplacian,

$$\Delta u = \lambda u$$

- Recall: Find eigenvalues via rayleigh quotients

$$\lambda_1 = \inf_{f \neq 0} \frac{\int |\nabla f|^2}{\int f^2}, \qquad \lambda_j = \inf_{f \perp \{0, f_1, \ldots, f_{j-1}\}} \frac{\int |\nabla f|^2}{\int f^2}$$

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- P_k, a k-plane of functions

$$P_k = \text{span}\{f_1, \ldots, f_k\}, \qquad f_i : M \to \mathbb{R}$$

$$\implies \lambda_k(M) = \inf_{P_{k+1}} \sup_{f \in P_{k+1} \setminus \{0\}} \frac{\int |\nabla f|^2}{\int f^2}$$

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- Compare

$$\omega_p := \inf_{\Phi \in \mathcal{P}_p} \sup_{x \in \text{Dom}(\Phi)} \mathcal{H}^{n-1}(\Phi(x))$$

Further Similarities

- Recall Weyl's law

Theorem (Weyl)

Let $\{\lambda_k\}$ *the eigenvalues of the laplacian, then*

$$\lim_{k\to\infty} \lambda_k k^{-2/n} = \frac{(2\pi)^{-n}}{\alpha_n Vol(M)}$$

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- Analogous law for ω_p!

Theorem (Gromov, Guth, Liokumovich–Marques–Neves)

There exists a constant $c(n) > 0$ such that for every (M^{n+1}, g) with boundary (possibly empty), we have

$$\lim_{p\to\infty} \omega_p(M) p^{\frac{-1}{n+1}} = a(n+1) vol(M)^{\frac{n}{n+1}}$$

Utility of p-widths

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1. For each p

$$\omega_p = \sum_{i=1}^{N_p} A(\Sigma_i)$$

with Σ_i connected, *distinct* (multiplicity one).

2. Counting arguments from non-linear growth

$$\omega_p \sim p^{1/(n+1)}$$

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We will apply the p-widths, $\{\omega_p\}$, to find related surfaces called surfaces with **prescribed mean curvature** (PMC).

Constant and Prescribed Mean Curvature Surfaces

Surfaces, Y, with

$$H = c \in \mathbb{R}, \quad \text{or} \quad H = h\Big|_{Y}, \qquad h \in C^{\infty}(M)$$

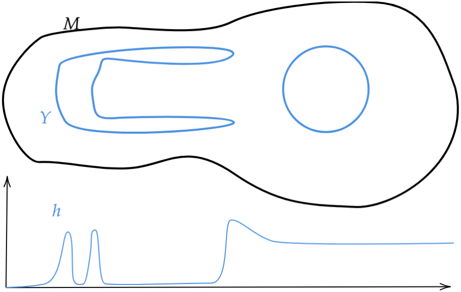

Figure: L: Bubbles as Constant Mean Curvature Surfaces, R: A prescribe mean curvature surface with prescribing function visualized

Definitions

$h : M \to \mathbb{R}$ smooth. $Y^n \subseteq M^{n+1}$ is a **prescribed mean curvature (PMC)** surface if

$$H\Big|_Y = h\Big|_Y$$

if h is constant, Y is a **constant mean curvature (CMC)** surface.

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<u>Motivation</u>:

- CMCs - bubbles
- PMCs - ovaloids (Minkowski problem), regularization/generalization of CMC surfaces (see Zhou), natural elliptic PDE problem

CMCs: Twin Bubble Conjecture

Conjecture (Arnold, Zhou)

On any closed manifold (M^{n+1}, g), there exist 2 hypersurfaces of constant mean curvature, c, for any $c > 0$.

- By Arnold, $n+1=2$, by Zhou for $n+1 \geq 3$
- Conjecture still totally open!

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Generalized Yau's conjecture

Conjecture

On any closed manifold (M^{n+1}, g) *and any* $h : M \to \mathbb{R}$ *smooth, there exist infinitely many hypersurfaces with prescribed curvature,* h.

Our work verifies the above conjecture for certain manifolds and prescribing functions.

Past progress

PMCs

- Existence of 1 PMC (Zhou–Zhu)
- Compactness of PMCs (Zhou–Zhu)
- Existence of PMCs in non-compact settings (Mazurowski, Stryker)

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CMCs

- Existence of 1 c-CMC (Zhou–Zhu)
- Existence of multiple c-CMCs for c large (Pacard–Xu)
- Compactness + Bubbling for c-CMCs of bounded index (Bourni–Sharp–Tinaglia/Sun/Zhou–Zhu)
- **Existence of many c-CMCs for c small (Dey)**

Dey's construction of many c-CMCS

Theorem (Dey 2019)

Let (M^{n+1}, g) *(*$3 \leq n+1 \leq 7$*) closed and* $c > 0$*. For each* $p \in \mathbb{N}$*, such that* $\omega_{p+1} - \omega_p > c \cdot Vol(M)$*, there exists a c-CMC,* Y *with*

$$\omega_k - c\,Vol(M) < Area(Y) < \omega_k + c\,Vol(M) + C$$

for C *independent of* k*.*

Theorem (Dey 2019)

There exists constants $c_0(M,g), \gamma_0(M,g) > 0$*, such that for all* $c < c_0$*, there are at least* $\gamma_0 c^{-1/(n+1)}$ *closed c-CMC hypersurfaces.*

Main Results: Finding Infinitely Many PMCs

Context: Extensions of Dey's work

- Given work of Zhou–Zhu, easy to extend Dey's construction to PMCs with $c\text{Vol}(M) \to ||h||_{L^1}$

- Number of c-CMCs limited by $\omega_{p+1} - \omega_p > c\text{Vol}(M)$, and Weyl law

$$\lim_{p\to\infty} \omega_{p+1} - \omega_p = 0$$

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$$\lim_{p\to\infty} \omega_{p+1} - \omega_p = 0$$

Q: What if we work on a manifold such that $\omega_{p+1} - \omega_p \geq C > 0$ for all C?

A: Manifolds with Cylindrical Ends

Cylindrical Ends in Song's Resolution of Yau's conjecture

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Song proved the existence of infinitely many minimal surfaces on non-generic metrics by

- considering manifolds with boundary and nice foliations
- constructing new minimal surfaces by attaching cylindrical ends, applying min-max using p-widths

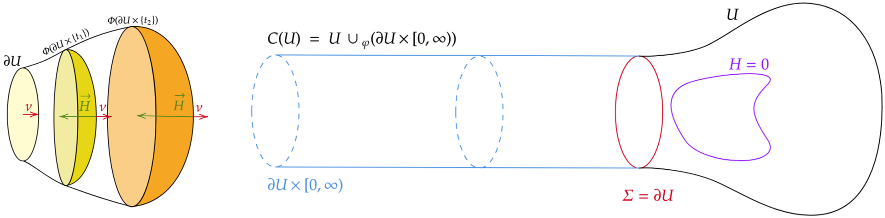
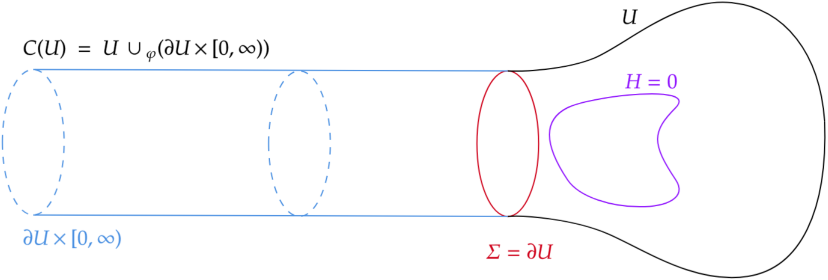


Figure: L: Nice foliations, R: Result of min-max in manifold with cylindrical end attached.

Cylindrical Weyl Law



For a manifold with cylindrical ends,

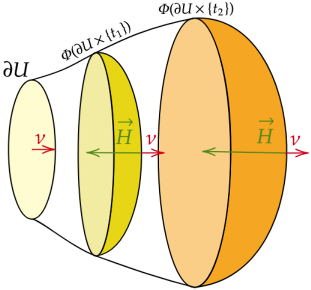
$$\omega_p \sim C \cdot p \implies \omega_{p+1} - \omega_p \geq C$$

so able to construct many more PMCs via Dey's construction!

Main Results

(M^{n+1}, g) manifold, $3 \leq n+1 \leq 7$, $\partial M = \Sigma$ minimal, and g a bumpy metric. Σ has a *contracting neighborhood* if there exists a $U \supseteq \Sigma$,

$$\Phi : \Sigma \times [0, \hat{t}] \xrightarrow{\cong} U$$
$$\Sigma_t := \Phi(\Sigma \times \{t\})$$
$$H_{\Sigma_t} < 0$$



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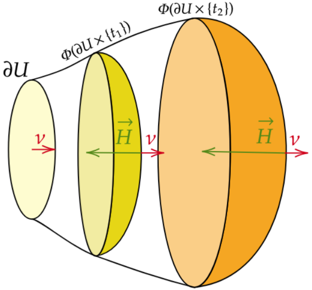


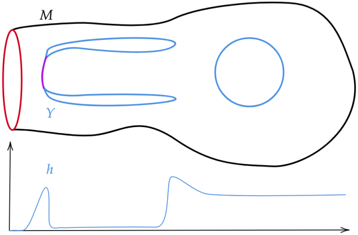
Remark: If Σ is non-degenerate, then such a contracting neighborhood always exists by the inverse function theorem.

Main Results

Theorem (Gaspar–MK)

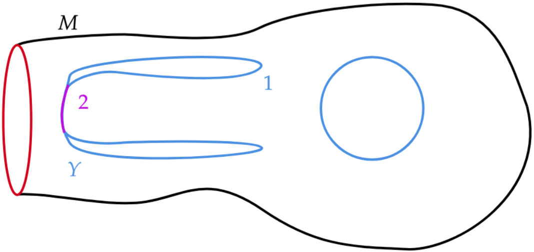
There exists a $C(M,\Sigma) > 0$, such that for $h \in C_c^\infty(M \backslash \Sigma)$, $||h||_{C^{3,\alpha}(M)} \leq C$, and h morse away from a neighborhood of Σ, there exist infinitely many multiplicity one, almost embedded, hypersurfaces, $\{Y_p\}$, with mean curvature $H = h$ and

$$(p+1)\cdot A(\Sigma) - 2||h||_{L^1(M)} \leq A(Y_p) \leq (p+1)\cdot A(\Sigma) + W_0 + Cp^{1/(n+1)} + 2||h||_{L^1(M)}$$



"Sticking"

Remark: Despite being multiplicity one, the PMCs may "stick" to themselves on large open sets, and hence "almost embedded" and have density 2 on this set.



Technical Assumptions, Stronger theorem

Suppose $h : M \to \mathbb{R}$ smooth and satisfies

1. $||h||_{L^1(M)} \leq A(\Sigma)/2$
2. $h\Big|_\Sigma = \partial_\nu h\Big|_\Sigma = \partial_\nu^2 h\Big|_\Sigma = \partial_\nu^3 h\Big|_\Sigma = 0$
3. $h\Big|_{M\setminus\Sigma}$ is a morse function, and $\{h = 0\} = \Sigma \cup \Sigma'$ where $\Sigma' \cap \Sigma = \emptyset$ and Σ' is a closed smoothly embedded hypersurface with mean curvature vanishing to at most finite order.

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For $h : M \to \mathbb{R}$ satisfying the above, there are infinitely many PMCs, and the sticking set is $(n-1)$ dimensional.

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Remark: Density 1 except on a set of dimension $(n-1)$ is generic.

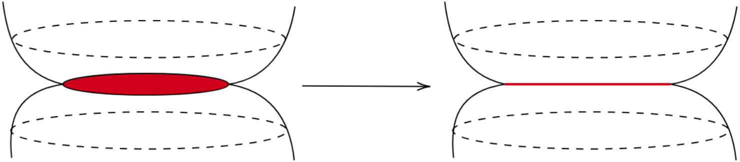

Figure: Density 2 on a codimension 1 (dimension $(n-1)$) set.

Corollaries: Closed setting

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If (M^{n+1}, g) closed and Σ is embedded strictly stable minimal surface, *unravel* $M \backslash \Sigma$ and apply previous theorems to *metric completion*, $N = \text{Comp}(M \backslash \Sigma)$.

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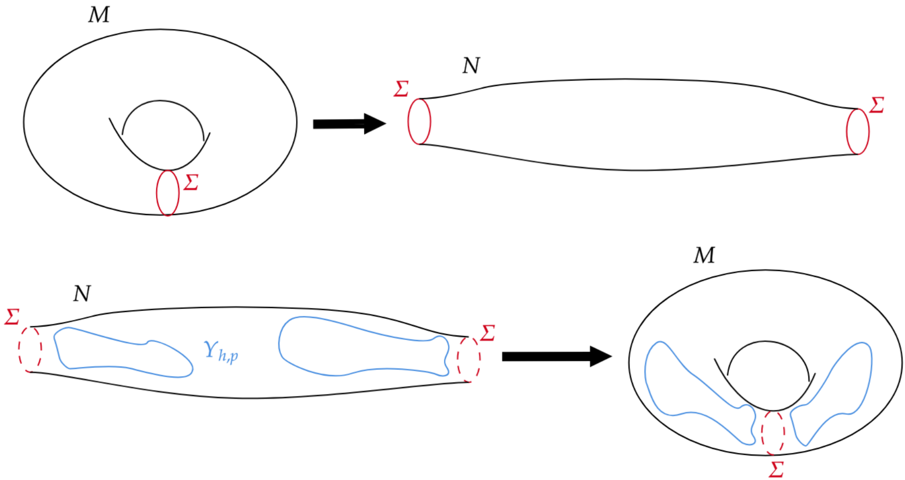
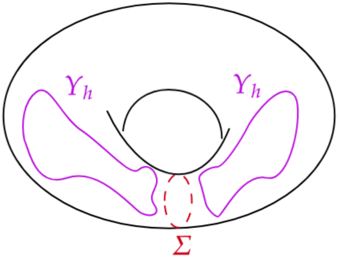


$H_n \neq 0$

Corollary (Gaspar–MK)

Suppose (M^{n+1}, g) closed, bumpy metric, and $H_n(M, \mathbb{Z}_2) \neq 0$. Then there exists a stable closed, embedded minimal surface Σ and a constant $C = C(M, \Sigma) > 0$ such that for any prescribing function $h \in C_c^\infty(M \backslash \Sigma)$ and $||h||_{C^{3,\alpha}(M)} \leq C$, there exist infinitely many PMCs.



Non-Frankel

Recall that a manifold (M^{n+1}, g) does *not* satisfy the Frankel property if there exist 2 distinct minimal surfaces which do not intersect.

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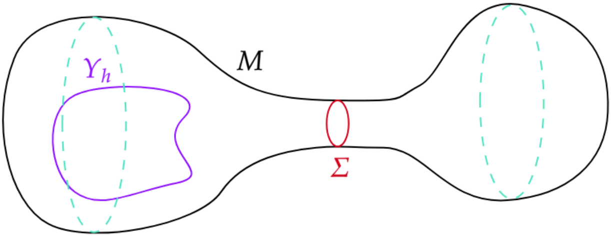
Corollary (Gaspar–MK)

Suppose (M^{n+1}, g) closed, bumpy, and not Frankel. Then there exists a stable closed, embedded minimal surface Σ and a constant $C = C(M, \Sigma) > 0$ such that for any prescribing function $h \in C_c^\infty(M \setminus \Sigma)$ and $||h||_{C^{3,\alpha}(M)} \leq C$, there exist infinitely many PMCs.



Main Ideas of Proof

Suppose $h \in C_c^\infty(M \backslash \Sigma)$. For each p (associated to ω_p)

1. Attach "approximate" cylindrical ends to $M \to (U_\epsilon, g_\epsilon)$.
2. Choose approximations, h_ϵ, on U_ϵ, to h .

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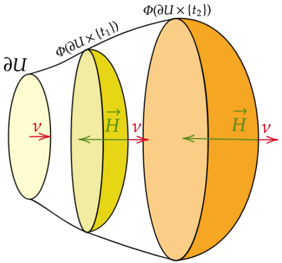
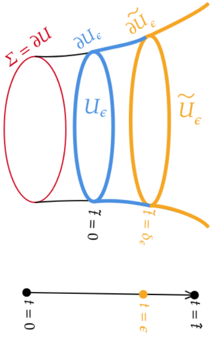
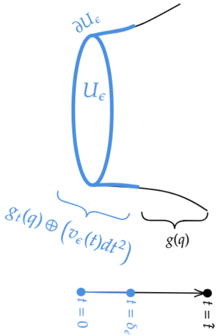
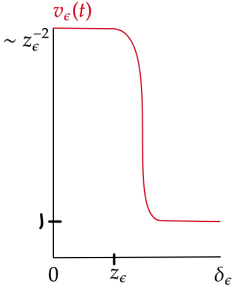
4. Send $\epsilon \to 0$, use (novel) diameter estimates to show that $Y_\epsilon \xrightarrow{\epsilon \to 0} V$, a varifold contained in M.
5. Show that $V = Y_h$, an almost embedded PMC with no component equal to Σ. Use morseness of h (away from Σ) and contracting neighborhood of Σ to prevent multiplicity of components.

Approximate Cylindrical Ends









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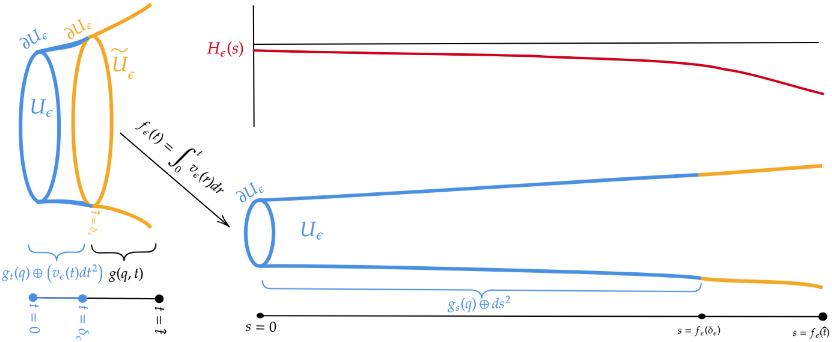
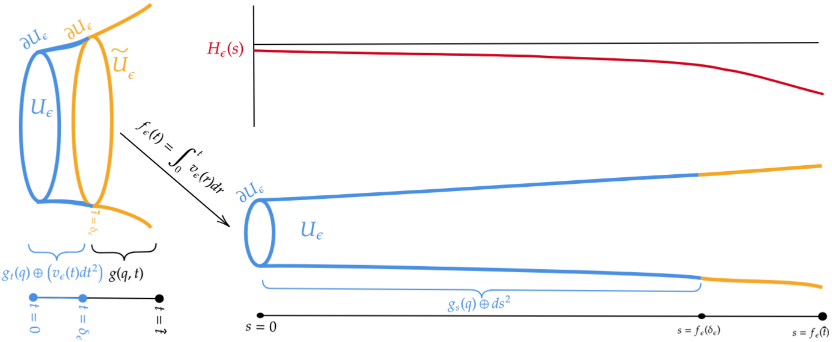


Approximate Cylindrical Ends



1. Slices, $\partial U_\epsilon \times \{s\}$ have non-zero mean curvature, H_s
2. $H_s \to 0$ as $\epsilon \to 0$
3. $(U_\epsilon, g_\epsilon) \xrightarrow{C^0} M \sqcup_\Sigma \Sigma \times [0,\infty)$, g_ϵ uniform $C^{0,1}$ bounds

Approximating h by h_ϵ

Approximate h by $h_\epsilon : U_\epsilon \to \mathbb{R}$, closely in C^1 and L^1, so that $|h_\epsilon|\Big|_{\partial U_\epsilon \times \{s\}} \leq H_s$

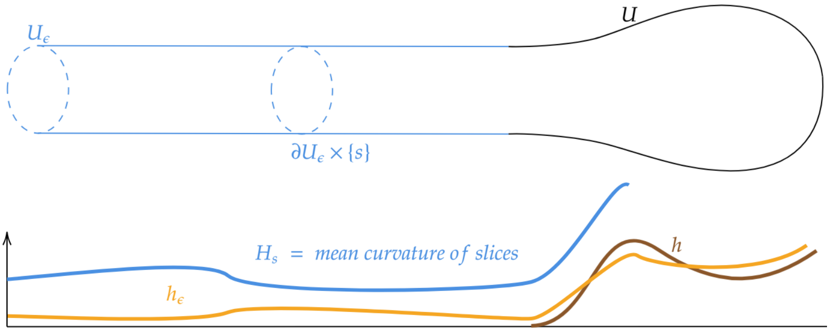


Figure: h is extended to 0 on the cylindrical end due to its compact support.

Dey's construction

- (Higher parameter) **mountain pass construction** for any $p > 0$.
- Utilizes $\omega_{p+1} > \omega_p$, to show the existence of a mountain pass

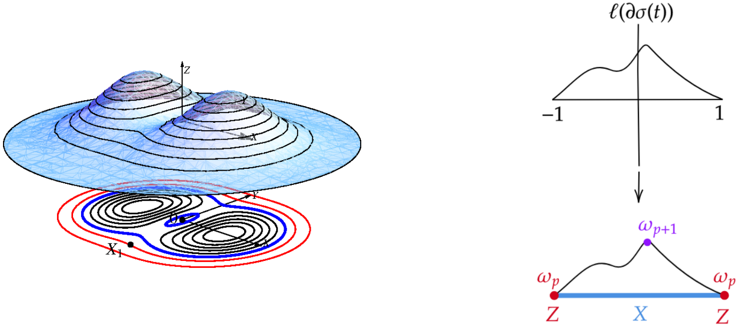
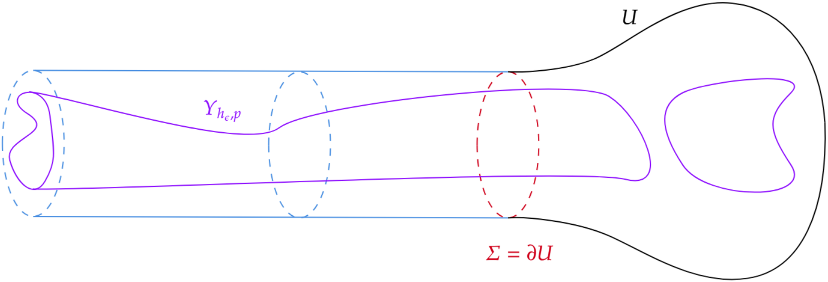


Figure: L: Classical mountain pass, R: Mountain pass via "relative homotopy class"

Dey's Construction

Yields, a PMC, $Y_{h_\epsilon,p}$ in (U_ϵ, g_ϵ):

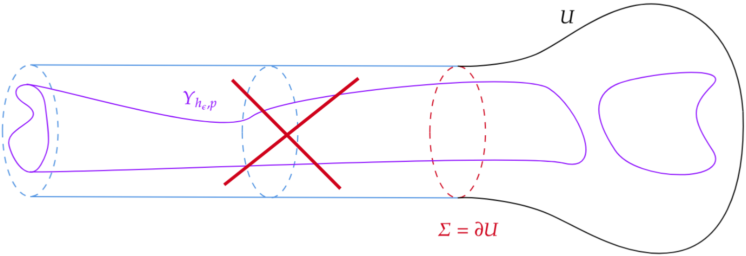


for p fixed, ϵ small, we have

$$(p+1)\cdot C - K \leq \text{Area}(Y_{h,p,\epsilon}) \leq (p+1)\cdot C + K$$

$\epsilon \to 0$

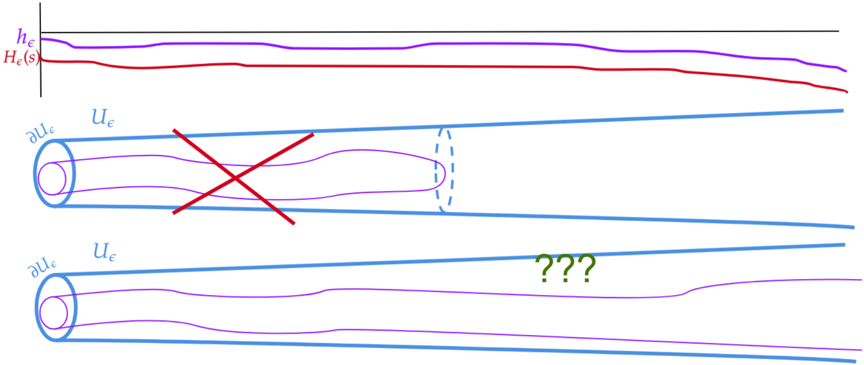
As $\epsilon \to 0$, we want $Y_{h_\epsilon,p}$ to converge to a PMC in U



Want to prevent our PMCs from touching ∂U_ϵ.

Maximum Principle and Tethering

By the maximum principle



Could have PMC with boundary and large diameter?
(<u>Need diameter estimates for PMCs</u>)

Diameter Estimates

Theorem (Chambers–MK)

Suppose $P^m \subseteq M^{n+1}$ and the ambient sectional curvature is bounded, $K_M \leq k_0$. If P is closed, then

$$diam_{int}(P) \leq C(m, k_0) \left[\int_P |H_P|^{m-1} + \max(\mathcal{H}^m(P), \mathcal{H}^m(P)^{1/m}) \right]$$

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Suppose $P^m \subseteq M^{n+1}$ and the ambient sectional curvature is bounded, $K_M \leq k_0$. Let $x \in \mathring{P}$, then

$$dist(x, \partial P) \leq C(m, k_0) \left[\int_P |H_P|^{m-1} + \max(\mathcal{H}^m(P), \mathcal{H}^m(P)^{1/m}) \right]$$

Remark: Interpolation of diameter bounds from monotonicity formula and work of Topping.

Diameter and Tethering

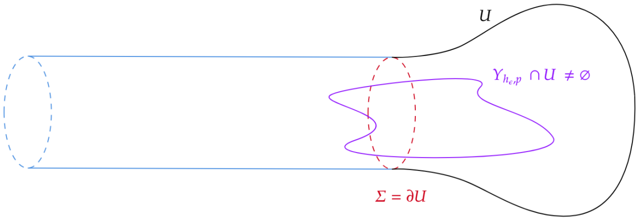
Because our PMCs satisfy $H = h_\epsilon$ which has uniform C^1 bounds, and

$$A(Y_{h_\epsilon,p}) \leq K \cdot (p+1) + C$$

we have

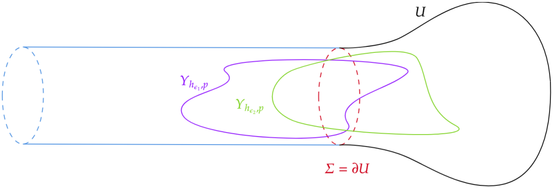
Proposition

$Y_{h_\epsilon,p}$, has finite diameter. Moreover, $Y_{h_\epsilon,p}$ is not a free boundary PMC, and "tethered to the core" of U_ϵ.



$Y_\epsilon \to V = Y$

By maximum principle, and vanishing mean curvature of leaves, $Y_\epsilon \to V$, a varifold supported in M



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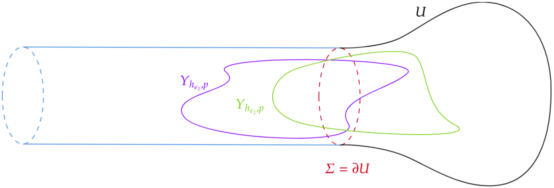
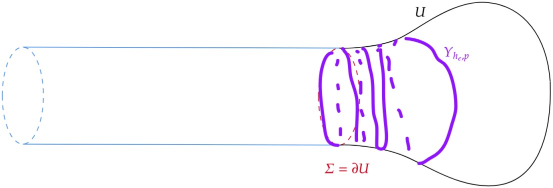


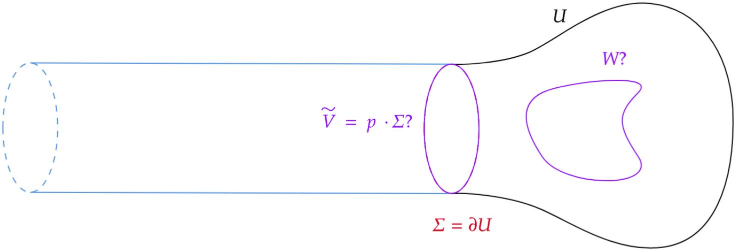
A priori, $Y_{h_\epsilon,p}$ can **accumulate** at the boundary!



Solomon-White Maximum Principle

Solomon–White maximum principle gives

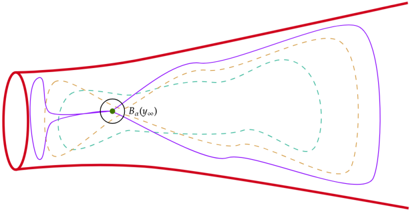
$$\lim_{\epsilon \to 0} Y_{h_\epsilon, p} = V = \tilde{V} + W, \qquad \mathrm{supp}(W) \cap \Sigma = \emptyset, \qquad \mathrm{supp}(\tilde{V}) \subseteq \Sigma$$



Accumulation at the boundary could lead to multiplicity!

"No Pinching"

- But "tethering" + "no pinching" argument implies $\tilde{V} = 0$



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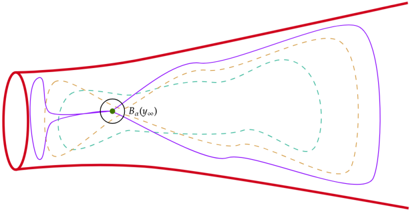

- Away from Σ, good regularity of convergence due to good convergence of metric + compactness of PMCs

Concluding theorem 1

Our construction for $h \in C_c^\infty(M \backslash \Sigma)$

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$$\text{Area}(Y_{h,p}) \geq (p+1) \cdot C$$
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for some $C > 0$.

Unbounded growth of area + multiplicity one implies there exist infinitely many PMCs!

Theorem 2

Recall $h \in C^\infty(M)$

1. $h\Big|_\Sigma = \partial_\nu h\Big|_\Sigma = \partial_\nu^2 h\Big|_\Sigma = \partial_\nu^3 h\Big|_\Sigma = 0$
2. $h\Big|_{M\setminus\Sigma}$ is a morse function, and $\{h = 0\} = \Sigma \cup \Sigma'$ where $\Sigma' \cap \Sigma = \emptyset$ and Σ' is a closed smoothly embedded hypersurface with mean curvature vanishing to at most finite order.

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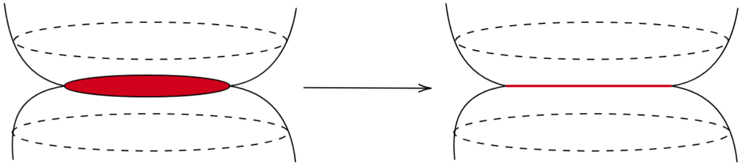
Want to show infinitely many almost embedded PMCs, density is 1 except on a small set (condition 2 ensures this)



Sketch of theorem 2

1. Approximate h by $h_i \in C_c^\infty(M \backslash \Sigma)$
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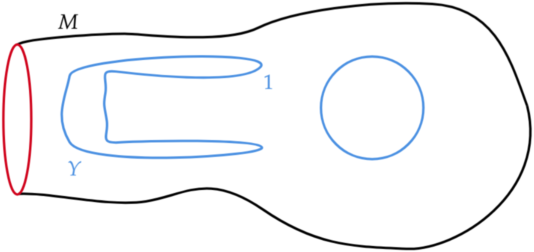
5. Because h is Morse away from Σ, "touching set" is $(n-1)$-dimensional



Future Work

- Lower regularity conditions, e.g. $h \in C^\infty(M)$, $h\Big|_\Sigma = 0$
- No restrictions on h?
- Infinitely many PMCs when $H_\Sigma = c > 0$

Thank You!