These notes are for a lecture for the algebraic geometry working seminar, 2015. The presentation is lifted from TA Springer.

\( G \) is a linear algebraic group over \( k \), \( k \) algebraically closed.

Let us recall a few facts about complete varieties. The proofs are relatively easy, we refer the reader to the text. We make heavy use of these facts in proving the main theorems.

**Definition 1.** A variety \( X \) is said to be complete if, for all varieties \( X \), the projection \( X \times Y \rightarrow Y \) is a closed mapping.

**Lemma 1.** Let \( X \) be a complete variety.

(i) Any closed subvariety of \( Y \) is complete.

(ii) If \( Y \) is also complete, then \( X \times Y \) is complete.

(iii) If \( \varphi: X \rightarrow Y \) is a morphism of varieties, then \( \varphi(X) \) is closed and complete subvariety of \( Y \).

(iv) If \( X \subset Y \), then \( X \) is closed.

(v) If \( X \) is irreducible, then the regular functions on \( X \) are all constant.

(vi) If \( X \) is affine, then \( X \) is finite.

**Definition 2.** A closed subgroup \( P \subseteq G \) is a parabolic subgroup if \( G/P \) is a complete variety.

Recall that for a closed subgroup \( H \), \( G/H \) is a quasi-projective variety of dimension \( \dim G - \dim H \), i.e. is an open subvariety of \( \mathbb{P}^n \). Since complete subvarieties are closed, this proves

**Lemma 2.** \( (6.2.2) \) \( G/P \) is a projective variety.

Parabolicity is transitive, in the following sense:

**Lemma 3.** Let \( P \) be parabolic in \( G \), \( Q \) parabolic in \( P \). Then \( Q \) is parabolic in \( G \).

**Proof.** We need to show \( \pi: G/Q \times X \rightarrow X \) is a closed mapping for any variety \( X \). To prove this, it suffices to show that, for closed \( A \subset G \times X \) such that

\[(g, x) \in A \iff (gQ, x) \subseteq A\]

we have that \( \pi(A) \) is closed in \( X \). Define

\[\alpha: P \times G \times X \rightarrow G \times X\]

\[\alpha(p, g, x) = (gp, x)\].
Then the set
\[ A' = \alpha^{-1}(A) = \{(p, g, x) \mid (gp, x) \in A\} = \{(p, gp^{-1}, x) \mid (g, x) \in A, p \in P\} \]
is closed in \( P \times G \times X \). Note that
\[ (p, g, x) \in A' \iff (gp, x) \in A \iff (gpQ, x) \subset A \iff (pQ, g, x) \subset A' \]
Thus, setting
\[ \varphi: P \times G \times X \to P/Q \times G \times X \]
\[ (p, g, x) \mapsto (pQ, g, x), \]
we observe that \( \varphi(A^{c}) = \varphi(A')^{c} \). Since \( \varphi \) is an open mapping (5.3.2(i)), this observation proves that \( \varphi(A') \) is closed. Let \( C \) denote the projection of \( \varphi(A') \) in \( G \times X \). Since \( P/Q \) is complete, the \( C \) is closed. Note that we can also write
\[ C = \cup_{(g, x) \in A} (gP, x). \]
Thus, we see that
\[ (g', x) \in C \iff (g'P, x) \subseteq C \]
By an analogous argument to the one in the previous paragraph, the projection of \( C \) in \( G/P \times X \) is closed, call this closed set \( C' \subseteq G/P \times X \). Since \( G/P \) is complete, it follows that the projection of \( C' \) to \( X \) is closed, but this projection is exactly \( \pi(A) \). \( \pi(A) \) is thus closed, which is what we wanted to prove. \( \square \)

As a partial converse, subgroups containing parabolic subgroups are also parabolic, as the lemma first part of the following lemma shows.

**Lemma 4. (6.2.4)**

(i) Let \( P \) be a parabolic subgroup of \( G \). If \( Q \) is a closed subgroup of \( G \) containing \( P \), then \( Q \) is parabolic in \( G \).

(ii) \( P \) is parabolic in \( G \) \iff \( P^{0} \) is parabolic in \( G^{0} \).

**Proof.** \( G/Q \) is complete since it is the image of the surjection \( G/P \to G/Q \), proving (i).

To prove (ii), note that \( G^{0} \) is parabolic in \( G \) and \( P^{0} \) is parabolic in \( P \) (remember \( G/G^{0} \) is just a finite set, so is a dimension zero projective variety). First suppose \( P \) is parabolic in \( G \). Then \( P^{0} \) is parabolic in \( G \) by the previous lemma, so \( G/P^{0} \) is projective. Since \( G^{0}/P^{0} \) is a closed subvariety of \( G/P^{0} \), it is also complete, so \( P^{0} \) is parabolic in \( G^{0} \). Conversely, if \( P^{0} \) is parabolic in \( G^{0} \), then \( P \) is parabolic in \( G \) by the previous lemma, so \( P \) is parabolic in \( G \) by (i). \( \square \)

**Lemma 5. (6.2.1)** If there is a bijective morphism of \( G \)-spaces \( X \to Y \), then \( X \) is complete if and only if \( Y \) is complete.

**Proof.** embedded in proof of next proposition \( \square \)

**Proposition 1.** A connected LAG \( G \) contains proper parabolic subgroups if and only if \( G \) is non-solvable.
Proof. First, suppose $G$ has no proper parabolic subgroups. As usual, we can assume $G$ is a closed subgroup of some $GL(V), V \cong k^n$. $G$ then naturally acts on $P(V)$. Let $X$ be a closed orbit for this action, which we know exists since closed orbits exist for $G$-spaces. Then $X$ is a complete, projective variety. For $x \in X$, we have that $G_x$, the isotropy group for $x$. The orbit-stabilizer theorem gives a bijective morphism of homogenous spaces:

$$G/P \rightarrow X$$

$$gG_x \mapsto g.x.$$ 

For any variety $Y$, then, we get a bijective morphism of $G$-spaces

$$G/G_x \times Y \rightarrow X \times Y$$

$$(gG_x, y) \rightarrow (g.x, y)$$

which is an open mapping by 5.3.2, and thus a homeomorphism. Thus, the projection $G/G_x \times Y \rightarrow Y$ is a closed mapping, because $X \times Y \rightarrow Y$ is closed mapping (as $X$ is complete). So $G_x$ is parabolic.

Write $P_0 = G_x$. In the case that $G_x = G$, set $V_1 = V/x$ (i.e. mod out by a 1-D subspace, since $x$ is a line in $V$). Then $G$ acts on $P(V_1)$. Take a closed orbit, and get a parabolic subgroup $P_1$ as before, by looking at some isotropy. If $P_1 = G$, mod out by another 1-D subspace corresponding to a point in the isotropy. Eventually, we will get a proper parabolic subgroup, or we will exhaust dimensions. If we exhaust all dimensions, then, geometrically, every element of $G$ fixes a chain of $n$ subspaces, one in each dimension. Thus, in a certain basis, every element of $G$ is upper triangular. Thus, $G$ is solvable (just start taking commutators of group of upper triangular matrix, and the first commutator makes all diagonal entries 1, and each successive kills a super diagonal, making all entries in that superdiagonal equal to 0).

Conversely, suppose $G$ is solvable. We show $G$ has no parabolic subgroup via induction on $\dim G$. Suppose for the sake of contradiction that $G$ has a proper parabolic subgroup. Let $P$ be a proper parabolic subgroup of maximal dimension. Since $G = G^0$, it suffices to show that the closed connected group $P^0$ is parabolic in $G$ in light of the last lemma. $[G, G]$ is a closed, connected subgroup of $G$. Let $Q = P^0 . [G, G]$. $Q$ is a group because $[G, G]$ is normal in $G$. Then $Q$ is a closed, connected group of $G$ containing $P^0$, so is parabolic. So either $Q = P^0$ or $Q = G$. If $Q = G$, then $G = P^0 \cdot [G, G]$, so the second isomorphism theorem says that the natural $G$-morphism

$$[G, G]/[G, G] \cap P^0 \rightarrow G/P^0$$

is bijective. Thus, $[G, G]/[G, G] \cap P^0$ is complete, so $[G, G] \cap P^0$ is parabolic in $[G, G]$. $[G, G]$ is of strictly smaller dimension than $G$, since $G$ is solvable. By our induction hypothesis, $[G, G] \cap P^0 = [G, G]$, so $[G, G] \subseteq P^0$, which contradicts $Q = G$. If, on the other hand, $Q = P^0$, then $[G, G] \subseteq P^0$, so $P^0 \leq G$ (basic group theory). Thus, from 5.5.10, $G/P^0$ is affine, but also complete, which means $G/P^0$ is finite and connected, so $G = P^0$. Contradiction.

The following theorem is due to Armand Borel, and is known as Borel’s Fixed Point Theorem.

**Theorem 1.** Let $G$ be a connected solvable LAG, and $X$ a complete $G$-variety. Then there is some $x \in X$ such that $G_x = G$.

Proof. There exists a closed orbit for $G$ in $X$. For some $x$ in the orbit, $G_x$ is parabolic (as we showed in the proof of the previous proposition). But we must have $G_x = G$, by the previous proposition.
Definition 3. A Borel subgroup of $G$ is a closed, connected, solvable subgroup of $G$, which is maximal for these properties.

Borel subgroups clearly exist. Just pick a closed, connected, solvable subgroup of maximal dimension.

The next theorem gives an alternate definition of a Borel subgroup: $B$ is Borel if and only if it is a minimal parabolic subgroup.

Theorem 2.  
(i) A closed subgroup of $G$ is parabolic if and only if it contains a Borel subgroup.

(ii) A Borel subgroup is parabolic.

(iii) Any two Borel subgroups are conjugate.

Proof. Any Borel subgroup is contained in $G^0$, and $P$ is parabolic in $G$ if and only if $P^0$ is parabolic in $G^0$. Thus, it suffices to assume $G$ is connected.

First we prove the "only if" direction for (i). Let $B$ be a Borel subgroup, $P$ a parabolic subgroup. The Borel fixed point theorem applied to the complete $B$-variety $G/P$, there is some $gP \in G/P$ such that $BgP = B$, i.e. $BgP \subset gP$, which shows that $P$ contains a conjugate of a Borel subgroup, which is also Borel.

We now prove (ii). Assume $G$ is non-solvable, or else (ii) is trivial as the only Borel subgroup is $G$ itself, which is clearly parabolic. Then $G$ has a proper parabolic subgroup $P$ of $G$. Clearly, $B$ is also Borel in $P$. Induction on dimension of $G$ shows that $B$ is parabolic in $P$. Thus, by transitivity of parabolicity, $B$ is parabolic in $G$.

The "if" direction is now done, using an earlier lemma.

Let $B, B'$ be two Borel subgroups. Then $B'$ is conjugate to a subgroup of $B$ and $B$ is conjugate to a subgroup of $B'$. Thus, $\dim B = \dim B'$, and we're done.

Corollary 1. (6.2.8) $\varphi : G \rightarrow G'$ surjective homomorphisms of LAGs. Let $P$ be a parabolic (resp. Borel) subgroup of $G$. Then $\varphi P$ is a subgroup of $G'$ of the same type.

Proof. By first part of theorem above, it suffices to consider the case of Borel subgroup $P$. Then $\varphi P$ is closed, connected, solvable. Since $G/P \rightarrow G'/\varphi P$ is surjective, $G'/\varphi P$ is complete, so $\varphi P$ is parabolic, and so $\varphi P$ contains a Borel subgroup of $G'$. Thus, $\varphi P$ is Borel.

A few remarks about the center of a Borel group. Here $B$ is a Borel group of $G$.

Corollary 2. $G$ connected, then $Z(G)^0 \subset Z(B) \subset Z(G)$

Proof. $Z(G)^0$ is closed and commutative (i.e. solvable) and connected, so is inside some Borel subgroup. Therefore, for some $g \in G$, $gZ(G)^0g^{-1} = Z(G)^0 \subseteq B$, hence the first inclusion.

For the other inclusion, if $g \in Z(B)$, the morphism $x \mapsto g\bar{x}g^{-1}\bar{x}^{-1}$ induces a morphism $G/B \rightarrow G$, since if $x = yb$, then $g\bar{y}^{-1}y^{-1} = gxb^{-1}g^{-1}bx^{-1} = gxg^{-1}x^{-1}$. Now $G/B$ is complete, so the image in $G$ is also complete, closed subvariety of $G$, i.e. a complete affine variety (since $G$ is affine). So the image of $G/B$ must be a point, $e$. Thus, $xg = gx$ for all $x \in G$, so $g \in Z(G)$, hence the second inclusion.