A Lower Bound for the Number of Finitely Maximal C_p-Actions on a Compact Oriented Surface

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Abstract. Let C_p denote a cyclic group of prime order $p \geq 7$. A topological action of C_p on a compact, oriented surface S of genus $\sigma \geq 2$ is said to be finitely maximal if there is no finite supergroup of homeomorphisms $G > C_p$. In the following, for sufficiently large genus σ, when non-zero, we show that the number of topologically distinct finitely maximal C_p-actions on a surface of genus σ is at least linear in σ.

1. Introduction

A finite subgroup H of the full group of orientation preserving homeomorphisms $\mathrm{Homeo}^+(S)$ of a compact oriented surface S of genus $\sigma \geq 2$ is said to be *finitely maximal* if there does not exist a finite $G < \mathrm{Homeo}^+(S)$ with $G > H$. Two subgroups $H_1, H_2 \leq \mathrm{Homeo}^+(S)$ are said to define topologically equivalent actions if they are conjugate in $\mathrm{Homeo}^+(S)$. For a given finite group G and genus $\sigma \geq 2$, let $N_{G,\sigma}$ denote the number of distinct topological G-actions on a surface of genus σ which are finitely maximal. In the following, we provide a lower bound for the number $N_{C_p,\sigma}$ for C_p a cyclic group of prime order $p \geq 7$. Specifically, for sufficiently large genus σ, when non-zero, we show that this number is at least linear in σ.

Though interesting in its own right, motivation for this work comes from a number of different places. For example, finitely maximal C_p-actions are in one-to-one correspondence with the isolated strata in the branch locus, $\mathcal{B}_\sigma$ of the moduli space $\mathcal{M}_\sigma$ of compact Riemann surfaces of genus σ. Providing bounds for $N_{C_p,\sigma}$ in turn provides bounds for the number of distinct isolated strata in $\mathcal{B}_\sigma$, and hence provides a bound for the total number of disconnected components in the branch locus. For further reading on the branch locus of moduli space, see also [**1**], [**2**], [**3**], [**4**], [**10**], [**11**], [**18**].

This work also has implications for counting conjugacy classes of finite subgroups of the mapping class group. Specifically, if $\mathfrak{M}_\sigma$ denotes the mapping class group in genus σ, then there is a natural one-to-one correspondence between conjugacy classes of finite subgroups of $\mathfrak{M}_\sigma$ and equivalence classes of finite topological group actions on a smooth oriented surface of genus σ. Moreover, if $H < G$ both act on a surface of genus σ, then we have the corresponding containment in $\mathfrak{M}_\sigma$. As such, our results provide a lower bound for the number of distinct conjugacy classes in $\mathfrak{M}_\sigma$ of subgroups isomorphic to C_p that are finitely maximal in $\mathfrak{M}_\sigma$. See [**6**], [**14**], and [**26**] for work in this area.

Classification and enumeration of group actions on compact oriented surfaces has spanned the literature for over a century dating back to Hurwitz, see for example [**15**], [**16**] and [**17**], and is still of current significant interest today. Due to the simplicity of their structure, cyclic group actions have been extensively studied. In [**13**], necessary and sufficient conditions for the existence of the action of a cyclic group are given. The problem of enumeration of classes of cyclic group actions was considered in [**20**] using the theory of generating functions, with an explicit generating function provided in the special case of a cyclic group of prime order. Other work, also using generating functions, to count classes of cyclic group actions appear in [**14**], and more recently, [**22**]. Similar results exist for other classes of groups. For example, conditions for the existence of an Abelian group action are considered in [**21**], and enumeration of such actions in [**22**] using generating functions, and [**6**] using more direct methods. There have also been significant contributions in the classification and enumeration of "large" automorphism groups and quasiplatonic groups. For example, in [**25**], it is shown that the number of classes of quasiplatonic Riemann surfaces of genus at most σ has growth type $\sigma^{\log(\sigma)}$. Using computers, other more direct methods have found explicit counts of classes of large automorphism groups for small (≤ 300) genus, see [**9**] for a recent survey of results in this area.

Our approach to the problem of enumeration is direct. A starting point for our work is the paper [**24**] in which it is shown that for sufficiently large genus, $N_{C_p,\sigma} = 0$ if and only if $\sigma \equiv (p-3)/2 \mod (p-1)/2$. The method used in [**24**] to prove this result is explicit – when $\sigma \equiv (p-3)/2 \mod (p-1)/2$, it is shown that any such action always has to extend (and in fact extend to the cyclic group of order $2p$) and outside of this sequence, to show $N_{C_p,\sigma} \neq 0$, an explicit action is constructed which cannot possibly extend to a larger finite group. Our general approach is to adapt and extend this method to construct additional finitely maximal actions, the number of which depend upon the number of fixed points of C_p. We shall then use this to construct a linear (in σ) lower bound.

Our work is outlined as follows. In Section 2, we provide all the necessary terminology and background results. Our approach to the problem is fairly standard, using the theory of Fuchsian groups and generating vectors. In Section 3, we develop the main results required to prove the result. The proof we offer is direct, providing explicit descriptions of these actions via generating vectors. We finish in Section 4 by proving the main result – that outside of an infinite sequence of genera, the number $N_{C_p,\sigma}$ is at least linear in σ.

2. Preliminaries

We approach the study of topological group actions via the theory of surface kernel epimorphisms and generating vectors as introduced in [**12**]. Since we are only considering actions of cyclic groups of prime order p, we simplify the notation, terminology and preliminary results to this case. For a more general approach see, for example [**5**].

A surface S of genus $\sigma \geq 2$ is topologically equivalent to a quotient of the upper half plane $\mathbb{H}/\Lambda$ where Λ is any torsion free Fuchsian group isomorphic to the fundamental group of S, also called a *surface group* for S. A cyclic group C_p

of prime order p acts on S if and only if $C_p = \Gamma/\Lambda$ for some Fuchsian group Γ containing such a Λ as a normal subgroup of index p. We call the map $\rho\colon \Gamma \to C_p$ a *surface kernel epimorphism*.

A presentation of Γ is completely determined by the genus h of the quotient surface S/C_p and the number r of fixed points of C_p on S and is given by

$$\Gamma = \left\langle a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r \mid c_i^p, \prod_{j=1}^{h} [a_j, b_j] \prod_{i=1}^{r} c_i \right\rangle \tag{1}$$

where

$$\sigma = 1 + p(h-1) + r\left(\frac{p-1}{2}\right).$$

Note that the map ρ is completely determined by the images of the generators of Γ so a convenient way of representing a surface kernel epimorphism is through a so-called generating vector, defined as follows:

DEFINITION 2.1. A vector of group elements $(\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \eta_1, \ldots, \eta_r)$ belonging to C_p is called a (h, r)-*generating vector* for C_p with genus σ if all of the following hold:

(1) $C_p = \left\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \eta_1, \ldots, \eta_r \right\rangle$
(2) $\prod_{j=1}^{r} \eta_j = 1$
(3) Each η_i is non-trivial
(4) The Riemann-Hurwitz formula holds:

$$\sigma = 1 + p(h-1) + r\left(\frac{p-1}{2}\right)$$

For conciseness, in a generating vector for C_p, we adopt the notation $(\alpha)^k$ to mean k copies of α and α^k to mean a single α raised to the kth power. Since it will be important later, we call the vector of group elements $(\eta_1, \ldots, \eta_r)$ containing the last r elements of a generating vector the the *tail* of the generating vector.

A topological group action gives rise to a generating vector via the corresponding surface kernel epimorphism. Likewise, a generating vector gives rise to a topological group action by defining a surface kernel epimorphism. Therefore, we shall often state that a generating vector defines a topological group action of C_p, and by this, we mean the group action determined by the corresponding surface kernel epimorphism.

Distinguishing between topological equivalence classes of group actions was first considered in [**23**] (see also [**12**] for cyclic prime group actions, [**8**, Theorem 7] for all cyclic groups and [**20**] for all groups.) When applied to the special case of a cyclic group of prime order p, this classification implies when $r = 0$, all generating vectors define topologically equivalent actions, and when $r > 0$, we have the following criteria to distinguish between topologically distinct actions in terms of their generating vectors:

THEOREM 2.2. *Fix a prime p. For $r > 0$ two (h, r)-generating vectors*

$$(\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \eta_1, \ldots, \eta_r) \text{ and } (\alpha_1', \beta_1', \ldots, \alpha_h', \beta_h', \eta_1', \ldots, \eta_r')$$

for C_p define topologically equivalent group actions if and only if there exists a permutation $\chi \in S_r$ and $\tau \in \mathrm{Aut}(C_p)$ such that

$$(\tau(\eta_{\chi(1)}), \dots, \tau(\eta_{\chi(r)})) = (\eta'_1, \dots, \eta'_r)$$

i.e. *the tails differ by permutation and/or automorphism of C_p.*

For brevity, given the tail $\mathcal{T} = (\eta_1, \dots, \eta_r)$ of a generating vector, a permutation $\chi \in S_r$ and an automorphism $\tau \in \mathrm{Aut}(C_p)$, we let $\tau(\mathcal{T}(\chi))$ denote the composition $(\tau(\eta_{\chi(1)}), \dots, \tau(\eta_{\chi(r)}))$.

Since our primary goal is to determine when a given C_p-action is finitely maximal we adopt this term for generating vectors themselves. That is, when we say a generating vector is (or is not) finitely maximal, it is understood that the corresponding topological group action is (or is not) finitely maximal.

In [**24**], it is shown the genus of the quotient surface S/C_p for a finitely maximal C_p-action satisfies $h < (p-3)/2$. The following result, also from [**24**], proved using the techniques employed in [**7**], provides necessary and sufficient conditions for when a given generating vector defines a finitely maximal C_p-action in terms of its tail:

THEOREM 2.3. *Let $\mathcal{V} = (\eta_1, \dots \eta_r)$ be the tail of a (h, r)-generating vector for C_p where $h < (p-3)/2$. If $\mathcal{V}$ defines an action which is not finitely maximal, then C_p is a subgroup of either $C_p \times C_p$, C_{pq} or $C_p \rtimes C_q$ for some prime q. Moreover:*

1. *$C_p < C_{pq}$ if and only if there exist integers $g, k, m \geq 0$ such that $h = gq + (k+m-2)\left(\frac{q-1}{2}\right)$ and after a reordering of the η_i's, $\mathcal{V}$ has the form $((c_1)^q, \dots, (c_n)^q, f_1, \dots, f_k)$ where $r = qn + k$.*
2. *$C_p < C_p \rtimes C_q$ if and only if there exist integers $g, m \geq 0$ such that $h = gq + (m-2)\left(\frac{q-1}{2}\right)$ and after a reordering of the η_i's, there exists an integer α with $\alpha^q \equiv 1 \mod (p)$ so $\mathcal{V}$ has the form $\left(c_1, c_1^{\alpha}, \dots, c_1^{\alpha^{q-1}}, c_2, c_2^{\alpha}, \dots c_n^{\alpha^{q-1}}\right)$ where $r = qn$.*
3. *$C_p < C_p \times C_p$ if and only if there exist integers $g, m \geq 0$ such that $h = gp + (m-2)\left(\frac{p-1}{2}\right)$ and after a reordering of the η_i's, $\mathcal{V}$ has the form $((c_1)^p, \dots (c_n)^p))$ where $r = np$.*

If none of these conditions is satisfied then $\mathcal{V}$ defines a finitely maximal C_p-action

The following Corollary is immediate.

COROLLARY 2.4. *Let $\mathcal{V} = (\eta_1, \dots \eta_r)$ be the tail of a (h, r)-generating vector for C_p and suppose $\eta \in C_p$ appears exactly $n > 1$ times in $\mathcal{V}$ i.e. there are exactly n η_i's equal to η. If no other element of C_p appears exactly n times, then C_p does not extend to $C_p \rtimes C_q$ or $C_p \times C_p$.*

Note that if $\mathcal{V}$ is the tail of a generating vector for $C_p = \langle x \rangle$, then by Theorem 2.2, it is equivalent to a tail of the form $((x)^{\alpha_1}, (x^2)^{\alpha_2}, \dots, (x^{p-1})^{\alpha_{p-1}})$ i.e. we simply permute all like powers to be consecutive in the generating vector. With this in mind, we have the following useful consequence of Theorem 2.3

COROLLARY 2.5. *Let $\mathcal{V} = ((x)^{\alpha_1}, (x^2)^{\alpha_2}, \dots, (x^{p-1})^{\alpha_{p-1}})$, be the tail of a (h, r)-generating vector for C_p which extends to C_{pq} for some prime q. Then for k defined in Theorem 2.3, we have $k \geq r_1 + r_2 + \cdots + r_{p-1}$ where r_i is the remainder of α_i after division by q.*

PROOF. Let $\alpha_i = q\beta_i + r_i$ where r_i is the remainder of α_i after division by q. By Theorem 2.3, the tail of a C_p-generating vector which extends to the action of

C_{pq} will be equivalent to a tail of the form $((c_1)^q, \dots, (c_n)^q, f_1, \dots, f_k)$. We rewrite the vector $\mathcal{V} = ((x)^{\alpha_1}, (x^2)^{\alpha_2}, \dots, (x^{p-1})^{\alpha_{p-1}})$ to be of this form. First, since there are $\beta_1 q + r_1$ total copies of x and $r_1 < q$, then there are at most β_1 repetitions of q-copies at the start of the tail. This leaves a minimum of r_1 copies of x which must appear as individual terms at the end of the tail. Using a similar argument with x^i for each i we see that are minimally r_i copies of x^i that must appear individually at the end of tail. Thus $k \geq r_1 + \cdots + r_{p-1}$. $\square$

3. Bounding Actions by the Length of the Tail

In order to determine a lower bound for the number of C_p actions, we shall first describe a method to create tails of generating vectors that do not satisfy any of the conditions of Theorem 2.3 necessary for extension to a larger group. We emphasis that our work in this section only considers tails of generating vectors, so in particular, the bound we develop will not be in terms of the genus of the corresponding surface on which it acts, but rather the length of the tail r. We shall consider how this relates to the genus in the next section.

Henceforth, let x denote a generator of C_p.

LEMMA 3.1. *Let* $\mathcal{T} = (x, x^2, \dots, x^{p-1}, (x)^i, (x^a)^j, (x^b)^l)$ *and* $r = i+j+l+(p-1)$ *for positive integers* i, j, l, a, b *satisfying:*

(1) $l \in \{4, 6\}$ *if* r *is even, and* $l \in \{1, 3\}$ *if* r *is odd*
(2) $ja \not\equiv -i \mod (p)$
(3) $lb \equiv -(i + ja) \mod (p)$
(4) $1 < a \leq p - 1$
(5) i *is even*
(6) $i > l$
(7) $j > i + l + 1$

Then provided $h < (p-3)/2$, $\mathcal{T}$ *defines the tail of a finitely maximal* (h, r)*-generating vector for* C_p.

PROOF. First we check $\mathcal{T}$ defines the tail of a generating vector. In order to do this, we need to show it satisfies the second condition of Definition 2.1 – that is, we need to show the product of the elements in $\mathcal{T}$ is the identity. Now we know $1 + 2 + \cdots + (p-1) \equiv 0 \mod (p)$, so the product of the first $p - 1$ elements is the identity. Therefore, we need to show the product of the remaining elements is also the identity, or equivalently that $i + aj + bl \equiv 0 \mod (p)$. This is ensured by the assumption $lb \equiv -(i + ja) \mod (p)$ provided we can solve for b. However, we are also assuming $ja \not\equiv -i \mod (p)$ and thus there is always a solution for b. Therefore, $\mathcal{T}$ defines the tail of a generating vector for C_p.

Next we check maximality. Since $j > i + l + 1$ and $a \neq 1$, x^a appears at least $j+1$ times (possibly more depending upon the value of b). No other element of C_p appears in $\mathcal{T}$ this many times and therefore containment in $C_p \times C_p$ or $C_p \rtimes C_q$ is not possible by Corollary 2.4. This leaves containment in C_{pq} as the only possibility.

Suppose that $\mathcal{T}$ does extend to C_{pq} for some q. Then there exists k and m such that $\mathcal{T}$ can be written in the form given in Theorem 2.3. Since x, x^a and x^b are the only elements to appear more than one time, by Corollary 2.5, we must have $k \geq (p-4)$. This means when $q \geq 5$ we have

$$h = gq + (k + m - 2)\left(\frac{q-1}{2}\right) \geq (p-6)\left(\frac{q-1}{2}\right) > \frac{p-5}{2}$$

contradicting our assumption that $h < (p-3)/2$. Therefore, we only need consider $q = 3$ and $q = 2$.

For $q = 3$, if $b = 1$ or $b = a$, then there are at most two distinct elements which appear more than once in the tail, and therefore by Corollary 2.5, we must have $k \geq (p-3)$. This means we have

$$h = 3g + (k + m - 2)\left(\frac{3-1}{2}\right) \geq (p-5) > \frac{p-5}{2}$$

contradicting our assumption that $h < (p-3)/2$. If b is distinct from 1 and a, then x^b appears precisely $l+1$ times. For each given value of l, $l+1$ is never a multiple of 3. Therefore, since there are $p-4$ elements which each appear precisely once, and x^b appears $l+1$ times which has a positive remainder after division by 3, we must again have $k \geq p-3$ and we obtain the same contradiction.

For $q = 2$, a similar argument holds. In this case, we first observe that since i is even, for each given choice of l, since $r = i + j + l + (p-1)$, j is also even. It follows that if 1, a and b are all distinct, then x appears $i+1$ times and x^a appears $j+1$ times. In particular, since $i+1$ and $j+1$ are odd, and there are at least $p-4$ elements which appear exactly once, by Corollary 2.5, we must have $k \geq (p-2)$. This means we have

$$h = 2g + (k + m - 2)\left(\frac{2-1}{2}\right) \geq \frac{(p-4)}{2} > \frac{p-5}{2}$$

again contradicting our assumption that $h < (p-3)/2$. If $b = a$ or $b = 1$, similar reasoning also implies $k \geq p-2$ and the result follows.

□

Next we shall provide conditions for when the tails defined in Lemma 3.1 define distinct C_p-actions on a given surface for a fixed h, the genus of the quotient surface, and r the number of fixed points of the action. In order to do this, we first need the following simple Lemma which provides bounds on j and i.

Lemma 3.2. *For any tail $\mathcal{T}$ satisfying Lemma 3.1, we have:*

(1) $j > \dfrac{r-p}{2} + 1$
(2) $i < \dfrac{r-p}{2} - l$

Proof. We know $r = i + j + l + (p-1)$, so we have $j = r - (i + l + (p-1))$. Therefore, the last condition of Lemma 3.1 implies

$$j = r - (i + l + (p-1)) = r - (i + l + 1 + (p-2)) > r - (j + (p-2)).$$

Therefore we get

$$2j > r - (p+2) \text{ and so } j > \frac{r-(p-2)}{2} = \frac{r-p}{2} + 1.$$

The inequality for i then follows from the last condition of Lemma 3.1.

□

Lemma 3.3. *For the tail defined in Lemma 3.1, for a given l and r, any choice of the ordered pair (a, i) completely determines the topological equivalence class of a C_p-action. Moreover, if two pairs (a_1, i_1) and (a_2, i_2) define the same action, then either $(a_1, i_1) = (a_2, i_2)$ or $|i_1 - i_2| = l$.*

PROOF. For a given l and r, given a pair (a,i), the remaining integers are found using the conditions of Lemma 3.1. Specifically, $j = r - i - l - (p-1)$ and b is the unique solution to $lb \equiv -(i + ja) \mod (p)$. Therefore, the generating vector and hence the topological equivalence class of a C_p-action is determined by the pair (a,i).

Next we show that when $(a_1, i_1) \neq (a_2, i_2)$, the tails

$$\mathcal{T}_1 = \left(x, x^2, \ldots, x^{p-1}, (x)^{i_1}, (x^{a_1})^{j_1}, (x^{b_1})^l\right)$$

and

$$\mathcal{T}_2 = \left(x, x^2, \ldots, x^{p-1}, (x)^{i_2}, (x^{a_2})^{j_2}, (x^{b_2})^l\right)$$

define topologically distinct actions with the possible exception of when $|i_1 - i_2| = l$. By Theorem 2.2, in order to do this, we need to show that there is no automorphism τ of C_p and permutation χ of r such that $\tau(\mathcal{T}_1(\chi)) = \mathcal{T}_2$, so we assume there is.

First note that any automorphism of the first $p-1$ elements of $\mathcal{T}_1$ is simply a permutation of those elements, and therefore, after an appropriate permutation can be put back in the same order. Secondly, observe that the number of elements repeated more than once and the occurrences of those repeated elements will be the same after application of an automorphism of C_p and a permutation. Therefore, since we are assuming $\tau(\mathcal{T}_1(\chi)) = \mathcal{T}_2$, we can actually assume

$$(x, x^2, \ldots, x^{p-1}, (\tau(x))^{i_1}, (\tau(x)^{a_1})^{j_1}, (\tau(x)^{b_1})^l) = (x, x^2, \ldots, x^{p-1}, (x)^{i_2}, (x^{a_2})^{j_2}, (x^{b_2})^l).$$

Now, if x, x^{a_1} and x^{b_1} are all distinct, then so are $\tau(x)$, $\tau(x)^{a_1}$ and $\tau(x)^{b_1}$. Therefore, since $l < i_1 < j_1$ and $l < i_2 < j_2$, it follows that $\tau(x) = x$, $i_1 = i_2$, $j_1 = j_2$, $a_1 = a_2$ and $b_1 = b_2$. In particular, $(a_1, i_1) = (a_2, i_2)$, a contradiction.

If x, x^{a_1} and x^{b_1} are not distinct, then neither are x, x^{a_2} and x^{b_2}, and we either have $b_1 = a_1$ or $b_1 = 1$ and similarly $b_2 = a_2$ or $b_2 = 1$. This means we have one of the four following possibilities:

(1) $(x, x^2, \ldots, x^{p-1}, (\tau(x))^{i_1+l}, (\tau(x)^{a_1})^{j_1}) = (x, x^2, \ldots, x^{p-1}, (x)^{i_2+l}, (x^{a_2})^{j_2})$
(2) $(x, x^2, \ldots, x^{p-1}, (\tau(x))^{i_1+l}, (\tau(x)^{a_1})^{j_1}) = (x, x^2, \ldots, x^{p-1}, (x)^{i_2}, (x^{a_2})^{j_2+l})$
(3) $(x, x^2, \ldots, x^{p-1}, (\tau(x))^{i_1}, (\tau(x)^{a_1})^{j_1+l}) = (x, x^2, \ldots, x^{p-1}, (x)^{i_2+l}, (x^{a_2})^{j_2})$
(4) $(x, x^2, \ldots, x^{p-1}, (\tau(x))^{i_1}, (\tau(x)^{a_1})^{j_1+l}) = (x, x^2, \ldots, x^{p-1}, (x)^{i_2}, (x^{a_2})^{j_2+l})$

By Lemma 3.2,

$$i_1 < i_1 + l < (r-p)/2 < (r-p)/2 + 1 < j_2 < j_2 + l,$$

so in particular, it follows that

(1) $i_1 + l < j_2$
(2) $i_1 + l < j_2 + l$
(3) $i_1 < j_2$
(4) $i_1 < j_2 + l$.

Therefore, for all four cases, we must have $\tau(x) = x$ and $i_1 + l = i_2 + l$ in case (1), $i_1 + l = i_2$ in case (2), $i_1 = i_2 + l$ in case (3) and $i_1 = i_2$ in case (4). In particular, $a_1 = a_2$ in all cases, and in cases (1) and (4) we have $i_1 = i_2$, a contradiction, and in cases (2) and (3) we have $|i_1 - i_2| = l$. $\square$

Though ideally we would like the topological equivalence class of an action to be uniquely determined by a pair (a,i), we note that, as Lemma 3.2 indicates, this is not always the case. For example, if l is even, there always exists equivalent actions given by different pairs provided there exists a tail in which $b = a$ or $b = 1$. Specifically, if $b = 1$, then

$$(x, x^2, \dots, x^{p-1}, (x)^i, (x^a)^j, (x)^l)$$

with pair (a, i) is equivalent to

$$(x, x^2, \dots, x^{p-1}, (x)^{i+l}, (x^a)^{j-l}, (x^a)^l)$$

with pair $(a, i + l)$. On the other hand, when $b = a$,

$$(x, x^2, \dots, x^{p-1}, (x)^i, (x^a)^j, (x^a)^l)$$

with pair (a, i) is equivalent to

$$(x, x^2, \dots, x^{p-1}, (x)^{i-l}, (x^a)^{j+l}, (x)^l)$$

with pair $(a, i - l)$.

We are now ready to provide a lower bound for the number of distinct actions using the tails defined in Lemma 3.1.

THEOREM 3.4. *When non-zero, the number $N_{C_p.\sigma}$ of topologically distinct finitely maximal C_p-actions with tail of length $r > p + 20$ on a surface S of genus $\sigma \geq 2$ satisfies*

$$N_{C_p,\sigma} \geq \frac{(p-3)(r-p-20)}{8}.$$

PROOF. In order to find a lower bound, we shall filter the set of tails of the form given in Lemma 3.1 to a subset where each define a topologically distinct action. In order to do this, we must first determine which tails of this form satisfy the seven conditions of Lemma 3.1.

Suppose $\mathcal{T} = (x, x^2, \dots, x^{p-1}, (x)^i, (x^a)^j, (x^b)^l)$ for positive integers i, j, l, a, b. Since we are assuming $r > p + 20$, all conditions of Lemma 3.1 can be imposed on $\mathcal{T}$ except possibly conditions (2) and (3). Also note that satisfaction of condition (2) guarantees that there is a choice of b that satisfies condition (3). Therefore, we need to filter our set of tails to a subset for which condition (2) is guaranteed to hold.

If condition (2) fails, then $i + aj \equiv 0 \mod (p)$. Since $\gcd(a, p) = 1$, it follows that either p divides both i and j, or p divides neither i nor j. In the latter case, we get $a \equiv j^{-1}(-i) \mod (p)$. In particular, a is completely determined by i and j, so for a fixed i and j, there is only one possibility for a. Therefore, for each i and j, we can simply exclude this value of a to ensure condition (2) holds.

If p divides both i and j then condition (2) never holds, so we shall impose conditions on the tail to avoid this happening. Now, if p divides both i and j, then it must divide $i + j$, so it suffices to provide conditions ensuring that p does not divide $i + j$. For a fixed r and l, since $r = (p - 1) + i + j + l$, we have $i + j = r - (p - 1) - l$ and so $i + j \equiv r - l + 1 \mod (p)$. Therefore, if r is even, we choose $l = 4$ when $r \not\equiv 3 \mod (p)$ and $l = 6$ else, and when r is odd, we choose $l = 1$ when $r \not\equiv 0 \mod (p)$ and $l = 3$ else. Through these choices of l, we never have $i + j \equiv 0 \mod (p)$ and so condition (2) will always be satisfied (except for the choice of a previously excluded).

Next we need to filter this set of tails further so that each one defines a unique topological equivalence class. By Lemma 3.3, we know that if two tails with pairs (a_1, i_1) and (a_2, i_2) define the same action, then either they are the same pair, or $|i_1 - i_2| = l$. Therefore, in order to ensure every tail defines a distinct action, we simply restrict the values of i to a maximal subset of integers so that the difference of two members is never l. Specifically, for l odd, since we have already restricted i to the even numbers, there are no values for which $|i_1 - i_2| = l$, so we use the same set. For l even, we use the set $\alpha l + \beta$ where α is odd and β runs over the even integers $\{0, 2, \ldots, l-2\}$. Note that in each case, this will always be at least half of the possible values of i.

We are now ready to count, and for this, we need to count the maximum number of pairs (a, i) with the restrictions we have imposed. Since $a \neq 1$ and for a given i and j, $a \not\equiv j^{-1}(-i) \mod (p)$, there are precisely $p-3$ possible choices for a. Now we are assuming $i > l$ and by Lemma 3.2 we have $i < (r-p)/2 - 2$. We are also assuming i is even so $i = 2t$ for some t. Since $l \leq 6$, this means $8 \leq 2t < (r-p)/2 - 2$, and so $4 \leq t < (r-p)/4 - 1$. Therefore, there are at most

$$\frac{r-p}{4} - 1 - 4 = \frac{r-p-20}{4}$$

choices for t, and hence for i. Filtering this set further to ensure that no two values differ by l leaves at least half of these. Thus the total number of possibilities for i is at least $\frac{r-p-20}{8}$, and therefore, the total number of actions is at least

$$\frac{(p-3)(r-p-20)}{8}.$$

□

4. A Lower Bounds for $N_{C_p,\sigma}$

When non-zero, Theorem 3.4 provides a lower bound for $N_{C_p,\sigma}$ in terms of r, the length of the tail of a generating vector. We shall now use this result to show that the number $N_{C_p,\sigma}$, when non-zero, is always bounded below by a linear function in σ for sufficiently high σ.

Theorem 4.1. *If $\sigma \equiv (p-3)/2 \mod (p-1)/2$, then $N_{C_p,\sigma} = 0$. Else, for $\sigma > p^2 + 7p - 9$, the number $N_{C_p,\sigma}$ of finitely maximal C_p actions on a surface of genus σ is bounded below by a linear function of σ.*

Proof. By [**24**, Corollary 1], if $\sigma \equiv (p-3)/2 \mod (p-1)/2$, then $N_{C_p,\sigma} = 0$. Therefore, we shall henceforth assume $\sigma \not\equiv (p-3)/2 \mod (p-1)/2$.

Now, if C_p acts on a surface S with h the genus of the quotient surface S/C_p and r the number of fixed points then the Riemann-Hurwitz formula holds:

$$\sigma = 1 + p(h-1) + r\left(\frac{p-1}{2}\right).$$

In order to apply Theorem 3.4, we need $r > p + 20$. Therefore, since for a finitely maximal action we know $0 \leq h \leq \frac{p-5}{2}$ and since the Riemann-Hurwitz formula is an increasing function in both r and h, if we restrict to

$$\sigma > 1 + p\left(\frac{p-3}{2} - 1\right) + (p+20)\left(\frac{p-1}{2}\right) = p^2 + 7p - 9,$$

then the condition $r > p + 20$ is guaranteed to hold. Therefore, we henceforth assume σ satisfies this bound.

Now, since $\sigma > p^2 + 7p - 9 > \frac{1}{2}((p(p-4)+1)$, there always exists a C_p-action on a surface S of genus σ, see [**19**, Corollary 5.4]. In addition, by [**24**, Theorem 4], provided $\sigma \not\equiv (p-3)/2 \mod (p-1)/2$, then there always exists a C_p-action with

S/C_p of genus h for some unique $h < (p-3)/2$ and r, the number of fixed points, which can be found using the Riemann-Hurwitz formula:

$$r = \frac{2(\sigma - 1 - p(h-1))}{p-1}.$$

Given the assumptions made on σ, we know $r > p + 20$ and so we can use the bound developed in Theorem 3.4 to bound $N_{C_p,\sigma}$ in terms of the length of the tail r. Specifically, we have:

$$\begin{aligned} N_{C_p,\sigma} \geq \frac{(p-3)(r-p-20)}{8} &= \frac{(p-3)(\frac{2(\sigma-1-p(h-1))}{p-1} - p - 20)}{8} \\ &= \frac{p-3}{4(p-1)}\sigma - \frac{((p-3)(p(2h+p+17)-18))}{8(p-1)} \end{aligned}$$

Since $0 \leq h \leq \frac{p-5}{2}$, this gives

$$\begin{aligned} N_{C_p,\sigma} &\geq \frac{p-3}{4(p-1)}\sigma - \frac{((p-3)(p(2h+p+17)-18))}{8(p-1)} \\ &\geq \frac{p-3}{4(p-1)}\sigma - \frac{(p-3)(p^2+6p-9)}{4(p-1)} = A\sigma + B \end{aligned}$$

for constants A and B dependent only on p. The result follows.

□

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