My research utilizes techniques from **geometric group theory** to understand groups from **low-dimensional topology**. The starting observation of geometric group theory is that the Cayley graphs of a finitely generated group with respect to different finite generating sets are all **quasi-isometric**, i.e., distances differ only by an additive and multiplicative constant. This allows any finitely generated group to be given a geometry up to quasi-isometry by identifying it with its Cayley graph. This point of view has revolutionized our understanding of finitely generated groups and produced deep applications in logic, topology, geometry, and dynamical systems.

Much of my work draws inspiration from Gromov's theory of **hyperbolic groups** to study the **mapping class group** of a surface. The mapping class group, $\mathrm{MCG}(S)$, of an orientable surface S is the group of homeomorphisms of S modulo isotopy. As the topological symmetry group of S, the mapping class group is a central object in mathematics, appearing in algebraic geometry, number theory, dynamics, geometric topology, complex analysis, and group theory. Hyperbolic groups possess a powerful, quasi-isometric invariant notion of negative curvature. While the mapping class group is not hyperbolic, a major theme of my research is building bridges between Gromov's hyperbolic groups and the mapping class group via common generalizations of both. This high level perspective allows me to pursue specific results for the mapping class group, while simultaneously producing advances is several other important classes of groups, such as the fundamental groups of 3-manifolds and CAT(0) groups.

I present my research program in three parts, which I address in detail below.

§1 **Convex cocompactness and surface group extensions.** I have resolved a special case of a question of Farb–Mosher on a dynamical characterization of convex cocompact subgroups of the mapping class group (Theorem 1), and answered a question of Dowdall–Durham–Leininger–Sisto on the geometry of surface group extension of multicurve stabilizers (Theorem 2).

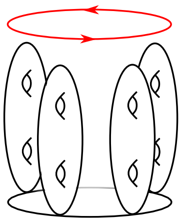
§2 **Morse local-to-global groups.** A key bridge my collaborators and I have built between hyperbolic groups and the mapping class group. This framework has yielded advances in subgroup structure (Theorem 7), a quasi-isometry invariant called the Morse boundary (Theorems 8, 9), and subgroup growth (Theorem 5). The later answers a question of Farb in the case of $\mathrm{MCG}(S)$.

§3 **Hierarchical hyperbolicity.** A generalization of Gromov's hyperbolicity introduced by Behrstock–Hagen–Sisto. Hierarchical hyperbolicity encompasses the mapping class group, most 3-manifold groups, and many Artin groups, while producing powerful geometric and algebraic consequences. My work has produced a foundational understanding of convexity properties in this class (Theorems 10, 11, 12), illuminated the the relationship between hierarchical and relative hyperbolicity (Theorems 14, 17), taken some of the first steps to understand the hierarchically hyperbolic boundary (Theorem 16), and resolved several conjectures of Behrstock–Hagen–Sisto on both graph products and 3-manifolds (Theorems 19, 20, 21, 22).

1. Convex cocompactness and surface group extensions

There is a long running and fruitful analogy between discrete groups of isometries of the hyperbolic n-space $\mathbb{H}^n$ and subgroups of $\mathrm{MCG}(S)$. Inspired by this analogy, Farb–Mosher transferred the notion of a **convex cocompact** subgroup of $\mathrm{Isom}(H^n)$ to subgroups of $\mathrm{MCG}(S)$. Work of Kent–Leininger and Hamenstädt has shown that is analogy is incredibly robust, with several different equivalent characterization of convex cocompactness in $\mathrm{MCG}(S)$ mirroring the several equivalent characterization of convex cocompactness in $\mathrm{Isom}(\mathbb{H}^n)$[55, 61, 62]. Combined with subsequent work from Bestvina, Bromberg, Taylor, Durham, and others, these results have established convex cocompact subgroups as the "best behaved" subgroups of $\mathrm{MCG}(S)$ [15, 40, 64].

Beyond this satisfying analogy, convex cocompact subgroups have a critical role in the study of $\pi_1(S)$-extensions and surface bundles. A $\pi_1(S)$-extension is any group E that fits into a short exact sequence $1 \to \pi_1(S) \to E \to G \to 1$. For closed surfaces, each $\pi_1(S)$-extension has a **monodromy homomorphism** $G \to \mathrm{MCG}(S)$. The best examples of these extensions come from topology: if M is a S-bundle over a base manifold B, then we have the short exact sequence of groups $1 \to \pi_1(S) \to \pi_1(M) \to \pi_1(B) \to 1$. The monodromy now comes from assigning each loop in B to the homeomorphism of S produced by moving the fibers along that loop. This topological monodromy exists even when S is not closed.



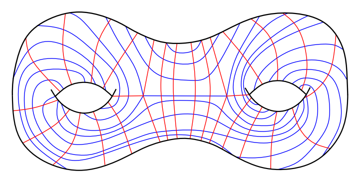
A surface bundle with base a circle

Each S-bundle or $\pi_1(S)$-extension is determined (up to homeomorphism/isomorphism) by these monodromy homomorphisms, and given a homomorphisms into $\mathrm{MCG}(S)$, there is a unique $\pi_1(S)$-extension with that monodromy. Understanding the still mysterious relationship between properties of bundles/extensions and properties of the monodromy is a fundamental problem in the study of the mapping class group.

Convex cocompactness connects to the geometry of $\pi_1(S)$-extensions by results of Farb–Mosher and Hamenstädt, which say for closed surfaces, a $\pi_1(S)$-extension is Gromov hyperbolic if and only if the monodromy has finite kernel and convex cocompact image [45, 55]. Hence, the study of convex cocompact subgroups of $\pi_1(S)$ is essentially the study of Gromov hyperbolic $\pi_1(S)$-extensions.

Here I will describe two components of my research on convex cocompact subgroups. Convex cocompactness will also appear in §2 and §3 as a source of both inspiration and applications of results obtained in much broader contexts than the mapping class group.

1.1. **Purely pseudo-Anosov subgroups.** A basic consequence of convex cocompactness is that every infinite order element of the subgroup is **pseudo-Anosov**. Informally, these are the elements of $\mathrm{MCG}(S)$ with the richest dynamics on the the surface. Formally, each pseudo-Anosov element stabilizes a pair of transverse singular foliations of the surface S and acts S by contracting along one foliation and expanding along the other. In their introductory paper on convex cocompactness, Farb–Mosher asked if the converse of this property is true, that is:



The foliations of a pseudo-Anosov

Question 1. If $H < \mathrm{MCG}(S)$ is finitely generated and every non-trivial element is pseudo-Anosov, is H convex cocompact?

Question 1 it one of the paramount questions in the study of convex cocompact subgroups, with either answer having significant implications. A "yes" answer would impose severe geometric limitations on the groups that could appear as convex cocompact subgroups of $\mathrm{MCG}(S)$, which in turn would limit the possible hyperbolic $\pi_1(S)$-extensions. On the other hand, a "no" answer would produce a $\pi_1(S)$-extension with interesting pathologies [45].

While Question 1 is still open, partial results have been obtained in several cases by Bestvina, Bromberg, Kent, Leininger, Dowdall, Koberda, Mangahas, Taylor, and Tshishiku [15, 35, 63, 65, 79]. In joint work with Leininger, I answer Question 1 in the positive for subgroups that live inside certain non-hyperbolic, fibered 3-manifold groups in the mapping class group.

Theorem 1 ([LR])**.** *The answer to Question 1 is yes, if H is a subgroup of $\pi_1(M)$, where M is a non-hyperbolic, non-virually $S \times \mathbb{S}^1$ 3-manifold so that $\pi_1(M)$ injects into the mapping class group via the Birman exact sequence.*

The cases of Theorem 1 where M is hyperbolic or virtually $S \times \mathbb{S}^1$ were previously established by Dowdall–Kent–Leininger [35] and Kent–Leininger–Schleimer [63] respectively[1]. Hence, Theorem 1 completes the resolution of Question 1 for this class of subgroups. Compared to the previous cases, the main difficulty in Theorem 1 was that there was no intrinsic hyperbolic geometry from the manifold to draw on. Instead we use a combination of **Bass–Serre theory** and Masur–Minsky's **subsurface projections** to achieve the required geometric control.

[1]In the $S \times \mathbb{S}^1$ case, $\pi_1(M)$ does not inject into $\mathrm{MCG}(S^\circ)$ and so the result is about the image of $\pi_1(M) \to \mathrm{MCG}(S^\circ)$.

Future work. There are several excellent candidate subgroups of $\mathrm{MCG}(S)$ for continued positive progress similar to Theorem 1. Candidates included the handlebody group, the stablizers of vector fields, and surface braid groups. These groups are promising because, like the 3-manifold groups in Theorem 1, they have additional topological structure as well as useful actions on natural graphs of curves that can be exploited to understand their purely pseudo-Anosov subgroups.

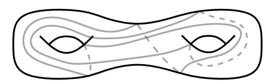
1.2. **Geometric finiteness.** Amoung Kleinian groups, convex cocompact groups are a special case of **geometrically finite** Kleinian groups. Unlike convex cocompactness, a robust definition of geometric finiteness in $\mathrm{MCG}(S)$ has yet to be established. Mosher proposed that such a definition should extend the equivalence between hyperbolic $\pi_1(S)$-extensions and convex cocompactness [71].

Question 2. Is there a definition of a geometrically finite subgroup of $\mathrm{MCG}(S)$ so that geometric finiteness characterizes some generalized hyperbolicity of $\pi_1(S)$-extensions?

Despite our lack of a good definition of geometric finiteness, there are several examples of subgroups of $\mathrm{MCG}(S)$ that ought to be included in any robust definition. The first such example are the **Veech subgroups**, which act geometrically finitely on an isometrically embedded copy of $\mathbb{H}^2$ inside of the Teichmüller space of S [71]. Dowdall–Durham–Leininger–Sisto recently proved that the $\pi_1(S)$-extensions corresponding to lattice Veech groups are **hierarchically hyperbolic** groups [38]. This inspired them to ask asked if the other groups that ought to be considered geometrically finite also have hierarchically hyperbolic extension. I showed that the answer is yes for the next best candidates of geometrically finite subgroups, the stabilizers of multicurves.

Theorem 2 ([Rus21])**.** *Let H be the stabilizer in* $\mathrm{MCG}(S)$ *of a multicurve on a closed surfaces S with genus at least 2. The $\pi_1(S)$-extension of H is hierarchically hyperbolic.*

Hierarchical hyperbolicity is a generalization of Gromov hyperbolic, that features heavily in my research; see §3. The combination of Theorem 2 with the result for lattice Veech groups strongly suggest that the "generalized hyperbolicity" in Question 2 should be hierarchical hyperbolicity.



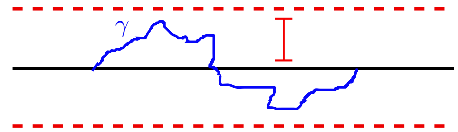
A multicurve on a surface

Future work. The project of understanding the relationship between geometric finiteness, surface groups extensions, and hierarchical hyperbolicity is just beginning, but is ripe for progress. In forthcoming work, I prove that the $\pi_1(S)$-extensions for **twist groups** (abelian subgroups generated by twists around disjoint curves) are also hierarchically hyperbolic. Leininger and I are currently working in the converse direction to Theorem 2, by investigating what sort of geometric consequences having a hierarchically hyperbolic extension has on a subgroup $H < \mathrm{MCG}(S)$. Another angle of attack is examining the geometry of $\pi_1(S)$-extensions for specific subgroups that should *not* be geometrically finite. The aim in these cases is showing that the extensions are not hierarchically hyperbolic. Understanding these "non-examples" will help us zero in on the correct notion of geometrically finite. Good candidates for bad examples are the kernel of the Birman exact sequence and the genus 2 handlebody subgroups.

2. The local-to-global property for Morse quasi-geodesics

The image of a geodesic under a quasi-isometry is a **quasi-geodesic**, a path that is a multiplicative and additive amount away from being distance minimizing. While quasi-geodesics are natural objects in geometric group theory, the quasi-geodesics in a space can be dramatically different from the geodesics in the space. This has attracted special attention to the **Morse** quasi-geodesics: a quasi-geodesic α is Morse if every other quasi-geodesic with endpoints on α is contained in a regular neighborhood of α.

Morse quasi-geodesics have been extensively studied by Charney, Druţu, Sapir, Mozes, Osin, Ol'Shanskii, and many others [3, 28, 36, 72]. Since Gromov's hyperbolic spaces are characterized by the fact that all quasi-geodesics are uniformly Morse, Morse quasi-geodesics capture the hyperbolic (or negatively curved) directions in a non-hyperbolic space. Accordingly, some results from the theory of hyperbolic groups have useful generalizations to Morse geodesics in non-hyperbolic groups; see for example [3, 30, 26].



Schematic of a Morse geodesic. The quasi-geodesic γ must stay uniformly close to the geodesic α.

Despite these results, examples from Fink [46] and Osin–Ol'Shanskii–Sapir [72] demonstrate obstructions to many result from hyperbolic groups having a generalization to Morse geodesics in full generality. Wanting to create a framework that was broad enough to be interesting, but restrictive enough to avoid these problematic examples, Spriano, Tran, and I introduced a local-to-global property for Morse quasi-geodesics that that allows for the detection of the Morse property at a small scale [RST22]. While not every space has this **Morse local-to-global** property, we show that many important groups and space do.

Theorem 3 ([RST22])**.** *The following groups and spaces have the Morse local-to-global property.*

- *All* CAT(0) *spaces and groups*
- *Hierarchically hyperbolic spaces, including the mapping class group and Teichmüller space*
- *All finitely generated virtually solvable groups*
- *Any group hyperbolic relative to subgroups with the Morse local-to-global property*
- *The universal cover and fundamental group of any closed 3-manifold*

Importantly, we also show that the pathological examples of Fink and Osin–Ol'Shanskii–Sapir do not have the Morse local-to-global property. Recent work of Sisto–Zalloum shows that injective metric spaces also have the Morse local-to-global property [76].

An example of a deep result from hyperbolic groups that cannot be generalized to Morse geodesics is Cannon's theorem that the geodesic in a hyperbolic group form a **regular language** [22]. However, Cordes, Spriano, Zalloum, and I showed that this obstruction vanishes when one restricts to the class of Morse local-to-global groups.

Theorem 4 ([CRSZ22])**.** *Morse local-to-global groups have regular languages of Morse geodesics.*

The geodesics of a group form a regular language when there is a **finite state automata** that can be used to determine whether or not a path in the Cayley graph of the group is a geodesic. For hyperbolic groups, this gives an incredibly powerful algorithmic tool to understand the group. The regular languages from Theorem 4 are more limited as the Morse geodesics can only access the negatively curved parts of the group. However, these regular languages give us powerful tools for studying stable subgroups.

A subgroup is **stable** if every pair of elements can be joined by a Morse geodesic in the Cayley graph that stays uniformly close to the subgroup. Stability is a convexity property that was introduced by Durham–Taylor and has been studied in a variety of settings by Hamenstädt, Hensel, Antolín, Mj, Sisto, and others [1, 2, 40, 56]. In several settings, stable subgroups coincide with other important subgroups. For example, the stable subgroups of right-angled Artin groups are precisely the purely loxodromic subgroups [65] while the stable subgroups of hyperbolic groups are the extensively studied **quasiconvex** subgroups. Most important for my work, Durham–Taylor showed that the stable subgroups of the mapping class group are precisely the convex cocompact subgroups from §1 [40]. Morse geodesics and stable subgroups therefore provide an avenue to prove vast generalization of results from hyperbolic groups, while simultaneously producing specific consequences of interest for convex cocompact subgroups of the mapping class group. I will highlight examples of my research in this direction.

2.1. **Growth.** Given a finite generating set A for a group G, the **growth function** of G is the function that counts the number of elements of G that are a product of at most n elements of A. The asymptotics of this growth function is a fundamental quasi-isometry invariant of group. For example, Gromov's polynomial growth theorem says a groups is virtually nilpotent if and only if the growth function is asymptotic to a polynomial [50]. We also have a growth function for each subgroup H, which counts the number of elements of H that are the product of n generators of G.

In the hyperbolic setting, Gersten–Short used regular languages to prove that the stable[2] subgroups have **rational growth** [48]. This means, the power series with coefficients given by the outputs of the growth function is equal to a rational function, which is equivalent to the growth function of the subgroup satisfying a linear recursive pattern. The regular languages of my coauthor's and I from Theorem 4, allow us to generalize the Gersten–Short result to the wide class of Morse local-to-global groups.

Theorem 5 ([CRSZ22]). *Let G be a Morse local-to-global group. If H is a stable subgroup of G, then the subgroup growth of H is rational with respect to any finite generating set.*

Farb asked which subgroups of the mapping class group have rational growth [43]. The equivalence of stability and convex cocompactness in $\mathrm{MCG}(S)$ means Theorem 5 answer's Farb's question in the positive for the convex cocompact subgroups.

Regular languages also give us access to tools from **Perron–Frobenius theory** to study the growth of stable subgroups. Using these methods, we show that stable subgroups of Morse local-to-global groups grow exponentially slower than the entire group in all known examples.

Theorem 6 ([CRSZ22]). *In all known examples[3], if G is a Morse local-to-global group and H is a stable subgroup, then H grows exponentially slower than G for any finite generating set of G. In particular, the convex cocompact subgroups of* $\mathrm{MCG}(S)$ *grow exponentially slower than* $\mathrm{MCG}(S)$.

Theorem 6 is a generalization of a theorem of Dahmani–Futer–Wise in the context of stable subgroups of hyperbolic groups [32], and says that any given stable subgroup takes up a vanishingly small percentage of the group as you travel farther from the identity. As with Theorem 5, this general theorem on stable subgroups, gave a new result for convex cocompact subgroups of $\mathrm{MCG}(S)$.

2.2. **Combination theorem.** The Morse local-to-global property arose from my collaborators and I trying to prove a combination theorem that would allow us to combine two stable subgroup into a new stable subgroup. This desire was largely motivated by the most important open question about convex cocompact subgroups of $\mathrm{MCG}(S)$ asked by Farb–Mosher [45].

Question 3. Is there a convex cocompact subgroup of the mapping class group that does not have a finite index free subgroup? Is there a convex cocompact closed surface group in $\mathrm{MCG}(S)$?

The importance of Question 3 follows directly from the correspondence between convex cocompact subgroups and hyperbolic $\pi_1(S)$-extension discussed in §1. For example, if all convex cocompact subgroups are free, then it would be impossible for a surface bundle over a surface to be a hyperbolic manifold. The existence of such a bundle is a major open question in 4-manifolds.

The hope of a combination theorem for convex cocompact subgroups is that two free examples could be combined to make a non-free example. The pathological example of Osin–Ol'shankii–Sapir demonstrate that a sensible combination theorem for stable subgroups is impossible in full generality, however, Spriano, Tran and I show that the addition of the Morse local-to-global property allows for the formulation of a combination theorem. This result is an archetypal example of how geometric properties can be used to produce algebraic consequences in groups.

[2]aka the quasiconvex subgroups.

[3]Specifically, we require that G is virtually torsion free or H is residually finite.

Theorem 7 ([RST22]). *Let G be a Morse local-to-global group and H, K be stable subgroups. There exists $C > 0$ so that if $H \cap K$ contains all element of $H \cup K$ whose length in G is at most C, then $\langle H, K \rangle$ is a stable subgroup isomorphic to $H *_{H \cap K} K$.*

The combined subgroup in Theorem 7 being an amalgamated product is promising for producing non-free groups from free groups. However, verifying the hypothesis on the intersection is difficult in practice. A successful attempt to use Theorem 7 to make non-free examples in the mapping class group will require an in-depth knowledge of the specific subgroups involved. However, we were able to use Theorem 7 to create non-obvious new examples of non-free stable subgroups of CAT(0) groups, another class of Morse local-to-global groups [RST22, Example 3.5].

2.3. **Morse boundaries.** Beyond stable subgroups, the Morse local-to-global property has applications to the **Morse boundary**, a topological space that collects the asymptotic behavior of Morse geodesic rays into a quasi-isometry invariant of the group. A group acts on its Morse boundary by homeomorphisms, and my collaborators and I used the regular languages from Theorem 4 to illuminate the dynamics of this action. The most interesting of these result is about the limit sets of normal subgroups.

Theorem 8 ([CRSZ22]). *If G is a Morse local-to-global group, then the limit set of any normal subgroup of G is the entire Morse boundary of G.*

A simple corollary of Theorem 8, is that if G is a non-hyperbolic Morse local-to-global group and N is a hyperbolic normal subgroup, then there is no continuous surjection from the Morse boundary of N to the Morse boundary of G (what geometric group theorists call a **Cannon–Thurston map**). This result was surprising because work of Cannon–Thurston [23] and later Mj [70] showed that a continuous surjection *does* exists when both G and N are hyperbolic.

I have also shown that several Morse local-to-global groups satisfy a criteria of Charney–Cordes–Sisto to have totally disconnected Morse boundaries [27]. As the Morse boundary is a quasi-isometry invariant, this helps distinguish theses groups up to quasi-isometry.

Theorem 9 ([Rus21]). *The following groups have totally disconnected Morse boundaries.*

(1) The $\pi_1(S)$-extensions of the stabilizers of multicurves discussed in §1.2.
(2) The genus 2 handlebody group.
(3) Any of the admissible groups defined by Croke–Kleiner in [31].

3. Hierarchically hyperbolic spaces

Hierarchically hyperbolicity is a generalization of Gromov's hyperbolicity introduced by Behrstock–Hagen–Sisto, and inspired by Masur–Minsky's subsurface projection machinery for the mapping class class group [10, 69]. The original examples of hierarchically hyperbolic spaces were Gromov hyperbolic spaces, the mapping class group and Teichmüller space of a surface, and the virtually special groups of Haglund–Wise [10]. The class has been enlarged to include a large class of Artin groups [53], the universal covers and fundamental groups of most 3–manifolds [8][HRSS], the genus 2 handlebody group [29], some $\pi_1(S)$-extensions [38] [Rus21], and several different ways of combining of these examples [8, 75, 13] [BR22]. Hierarchically hyperbolicity provides powerful machinery for understanding the geometry of a wide variety of spaces simultaneously, including significant advances in understanding the asymptotic dimension [9], quasi-flats [12], and metric geometry [8, 51]. At times, this unified framework has allowed for the development of techniques that would have been inaccessible in any of the individual settings [7, 9].

A **hierarchically hyperbolic space** (HHS) **structure** on a metric space is a collection of projections onto hyperbolic spaces along with three combinatorial relations between these projections. The projection maps encode the "hyperbolic parts" of the space while the relations dictate how these hyperbolic pieces fit together to build the geometry of the entire space. A **hierarchically**

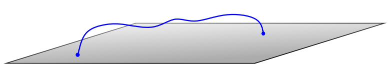
hyperbolic space is a metric space that admits an HHS structure and a **hierarchically hyperbolic group** (HHG) is a finitely generated group whose Cayley graph admits an HHS structure that is equivariant with respect to the multiplication in the group.

My work on hierarchically hyperbolic spaces includes establishing a foundational understanding of convexity properties, illuminating the relationship between relative hyperbolicity and hierarchical hyperbolicity, progress on understanding the HHS boundary, and producing new examples.

3.1. **Convexity properties.** Spriano, Tran, I have provided the bedrock understanding of convex subsets in hierarchically hyperbolic spaces [RST23]. These results have become fundamental tools in the theory, appearing in subsequent work of Hagen, Petyt, Hoda, Haettel, Durham, Zalloum and others [75, 54, 51, 74, 24, 41]. Questions asked in this work have inspired work of Genevois [47], Cashen [25], and Karrer [60].

In hierarchically hyperbolic spaces, one wants a notion of a "convex" subset that in some way agrees with the hierarchically hyperbolic structure. At the onset, there were three different candidates for what this notion of convexity could be. A key result of my work with Spriano and Tran is that these three notions are in fact all equivalent; we call the subsets that satisfy any of these equivalent definitions a **hierarchically quasiconvex** subset.

Theorem 10 ([RST23])**.** *The three possible definitions of hierarchically quasiconvex are equivalent.*



The blue hierarchy path says uniformly close to the subset

The simplest description of hierarchically quasiconvexity is the following: every pair of points in an HHS can be join by a **hierarchy path**, a special quasi-geodesic[4] that is well understood using the hierarchy structure. A subset is then hierarchically quasiconvex if every hierarchy path between points in the subset stays uniformly close to the subset; see the schematic to the left.

The main technical tool we develop to understand hierarchical quasiconvexity is the construction of a **hierarchically quasiconvex hull** for any subset using hierarchy paths. This construction produces a minimal hierarchically quasiconvex subset containing a given subset.

Theorem 11 ([RST23])**.** *The hierarchically quasiconvex hull of a set in an HHS can be constructed in a uniformly finite number of steps by iteratively connecting pairs of points by hierarchy paths*

The definition of hierarchical quasiconvexity is a direct generalization of the influential quasiconvex subsets of hyperbolic spaces, and our construction of hulls using hierarchy paths is a analogous of the construction of quasiconvex hulls in hyperbolic spaces via geodesics.

Spriano, Tran, and I also studied **strongly quasiconvex** subsets of hierarchically hyperbolic spaces. These are subsets where *every* quasi-geodesic between points stays uniform close to the subset (as opposed to just the hierarchy paths). The Morse quasi-geodesics from §2 are examples of strongly quasiconvex subsets. While strongly quasiconvex subsets are more restrictive than hierarchically quasiconvex subsets, they have the benefit of being quasi-isometry invariants. Genevois [47], Behrstock [4], and my collaborators and I [RST23] have used this invariance to distinguish different quasi-isometry classes of groups.

In the case of hyperbolic spaces, strongly quasiconvex subsets are characterized by being the image of a contracting retraction of the entire space. We prove this equivalence extends to all hierarchically hyperbolic spaces. The key to this proof is creating an explicit characterization of strong quasiconvexity in terms of the hierarchy structure.

Theorem 12 ([RST23])**.** *A subset of an HHS space is strongly quasiconvex if and only if it is contracting. Moreover, strong quasiconvexity can be detected using the hierarchy structure.*

[4]Recall, quasi-geodesics are paths that are a multiplicative and additive amount away from being distance minimizing.

Using Theorem 12, my collaborators and I characterized the **hyperbolically embedded subgroups** in hierarchically hyperbolic groups. These subgroups were introduced by Dahmani–Guirardel–Osin [34] and are important in the study of **acylindrically hyperbolic groups** and **group theoretic Dehn filling**. Bowditch showed the hyperbolically embedded subgroups of hyperbolic groups are exactly those that are almost malnormal and strongly quasiconvex [16]. We show Bowditch's result is a special case of all hierarchically hyperbolic groups. Examples of Ol'shanskii–Osin–Sapir demonstrate that this characterization fails to hold amongst all finitely generated groups [72].

Theorem 13 ([RST23])**.** *A subgroup of a hierarchically hyperbolic group is hyperbolically embedded if and only if it is strongly quasiconvex and almost malnormal.*

A particularly interesting case of Theorem 13 is the mapping class group, where an equivalence between strong quasiconvexity and the convex cocompact subgroups of §1 says that Theorem 13 proves the hyperbolically embedded subgroups of $\mathrm{MCG}(S)$ are precisely the almost malnormal and convex cocompact subgroups.

3.2. **Relative hyperbolicity.** Relative hyperbolicity is an older generalization of Gromov hyperbolicity introduced independently by Farb and Bowditch and extensively studied by many authors including Osin, Druţu, Sapir, Groves, and Dahmani [16, 33, 37, 42, 73]. A **relatively hyperbolic space** is a metric space that is hyperbolic outside a collection of isolated, strongly quasiconvex subsets, which we call the **peripheral subsets**. Naturally occurring examples of relatively hyperbolic spaces include the fundamental groups of finite volume, non-compact hyperbolic 3-manifolds, free products of any finitely generated groups, and the "limit groups" of Sela.

Utilizing the understanding of strongly quasiconvex subsets gained from Theorem 12, I characterized when a hierarchically hyperbolic space is also relatively hyperbolic using a combinatorial condition, called **isolated orthogonality**, on HHS structures.

Theorem 14 ([Rus22])**.** *A hierarchically hyperbolic space is relatively hyperbolic if and only if it admits a hierarchically hyperbolic space structure with isolated orthogonality.*

Theorem 14 allows one to "read off" the presence of relative hyperbolicity from the HHS structure. This has been used by Berlyne to prove the relative hyperbolicity of some graph braid groups [14] and by Behrstock–Martin–Hagen–Sisto to show certain quotients of the mapping class group are relatively hyperbolic [7].

Vokes and I applied Theorem 14 to **hierarchical graphs of multicurves**, a large class of graphs associated to a surface S whose vertices are collections of disjoint curves on S. Examples of hierarchical graphs of multicurves in the literature include the curve graph used by Ivanov, Masur–Minsky, and many others to study the mapping class group and 3-manifolds [58, 68]; the cut-system graph introduced by Hatcher–Thurston to prove finite presentability of the mapping class group [57]; the pants graph used by Brock to study the coarse geometry of the Weil–Petersson metric on Teichmüller space [20]; and the Torelli and separating curve graphs use by Farb–Ivanov and Brendle–Margelit to study the algebra of the Torelli group and the Johnson kernel [44, 17].

Vokes proved that every hierarchical graph of multicurves is a hierarchically hyperbolic space [80]. Subsequently, Vokes and I used Theorem 14 along side more topological techniques to classify when exactly a hierarchical graph of multicurves is relatively hyperbolic. This produced a new geometric understanding of the separating curve graph, the Torelli graph, and the cut-system graph.

Theorem 15 ([Rus22, RV22])**.** *If $\mathcal{G}$ is a hierarchical graph of multicurves on S, then $\mathcal{G}$ is either hyperbolic, relatively hyperbolic, or thick. Further, which of these is true is determined by the set of subsurface of S that intersect every curve in the vertex set of $\mathcal{G}$.*

Theorem 15 says that if a hierarchical graph of multicurves is not entirely negatively curved (hyperbolic), then the non-negatively curved parts of the metric space must be organized in one of two incompatible ways: they are either isolated away each other (relative hyperbolicity) or they intersect in an organized network of subspace that covers the entire space (thickness). This sort of trichotomy was purposed by Behrstock–Druţu–Mosher who established it for Artin groups and the universal covers of 3–manifolds [5]. The same trichotomy has been established for Teichmüller space by Brock–Farb, Brock–Masur, and Behrstock–Druţu–Mosher [5, 18, 19]; for Coxeter groups by Behrstock–Hagen–Sisto–Caprace [11]; and for free-by-cyclic groups by Hagen using work of Macura and others [52]. This trichotomy produces strong geometric consequences including the existence of minimal relatively hyperbolic structures and information about a powerful quasi-isometry invariant called divergence [6, 11, 66].

Future work. The geometric insight gained from Theorem 15 opens the door to understanding the **quasi-isometric rigidity** of a hierarchical graphs of multicurves. The main question is a quasi-isometry version of Ivanov's meta-conjecture [59].

Question 4. If $\mathcal{G}$ is a hierarchical graph of multicurves on a surface S, is every quasi-isometry of $\mathcal{G}$ bounded distance from an element of $\mathrm{MCG}(S)$?

Recent work of Goldsborough–Hagen–Petyt–Sisto combines with Theorem 15 to reduce Question 4 to the case $\mathcal{G}$ is Gromov hyperbolic [49]. In this case, there is description of the Gromov boundary $\partial\mathcal{G}$ in terms of the boundaries of the curve graphs of subsurfaces of S that intersect every vertex of $\mathcal{G}$ [39]. In particular, the boundary of the curve graph of the whole surface, $\partial\mathcal{C}(S)$, topologically embeds into $\partial\mathcal{G}$. One avenue to resolve Question 4 would be to show that quasi-isometries of $\mathcal{G}$ preserve this copy of $\partial\mathcal{C}(S)$ inside $\partial\mathcal{G}$. One could then use quasi-möbius techniques as I did in [MR19] or [Rus] to produce a quasi-isometry of $\mathcal{C}(S)$ from this map on the boundary. This would produce an element of $\mathrm{MCG}(S)$ that is close to the original quasi-isometry of $\mathcal{G}$ as Rafi–Schliemer have shown the answer to Question 4 is yes for $\mathcal{C}(S)$.

3.3. **Maximization and the boundary.** Inspired by work in the case of hyperbolic and CAT(0) groups, Durham–Hagen–Sisto defined a **boundary** for hierarchically hyperbolic groups [39]. This attempts to organizes the geometry of the group "at infinity" into to a topological space that illuminates the geometry of the group and the dynamics of its elements.

It is possible that a group G has two different HHG structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$. While the definition of the boundary depends on the specific structure, Durham–Hagen–Sisto asked whether or not the boundary $\partial(G, \mathfrak{S}_1)$ with respect to $\mathfrak{S}_1$ must be to homeomorphic to the boundary $\partial(G, \mathfrak{S}_2)$ with respect to $\mathfrak{S}_2$. This is the most fundamental open question about the boundary.

Question 5. Does the hierarchically hyperbolic boundary depend on the specific HHG structure?

Abbott, Behrstock and I have recently made the first progress on Question 5, by studying how the boundary behaves under a "maximization" procedure. This procedure takes one HHG structure $\mathfrak{S}$ and produces a related structure $\mathfrak{S}'$ with certain universal (or maximal) properties [1]. We show that the maximized structure $\mathfrak{S}'$ produces the same boundary as the original structure $\mathfrak{S}$.

Theorem 16 ([ABR22]). *The HHS boundary is invariant under the maximization process.*

The universal properties of the maximized structure are strong enough that we can use Theorem 16 to prove that some topological and dynamical properties of the boundary are independent of the specific HHG structure. For example, we show that the the elements of the group that act with **north-south dynamics** are independent of the structure [ABR22].

In a second paper, Abbott, Behrstock, and I use Theorem 16 to relate the boundary to my previous work on relative hyperbolicity (Theorem 14). We establish that relative hyperbolicity can be characterized in terms of the boundary and that the **Bowditch boundary** of a relatively hyperbolic HHG is a quotient of the HHG boundary.

Theorem 17 ([ABR]). *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group.*

(1) *Whether or not G is relatively hyperbolic is characterized by the HHS boundary $\partial(G, \mathfrak{S})$.*
(2) *If G is relatively hyperbolic, then the Bowditch boundary of G is a quotient of $\partial(G, \mathfrak{S})$.*

Theorem 17(1) is analogous to a characterization Behrstock–Hagen showed in the case of CAT(0) cube complexes, while Theorem 17(2) generalizes work of Spriano [77], Tran [78], and Manning [67] in the case of hyperbolic and CAT(0) groups.

Future work. The next step in understanding the HHS boundary is trying to resolve Question 5 in some special cases. The best candidate is the relatively hyperbolic case. Here, the Bowditch boundary is independent of the specific HHS structure, so this independence should be able to be pulled back under the quotient map from Theorem 17 as long as the peripheral subgroups are known to satisfy Question 5.

Another test case is **rank 2** HHGs. In this case, work of Behrstock–Hagen–Sisto on maximal quasi-flats in HHGs [12] gives strong control over the parts of the boundary that are not controlled by the maximization process. Abbott, Behrstock, and I plan to combine these techniques with those from Theorem 16 to show that the boundary does not depend on the structure in this case. This would demonstrate a stark difference between the HHG boundary and the CAT(0) boundary, as the latter is not well behaved even in the rank 2 case.

Beyond Question 5, another promising direction arises from one of the corollaries to Theorem 16. We show that the limit set of any normal subgroup is the entire boundary.

Corollary 18 ([ABR22])**.** *Let $(G, \mathfrak{S})$ an HHG and $N < G$ a normal subgroup. The limit set of N in $\partial(G, \mathfrak{S})$ is all of $\partial(G, \mathfrak{S})$.*

This is a direct analogue of what happens in the case of normal subgroups of hyperbolic groups, and begs the question of when the Cannon–Thurston maps that exist in the case of hyperbolic groups also exist in the case of hierarchically hyperbolic groups. Natural test cases for this question are the kernel of the Birman exact sequence or special cases of the Bestvina–Brady subgroups of right angled Artin groups.

3.4. **New examples.** Finding new examples of hierarchically hyperbolic groups is a central aspect of my research program. One direction of this research are the $\pi_1(S)$-extensions of curve stabilizers and other "geometrically finite" subgroups of $\mathrm{MCG}(S)$ discussed in §1.2. In different directions, I have expanded the world of hierarchically hyperbolic groups and spaces to include graph products, 3-manifold groups, and the admissible curve graph associated to a vector field on a surface.

3.4.1. *Graph products.* A **graph product** of a finite collection of groups $G_1, \dots, G_n$ is a group combination technique that interpolates between the free product and direct product of the G_i. Right-angled Artin groups and Coxeter groups are examples of graph products where the vertex groups are all $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ respectively.

Berlyne and I answered two questions of Behrstock–Hagen–Sisto about graph products [8]. First we affirmed that if each G_i is an HHG, then the graph product will be an HHG.

Theorem 19 ([BR22])**.** *If each G_i is an HHG, then the graph product is an HHG.*

Berlai–Rubbio had previous given an affirmative answer when the G_i satisfied some additional hypotheses [13]. Berlyne and I used different techniques that avoid these extra conditions and produce a more explicit HHS structure. This explicit structure allowed Berlyne and I to also answer two questions of Genevois about the geometry of the **electrification** of a right-angled Coxeter group [47]. This electrification is a metric space associated to every right angled Coxeter group that is a quasi-isometry invariant. Berlyne and I confirmed Genevois' conjectures about when the electrification was a point or a line [BR22].

The techniques employed in Theorem 19 allowed Berlyne and I to show that the **syllable metric** on any graph product is an HHS regardless of what the groups G_i are. This gave an affirmative answer to the second question of Behrstock–Hagen–Sisto [8], which was motivated by the fact that the syllable metric on a graph product is *not* quasi-isometric to standard metric on the group, but is instead a graph product analogue of the Weil–Petersson metric on Teichmüller space.

Theorem 20 ([BR22])**.** *The syllable metric on a graph product of any collection of groups is a hieraricharically hyperbolic space.*

3.4.2. 3-*manifolds.* Behrstock–Hagen–Sisto showed that the fundamental groups of most 3-manifolds (those without Nil or Sol geometry) are hierarchically hyperbolic *spaces* [8]. However, the hierarchically hyperbolic structures they created were not always equivariant with respect to the group action. That is, these 3-manifold groups where HHSs but not necessarily *HHGs*. Behrstock–Hagen–Sisto conjectured that in many cases—specifically most 3-dimensional graph manifolds—that there would never exist equivariant HHS structures. However, Hagen, Sisto, Spriano, and I have shown this conjecture is completely false.

Theorem 21 ([HRSS])**.** *If M is a 3-dimensional graph manifold, then $\pi_1(M)$ is a hierarchically hyperbolic group. That is, there exists a $\pi_1(M)$-equivariant HHS structure for $\pi_1(M)$.*

By combining Theorem 21 with the Perelman–Thurston geometric decomposition of 3-manifolds, we characterize precisely when a 3-manifold groups is an HHG and not just an HHS.

Theorem 22 ([HRSS])**.** *If M is a compact 3-manifold, then $\pi_1(M)$ is an HHG if and only if there are no Nil, Sol, or non-octahedral flat manifolds in the geometric decomposition of M.*

Future work. The techniques used to prove Theorem 21 are more general than the setting of graph manifolds. For example, we use them to prove the hierarchical hyperbolicity of any central $\mathbb{Z}$-extension of a hyperbolic group as well as any group with a certain "admissible" combinatorial decomposition introduced by Croke–Kleiner [31]. The ideas have also been employed by Hagen–Martin–Sisto and Dowdall–Durham–Leininger–Sisto in their respective proofs of the hierarchical hyperbolicity of extra large type Artin groups [53] and extensions of lattice Veech groups [38].

A variant of these techniques should prove useful in establishing the hierarchical hyperbolicity of other interesting groups. **Free-by-cyclic groups** are promising candidates as they have a long established analogy with 3-manifold groups and admit a rich combinatorial decomposition that may be amenable to the techniques from Theorem 21 in certain cases. The first examples to consider are the **linearly growing** free-by-cyclic groups, which are the closest to the case of graph manifold groups considered in Theorem 21.

3.4.3. *The admissible curve graph.* A **framing** ϕ of a surface S with punctures or boundary is a non-vanishing vector field on the surface, and the **framed mapping class group**, $\mathrm{MCG}(S;\phi)$, is the infinite index subgroup that stabilizes ϕ up to isotopy. For surfaces with enough genus, Calderon–Salter have identified $\mathrm{MCG}(S;\phi)$ as the image of the monodromy of the stratum in the space of abelian differential associated to the framing ϕ [21]. Central to Calderon–Salter's techniques to understand $\mathrm{MCG}(S;\phi)$ is the action on the **admissible curve graph**, $\mathcal{C}_\phi(S)$. This is the graph of curves whose winding number with respect to the vector field ϕ is 0, with edges corresponding to disjointness. The admissible curve graph appears to be the framed analogue of the curve graph, $\mathcal{C}(S)$, that is extremely influential in the study of the full mapping class group. One of the most important results for the curve graph is Masur–Minsky's proof that $\mathcal{C}(S)$ is Gromov hyperbolic [68]. In forth coming work, Calderon and I show that the admissible curve graph $\mathcal{C}_\phi(S)$ is not Gromov hyperbolic, but is hierarchically hyperbolic.

Theorem 23. [CR] *For surfaces with genus at least 3 and framings of holomorphic type, the admissible curve graph is hierarchically hyperbolic, but not Gromov hyperbolic.*

The hyperbolicity of the curve graph is an essential component of understanding the geometric group theory of the mapping class group, and Calderon and I hope that the hierarchically hyperbolicity of $\mathcal{C}_\phi(S)$ will similarly be a gateway to the geometric group theory of $\mathrm{MCG}(S;\phi)$.

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