This is a nice theorem to state:

“A variety is completely recovered by its category of coherent sheaves.”

Let’s be more precise:

**Theorem** (Gabriel) Let $X, Y$ be two varieties, then $X$ is isomorphic to $Y$ if and only if the category $\text{Coh}(X)$ is equivalent to $\text{Coh}(Y)$.

This theorem has seen many extensions. The most general* form works for any quasi-separated scheme [Gabriel, Rosenberg, Gabber/Brandenburg].

*If one is willing to think of $\text{Coh}(X)$ together with its tensor product, i.e. if one considers the *monoidal* category $(\text{Coh}(X), \otimes)$, then one has a corresponding “Tannakian-flavored” variant of the reconstruction theorem [Ballman, Lurie, Brandenburg]. This version needs much more information from $\text{Coh}(X)$ but has the big advantage of working even for stacks (unlike the ordinary one).
The idea is to extract a scheme $Z(\mathcal{A})$ out of an abelian category $\mathcal{A}$, such that $Z(\text{Coh}(X)) = X$ (by default, if $\mathcal{A} = \mathcal{B}$ then $Z(\mathcal{A}) = Z(\mathcal{B})$).

The original approach was to start by first producing a topological space $|\mathcal{A}|$, so that $|\text{Coh}(X)| = |X|$ - the topological space underlying $X$.

The key fact is the bijection:

$$
\{\text{closed irreducible subsets of } |X| \} \leftrightarrow \{\text{Serre subcategories of } \text{Coh}(X)\}
$$

$$
Y \subset |X| \leftrightarrow \text{Coh}_Y(X)
$$

where *Serre* means “closed under subobjects, quotients and extensions” and $\text{Coh}_Y(X)$ is the category of sheaves supported on $Y$.

The point is that Serre subcategories make sense in *any* abelian category.

With extra work, one produces the structure sheaf $\mathcal{O}_X$, thus recovering the whole scheme $X$. 
But there should be another way to prove this theorem:

\[ X = \text{Hilb}^1(X) = \text{moduli space of point-looking sheaves}. \]

Nowadays we are all into moduli, so it would be nice if we could write:

“\( X \) is the moduli space of points of \( \text{Coh}(X) \).”

Indeed, this is possible. Here are some of the advantages:

- a new paper on the arXiv,
- works for algebraic spaces and not just schemes,
- works for twisted varieties - where the twist comes from any class in \( H^2(X, \mathcal{O}_X) \).

Main disadvantage: works at most for quasi-compact and separated spaces (it is very likely that this approach simply does not apply to the non-separated world).
So, what is a pointlike object of an abelian category? How do we build a moduli space of them?

Let’s take a step back and review something we know well. There are at least two ways to present a scheme:

- $X = \text{ringed space} = \text{topological space} + \text{sheaf of rings} = (|X|, \mathcal{O}_X)$
- $X = \text{moduli space} = \text{collection of maps } S \to X, \text{ with } S \text{ affine scheme} = \text{the functor } \text{Hom}(-,X)$.

As we like moduli spaces we prefer the second approach, by default.*

*Hilbert schemes illustrate this principle quite well. Hilb(X) is naturally a functor and for good reason. Hironaka gave an example of a non-projective threefold whose Hilbert scheme of points is not a scheme, but rather an algebraic space. Algebraic spaces sit in between schemes and stacks and they do not admit a description in terms of ringed spaces, so we are stuck with functors (or maybe topoi, but no one wants to work with those).
The idea is to extract a moduli space $\text{Pt}_\mathcal{A}$ from an abelian category $\mathcal{A}$, such that $\text{Pt}_{\text{Coh}(X)}=X$.

How? One needs to define what a family of pointlike objects is.

Let’s start with $\mathcal{A}=\text{Coh}(X)$ and do some reverse engineering.

As $\text{Pt}_{\text{Coh}(X)}=X$, an $S$-family of pointlike objects in $\text{Coh}(X)$ is nothing but a morphism $S\to X$.

A morphism is equivalent to its graph $\Gamma\subset S\times X$, which is a closed* subscheme of $S\times X$.

In turn, this is equivalent to the structure sheaf $\mathcal{O}_\Gamma$.

A “family of points” is the structure sheaf a graph.

*Here is where separatedness of $X$ separated creeps in.
So, \( \text{Pt}_{\text{Coh}(X)}(S) = \{ \mathcal{O}_\Gamma, \text{for } \Gamma \text{ the graph of a morphism } S \to X \} \).

Can “being a graph” be phrased categorically? Yes - diagram please.

\[
\begin{array}{ccc}
S & \leftarrow & S \times X \\
& & \to X \\
\end{array}
\]

Given a morphism \( S \to X \), the key property is the bijection

\[
\{ \text{subschemes of } S \} \leftrightarrow \{ \text{quotients of } \mathcal{O}_\Gamma \}
\]

given by \( \text{pr}_S^*(-) \otimes \mathcal{O}_\Gamma \). Why?

Let’s denote by \( \text{gr}: S \to S \times X \) the graph morphism corresponding to \( S \to X \). The bijection follows once we notice that

\[
\text{pr}_S \circ \text{gr} = \text{id}_S, \quad \mathcal{O}_\Gamma = \text{gr}^* \mathcal{O}_S, \quad \text{gr}^*(-) = \text{pr}_S^*(-) \otimes \mathcal{O}_\Gamma.
\]
Now is time for the general definition. A quasi-coherent sheaf \( F \) on \( S \times X \) is a graph\(^*\) if and only if:

1. \( \text{pr}_S^*(-) \otimes F \) induces a bijection between subschemes of \( S \) and quotients of \( F \);
2. \( F \) is flat over \( S \) and of finite type;
3. for all \( G \in \text{Coh}(X) \) we have \( \text{Hom}(\text{pr}_X^*(G),F) \in \text{Coh}(S) \);
4. \( M \rightarrow \text{Hom}(F, F \otimes \text{pr}_S^*M) \) is an isomorphism, for all \( M \in \text{Coh}(S) \).

All these properties make sense in any abelian category. So, \( \text{Pt}_{\mathcal{A}} \) is well defined and \( \text{Pt}_{\text{Coh}(X)} = X \). Hence, we have reproved Gabriel’s theorem.

\(^*\)This definition is technical, but the main bit is property 1. To deal with non-noetherian spaces one needs to slightly modify 3.
Actually, I lied: an $F$ satisfying 1-4 is only a graph up to a twist of a line bundle $L \in \text{Pic}(S)$.

$\text{Pt}_{\text{Coh}(X)}$ is not $X$, but rather the trivial $\mathbb{G}_m$-gerbe on $X$.

Any $\alpha \in H^2(X, \mathbb{G}_m)$, defines a category $\text{Coh}(X, \alpha)$ of $\alpha$-twisted sheaves.

It turns out that $\text{Pt}_{\text{Coh}(X, \alpha)} = \alpha$, i.e. the moduli of points of $\alpha$-twisted sheaves is the gerbe corresponding to the twist $\alpha$.

**Theorem** Let $X, Y$ be two varieties and let $\alpha, \beta$ be two classes in $H^2$. Then $\text{Coh}(X, \alpha)$ is equivalent to $\text{Coh}(Y, \beta)$ if and only if there exists an isomorphism $g: X \to Y$, such that $g^* \beta = \alpha$.

*This twisted reconstruction theorem was already studied by Perego, Canonaco-Stellari and Antieau.*
Of course a question remains: if $\mathcal{A}$ is some other abelian category, what is $\text{Pt}_{\mathcal{A}}$?

Anyway, that’s all. This wasn’t really a poster, but rather a bunch of slides next to each other (at least it had lots of colors).

If you are interested, the relevant paper is:
*Moduli Problems in Abelian Categories and the Reconstruction Theorem*

John Calabrese (Rice)
Michael Groechenig (Imperial College).