1

UNIFORM DISTRIBUTION MOD 1

In this chapter, we develop the classical part of the theory of uniform distribution. We disregard quantitative aspects, which will be considered separately in Chapter 2.

1. DEFINITION

Uniform Distribution Modulo 1

For a real number x, let $[x]$ denote the *integral part* of x, that is, the greatest integer $\leq x$; let $\{x\} = x - [x]$ be the *fractional part* of x, or the residue of x modulo 1. We note that the fractional part of any real number is contained in the *unit interval* $I = [0, 1)$.

Let $\omega = (x_n)$, $n = 1, 2, \ldots$, be a given sequence of real numbers. For a positive integer N and a subset E of I, let the counting function $A(E; N; \omega)$ be defined as the number of terms x_n, $1 \leq n \leq N$, for which $\{x_n\} \in E$. If there is no risk of confusion, we shall often write $A(E; N)$ instead of $A(E; N; \omega)$. Here is our basic definition.

Definition 1.1. The sequence $\omega = (x_n)$, $n = 1, 2, \ldots$, of real numbers is said to be *uniformly distributed modulo 1* (abbreviated u.d. mod 1) if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N \to \infty} \frac{A([a, b); N; \omega)}{N} = b - a. \tag{1.1}$$

Thus, in simple terms, the sequence (x_n) is u.d. mod 1 if every half-open subinterval of I eventually gets its "proper share" of fractional parts. In the course of developing the theory of uniform distribution modulo 1 (abbreviated u.d. mod 1), we shall encounter many examples of sequences that enjoy this property (for an easy example, see Exercise 1.13).

Let now $c_{[a,b)}$ be the characteristic function of the interval $[a, b) \subseteq I$. Then (1.1) can be written in the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[a,b)}(\{x_n\}) = \int_0^1 c_{[a,b)}(x)\, dx. \tag{1.2}$$

This observation, together with an important approximation technique, leads to the following criterion.

THEOREM 1.1. The sequence (x_n), $n = 1, 2, \ldots,$ of real numbers is u.d. mod 1 if and only if for every real-valued continuous function f defined on the closed unit interval $\bar{I} = [0, 1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\, dx. \tag{1.3}$$

PROOF. Let (x_n) be u.d. mod 1, and let $f(x) = \sum_{i=0}^{k-1} d_i c_{[a_i,a_{i+1})}(x)$ be a step function on $\bar{I}$, where $0 = a_0 < a_1 < \cdots < a_k = 1$. Then it follows from (1.2) that for every such f equation (1.3) holds. We assume now that f is a real-valued continuous function defined on $\bar{I}$. Given any $\varepsilon > 0$, there exist, by the definition of the Riemann integral, two step functions, f_1 and f_2 say, such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in \bar{I}$ and $\int_0^1 (f_2(x) - f_1(x))\, dx \leq \varepsilon$. Then we have the following chain of inequalities:

$$\begin{aligned}\int_0^1 f(x)\, dx - \varepsilon &\leq \int_0^1 f_1(x)\, dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(\{x_n\}) \\ &\leq \varliminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) \leq \varlimsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) \\ &\leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_2(\{x_n\}) = \int_0^1 f_2(x)\, dx \leq \int_0^1 f(x)\, dx + \varepsilon,\end{aligned}$$

so that in the case of a continuous function f the relation (1.3) holds.

Conversely, let a sequence (x_n) be given, and suppose that (1.3) holds for every real-valued continuous function f on $\bar{I}$. Let $[a, b)$ be an arbitrary subinterval of I. Given any $\varepsilon > 0$, there exist two continuous functions, g_1

and g_2 say, such that $g_1(x) \le c_{[a,b)}(x) \le g_2(x)$ for $x \in \bar{I}$ and at the same time $\int_0^1 (g_2(x) - g_1(x))\,dx \le \varepsilon$. Then we have

$$b - a - \varepsilon \le \int_0^1 g_2(x)\,dx - \varepsilon \le \int_0^1 g_1(x)\,dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_1(\{x_n\})$$
$$\le \varliminf_{N\to\infty} \frac{A([a,b);N)}{N} \le \varlimsup_{N\to\infty} \frac{A([a,b);N)}{N} \le \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_2(\{x_n\})$$
$$= \int_0^1 g_2(x)\,dx \le \int_0^1 g_1(x)\,dx + \varepsilon \le b - a + \varepsilon.$$

Since ε is arbitrarily small, we have (1.1). ∎

COROLLARY 1.1. The sequence (x_n) is u.d. mod 1 if and only if for every Riemann-integrable function f on $\bar{I}$ equation (1.3) holds.

PROOF. The sufficiency is obvious, and the necessity follows as in the first part of the proof of Theorem 1.1. ∎

COROLLARY 1.2. The sequence (x_n) is u.d. mod 1 if and only if for every complex-valued continuous function f on $\mathbb{R}$ with period 1 we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx. \tag{1.4}$$

PROOF. By applying Theorem 1.1 to the real and imaginary part of f, one shows first that (1.3) also holds for complex-valued f. But the periodicity condition implies $f(\{x_n\}) = f(x_n)$, and so we arrive at (1.4). As to the sufficiency of (1.4), we need only note that in the second part of the proof of Theorem 1.1 the functions g_1 and g_2 can be chosen in such a way that they satisfy the additional requirements $g_1(0) = g_1(1)$ and $g_2(0) = g_2(1)$, so that (1.4) can be applied to the periodic extensions of g_1 and g_2 to $\mathbb{R}$. ∎

Some simple but useful properties may be deduced easily from Definition 1.1. We mention the following results.

LEMMA 1.1. If the sequence (x_n), $n = 1, 2, \ldots$, is u.d. mod 1, then the sequence $(x_n + \alpha)$, $n = 1, 2, \ldots$, where α is a real constant, is u.d. mod 1.

PROOF. This follows immediately from Definition 1.1. ∎

THEOREM 1.2. If the sequence (x_n), $n = 1, 2, \ldots$, is u.d. mod 1, and if (y_n) is a sequence with the property $\lim_{n\to\infty}(x_n - y_n) = \alpha$, a real constant, then (y_n) is u.d. mod 1.

PROOF. Because of Lemma 1.1 it suffices to consider the case $\alpha = 0$. Set $\varepsilon_n = x_n - y_n$ for $n \ge 1$. Let $0 < a < b < 1$, and choose ε such that

$$0 < \varepsilon < \min\left(a, 1 - b, \frac{b-a}{2}\right).$$

There exists an $N_0 = N_0(\varepsilon)$ such that $-\varepsilon \leq \varepsilon_n \leq \varepsilon$ for $n \geq N_0$. Let $n \geq N_0$, then $a + \varepsilon \leq \{x_n\} < b - \varepsilon$ implies $a \leq \{y_n\} < b$, and on the other hand $a \leq \{y_n\} < b$ implies $a - \varepsilon \leq \{x_n\} < b + \varepsilon$. Hence, if $\sigma = (x_n)$ and $\omega = (y_n)$, then

$$\begin{aligned} b - a - 2\varepsilon &= \lim_{N\to\infty} \frac{A([a+\varepsilon, b-\varepsilon); N; \sigma)}{N} \leq \varliminf_{N\to\infty} \frac{A([a, b); N; \omega)}{N} \\ &\leq \varlimsup_{N\to\infty} \frac{A([a, b); N; \omega)}{N} \leq \lim_{N\to\infty} \frac{A([a-\varepsilon, b+\varepsilon); N; \sigma)}{N} \\ &= b - a + 2\varepsilon. \end{aligned}$$

Since ε can be taken arbitrarily small, the sequence ω satisfies (1.1) for all a and b with $0 < a < b < 1$. To complete the proof, one uses the result in Exercise 1.2. ∎

Uniform Distribution Modulo a Subdivision

We mention briefly one of the many variants of the definition of u.d. mod 1. Let Δ: $0 = z_0 < z_1 < z_2 < \cdots$ be a subdivision of the interval $[0, \infty)$ with $\lim_{k\to\infty} z_k = \infty$. For $z_{k-1} \leq x < z_k$ put

$$[x]_\Delta = z_{k-1} \qquad \text{and} \qquad \{x\}_\Delta = \frac{x - z_{k-1}}{z_k - z_{k-1}},$$

so that $0 \leq \{x\}_\Delta < 1$.

Definition 1.2. The sequence (x_n), $n = 1, 2, \ldots$, of nonnegative real numbers is said to be *uniformly distributed modulo* Δ (abbreviated u.d. mod Δ) if the sequence $(\{x_n\}_\Delta)$, $n = 1, 2, \ldots$, is u.d. mod 1.

If Δ is the subdivision Δ_0 for which $z_k = k$, this concept reduces to that of u.d. mod 1. An interesting case occurs if (x_n) is an increasing sequence of nonnegative numbers with $\lim_{n\to\infty} x_n = \infty$. Then we let $A(x, \alpha)$ be the number of $x_n < x$ with $\{x_n\}_\Delta < \alpha$, and we set $A(x) = A(x, 1)$. Clearly, the sequence (x_n) is u.d. mod Δ if and only if for each $\alpha \in (0, 1)$,

$$\lim_{x\to\infty} \frac{A(x, \alpha)}{A(x)} = \alpha. \tag{1.5}$$

The following remarkable result can be shown.

THEOREM 1.3. Let (x_n) be an increasing sequence of nonnegative numbers with $\lim_{n\to\infty} x_n = \infty$. A necessary condition for (x_n) to be u.d. mod Δ is that

$$\lim_{k\to\infty} \frac{A(z_{k+1})}{A(z_k)} = 1. \tag{1.6}$$

PROOF. Suppose (x_n) is u.d. mod Δ. Since

$$A(\tfrac{1}{2}(z_k + z_{k+1}), \tfrac{1}{2}) - A(z_k, \tfrac{1}{2}) = A(\tfrac{1}{2}(z_k + z_{k+1})) - A(z_k),$$

we have

$$\frac{A(\frac{1}{2}(z_k + z_{k+1}), \frac{1}{2})}{A(\frac{1}{2}(z_k + z_{k+1}))} = \frac{A(z_k, \frac{1}{2})}{A(\frac{1}{2}(z_k + z_{k+1}))} + \frac{A(\frac{1}{2}(z_k + z_{k+1})) - A(z_k)}{A(\frac{1}{2}(z_k + z_{k+1}))}$$

$$= \frac{A(z_k, \frac{1}{2})}{A(z_k)} \cdot \frac{A(z_k)}{A(\frac{1}{2}(z_k + z_{k+1}))} + 1 - \frac{A(z_k)}{A(\frac{1}{2}(z_k + z_{k+1}))}$$

$$= 1 + \frac{A(z_k)}{A(\frac{1}{2}(z_k + z_{k+1}))}\left(\frac{A(z_k, \frac{1}{2})}{A(z_k)} - 1\right). \tag{1.7}$$

Now the extreme left member of (1.7) goes to $\frac{1}{2}$ as $k \to \infty$, according to (1.5) and the assumption about the sequence (x_n). Similarly, the second factor of the second term of the extreme right member of (1.7) goes to $-\frac{1}{2}$ as $k \to \infty$. Hence, (1.7) implies

$$\lim_{k\to\infty} \frac{A(z_k)}{A(\frac{1}{2}(z_k + z_{k+1}))} = 1. \tag{1.8}$$

In a similar way, it can be shown that

$$\lim_{k\to\infty} \frac{A(z_{k+1})}{A(\frac{1}{2}(z_k + z_{k+1}))} = 1. \tag{1.9}$$

From (1.8) and (1.9) we obtain (1.6). ∎

Notes

The formal definition of u.d. mod 1 was given by Weyl [2, 4]. The distribution mod 1 of special sequences was already investigated earlier (see the notes in Section 2). Theorem 1.1 and its corollaries also come from Weyl [2, 4]. The notion of u.d. mod Δ was introduced by LeVeque [4] and was studied further by Cigler [1], Davenport and LeVeque [1], Erdös and Davenport [1], W. M. Schmidt [10], and Burkard [1, 2].

A detailed survey of the results on u.d. mod 1 prior to 1936 can be found in Koksma [4, Kap. 8, 9]. The period from 1936 to 1961 is covered in the survey article of Cigler and Helmberg [1]. An expository treatment of some of the classical results is given in Cassels [9, Chapter 4]. The survey article of Koksma [16] touches upon some of the interesting aspects of the theory.

Let λ be the Lebesgue measure in I. If (x_n) is u.d. mod 1, the limit relation

$$\lim_{N\to\infty} \frac{1}{N} A(E; N) = \lambda(E)$$

will still hold for all Jordan-measurable (or λ-continuity) sets E in I (see Chapter 3, Theorem 1.2) but not for all Lebesgue-measurable sets E in I (see Exercise 1.9). See also Section 1 of Chapter 3 and Rimkevičiūtė [1]. Similarly, the limit relation (1.3) cannot hold for all

Lebesgue-integrable functions f on $\bar{I}$. See Koksma and Salem [1] for strong negative results. The following converse of Theorem 1.1 was shown by de Bruijn and Post [1]: if f is defined on $\bar{I}$ and if $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N} f(\{x_n\})$ exists for every (x_n) u.d. mod 1, then f is Riemann-integrable on $\bar{I}$. Binder [1] gives an alternative proof and a generalization. See also Bass and Couot [1]. Rudin [2] discusses a related question.

Elementary criteria for u.d. mod 1 have been given by O'Neil [1] (see also the notes in Section 3 of Chapter 2) and Niederreiter [15]. Sequences of rationals of the type considered in Exercise 1.13 were studied by Knapowski [1] using elementary methods.

In the sequel, we shall discuss many variants of the definition of u.d. mod 1. One rather special variant was introduced by Erdös and Lorentz [1] in the context of a problem from probabilistic number theory. A sequence (x_n) is called *homogeneously equidistributed mod 1* if $((1/d)x_{nd})$, $n = 1, 2, \ldots$, is u.d. mod 1 for every positive integer d. This notion was also studied by Schnabl [1].

For the result in Exercise 1.14, see Pólya and Szegö [1, II. Abschn., Aufg. 163].

Exercises

1.1. A definition equivalent to Definition 1.1 is the following: A sequence (x_n) of real numbers is u.d. mod 1 if $\lim_{N\to\infty} A([0, c); N)/N = c$ for each real number c with $0 \leq c \leq 1$.

1.2. If (1.1) holds for all a, b with $0 < a < b < 1$, then it holds for all a, b with $0 \leq a < b \leq 1$.

1.3. A sequence (x_n) is u.d. mod 1 if and only if (1.1) is satisfied for every interval $[a, b) \subseteq I$ with rational end points.

1.4. A sequence (x_n) is u.d. mod 1 if and only if $\lim_{N\to\infty} A([a, b]; N)/N = b - a$ for all a, b with $0 \leq a \leq b \leq 1$.

1.5. A sequence (x_n) is u.d. mod 1 if and only if $\lim_{N\to\infty} A((a, b); N)/N = b - a$ for all a, b with $0 \leq a < b \leq 1$.

1.6. If (x_n) is u.d. mod 1, then the sequence $(\{x_n\})$ of fractional parts is everywhere dense in $\bar{I}$.

1.7. If we leave out finitely many terms from a sequence that is u.d. mod 1, the resulting sequence is still u.d. mod 1. What additional condition is needed if "finitely" is replaced by "infinitely"?

1.8. If finitely many terms of a sequence that is u.d. mod 1 are changed in an arbitrary manner, the resulting sequence is still u.d. mod 1. Generalize as in Exercise 1.7.

1.9. Let (x_n) be an arbitrary sequence of real numbers. Construct a Lebesgue-measurable subset E of I with $\lambda(E) = 1$ and $\lim_{N\to\infty} A(E; N)/N = 0$. *Hint:* Consider the complement in I of the set determined by the range of the sequence $(\{x_n\})$.

1.10. Let (x_n) be u.d. mod 1. Then the relation (1.3) is not valid for every Lebesgue-integrable function f on $\bar{I}$.

1.11. Let (x_n) and (y_n) be u.d. mod 1. Then the sequence $x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots$ is u.d. mod 1.

1.12. If r is a rational number, then the sequence (nr), $n = 1, 2, \ldots$, is not u.d. mod 1. Is there a nonempty proper subinterval $[a, b)$ of I for which (1.1) holds?

1.13. Prove that the sequence $0/1, 0/2, 1/2, 0/3, 1/3, 2/3, \ldots, 0/k, 1/k, \ldots, (k-1)/k, \ldots$ is u.d. mod 1.

1.14. Let (x_n) be a sequence in I. For a subinterval $[a, b)$ of I and $N \geq 1$, let $S([a, b); N)$ be the sum of the elements from $x_1, x_2, \ldots, x_N$ that are in $[a, b)$. Then (x_n) is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \frac{S([a, b); N)}{N} = \tfrac{1}{2}(b^2 - a^2)$$

for all subintervals $[a, b)$ of I.

2. THE WEYL CRITERION

The Criterion

The functions f of the form $f(x) = e^{2\pi i h x}$, where h is a nonzero integer, satisfy the conditions of Corollary 1.2. Thus, if (x_n) is u.d. mod 1, the relation (1.4) will be satisfied for those f. It is one of the most important facts of the theory of u.d. mod 1 that these functions already suffice to determine the u.d. mod 1 of a sequence.

THEOREM 2.1: Weyl Criterion. The sequence (x_n), $n = 1, 2, \ldots$, is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \qquad \text{for all integers } h \neq 0. \tag{2.1}$$

PROOF. The necessity follows from Corollary 1.2. Now suppose that (x_n) possesses property (2.1). Then we shall show that (1.4) is valid for every complex-valued continuous function f on $\mathbb{R}$ with period 1. Let ε be an arbitrary positive number. By the Weierstrass approximation theorem, there exists a trigonometric polynomial $\Psi(x)$, that is, a finite linear combination of functions of the type $e^{2\pi i h x}$, $h \in \mathbb{Z}$, with complex coefficients, such that

$$\sup_{0\leq x\leq 1} |f(x) - \Psi(x)| \leq \varepsilon. \tag{2.2}$$

Now we have

$$\begin{aligned}\left| \int_0^1 f(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \leq {} & \left| \int_0^1 (f(x) - \Psi(x))\,dx \right| \\ & + \left| \int_0^1 \Psi(x)\,dx - \frac{1}{N}\sum_{n=1}^{N} \Psi(x_n) \right| \\ & + \left| \frac{1}{N}\sum_{n=1}^{N} (f(x_n) - \Psi(x_n)) \right|.\end{aligned}$$

The first and the third terms on the right are both $\leq\varepsilon$ whatever the value of N, because of (2.2). By taking N sufficiently large, the second term on the right is $\leq\varepsilon$ because of (2.1). ■

Applications to Special Sequences

EXAMPLE 2.1. Let θ be an irrational number. Then the sequence $(n\theta)$, $n = 1, 2, \ldots$, is u.d. mod 1. This follows from Theorem 2.1 and the inequality

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h n\theta}\right| = \frac{|e^{2\pi i h N\theta} - 1|}{N\,|e^{2\pi i h\theta} - 1|} \leq \frac{1}{N\,|\sin \pi h\theta|}$$

for integers $h \neq 0$. ■

EXAMPLE 2.2. Let $\theta = 0.1234567891011121314\cdots$ in decimal notation. Then θ is irrational. Therefore, the sequence $(n\theta)$ is u.d. mod 1 by Example 2.1. It follows that the sequence $(\{n\theta\})$ is dense in $\bar{I}$ (see Exercise 1.6). One can even show that the subsequence $(\{10^n\theta\})$ is dense in $\bar{I}$. For let $\alpha = 0\,.\,a_1a_2\cdots a_k$ be a decimal fraction in I. One chooses n such that $\{10^n\theta\}$ begins with the digits $a_1, a_2, \ldots, a_k$ followed by r zeros. Then we have $0 < \{10^n\theta\} - \alpha < 10^{-k-r}$. ■

EXAMPLE 2.3. The sequence $(\{ne\})$, $n = 1, 2, \ldots$, is u.d. mod 1 according to Example 2.1. However, the subsequence $(\{n!e\})$ has 0 as the only limit point. We have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^{\alpha}}{(n+1)!}, \qquad 0 < \alpha < 1,$$

so that $n!e = k + e^{\alpha}/(n+1)$, where k is a positive integer. Hence, for $n \geq 2$ we get $\{n!e\} = e^{\alpha}/(n+1) < e/(n+1)$. ■

EXAMPLE 2.4. The sequence $(\log n)$, $n = 1, 2, \ldots$, is not u.d. mod 1. In order to show this we use the Euler summation formula. If $F(t)$ is a complex-valued function with a continuous derivative on $1 \leq t \leq N$, where $N \geq 1$ is an integer, then

$$\sum_{n=1}^{N} F(n) = \int_1^N F(t)\,dt + \tfrac{1}{2}(F(1) + F(N)) + \int_1^N (\{t\} - \tfrac{1}{2})F'(t)\,dt. \qquad (2.3)$$

Let $F(t) = e^{2\pi i \log t}$, and divide both sides of (2.3) by N. Then the first term on the right of (2.3) is equal to

$$\frac{Ne^{2\pi i \log N} - 1}{N(2\pi i + 1)}$$

and this expression does not converge as $N \to \infty$. The second term on the right of (2.3), divided by N, tends to zero as $N \to \infty$, as does the third term on the right of (2.3) divided by N, as follows from

$$\left| \int_1^N (\{t\} - \tfrac{1}{2}) F'(t)\, dt \right| \leq \pi \int_1^N \frac{dt}{t}.$$

Hence, (2.1) with $x_n = \log n$ and $h = 1$ is not satisfied. ■

EXAMPLE 2.5. In the previous example it was shown that $(\log n)$ is not u.d. mod 1. This general statement, however, does not describe the behavior of $(\log n)$ mod 1 with regard to a particular interval $[a, b) \subseteq I$. In the following we show in an elementary way that for every proper nonempty subinterval $[a, b)$ of I the sequence $(\log n)$ fails to satisfy (1.1). Consider first an interval $[a, b)$ with $0 \leq a < b < 1$. Choose a sequence of integers (N_m), $m \geq m_0$, such that $e^{m+b} < N_m < e^{m+1}$ for $m \geq m_0$. Now the number of indices $n = 1, 2, \ldots, N_m$ for which $a \leq \{\log n\} < b$, or $k + a \leq \log n < k + b$, or $e^{k+a} \leq n < e^{k+b}$, $k = 0, 1, \ldots, m$, is given by the expression

$$\sum_{k=0}^{m} (e^{k+b} - e^{k+a} + \theta(k)) = (e^b - e^a) \frac{e^{m+1} - 1}{e - 1} + \sum_{k=0}^{m} \theta(k),$$

$$\text{with } 0 \leq \theta(k) \leq 1.$$

Now it is clear that the fraction obtained by dividing this expression by N_m, which was chosen between e^{m+b} and e^{m+1}, is not convergent as $m \to \infty$ for every choice of the sequence (N_m). If $0 < a < b = 1$, one works in a similar way with a sequence (N_m) satisfying $e^m < N_m < e^{m+a}$ for $m \geq m_0$. With a slight modification of the calculation, one may show the same with respect to the sequence $(c \log n)$, $c \in \mathbb{R}$, $n = 1, 2, \ldots$. ■

EXAMPLE 2.6. Suppose we are given an infinitely large table of the Brigg logarithms $({}^{10}\log n)$, $n = 1, 2, \ldots$, in decimal representation, and consider the sequence of the consecutive digits in the kth column after the decimal point for some fixed $k \geq 1$. Let $c = 10^{k-1}(\log 10)^{-1}$; then

$$c \log n = 10^{k-1}\, {}^{10}\log n.$$

If for some n we have $\{{}^{10}\log n\} = 0.b_1 b_2 \cdots b_k \cdots$, then $\{c \log n\} = 0.b_k b_{k+1} \cdots$. Thus, we observe that the digit b_k at the kth decimal place of ${}^{10}\log n$ is equal to g, $g = 0, 1, \ldots, 9$, if and only if $g/10 \leq \{c \log n\} < (g + 1)/10$. Now the sequence $(c \log n)$ has the property described in Example 2.5. This implies that the relative frequency with which the digit g appears in the kth column of the table is not equal to $1/10$. ■