Continued Fractions and Square Packings

Dr. Morgan Weiler

Rice University

Lanier Middle School Math Club
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Who am I?

I'm a mathematician and postdoctoral fellow at Rice University.

My field is *contact geometry*. I investigate mathematical abstractions of topics such as:

- planetary motion
- electromagnetic flow
- circle and sphere packings (how many circles of radius 1 fit inside a circle of radius 9?)

In the picture on the right, you can see me with the clock tower at UC Berkeley, where I studied.

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During a Ph.D., you learn the state of the art in a small (small!) part of the research community. Then you write an original dissertation, i.e., a very long paper – 50+ pages.

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My job has three main components:

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Community: I work to better the mathematical community and the community mathematically! For example, I have organized mentorship programs, jobs panels, and conferences.

What's to like about being a mathematician?

Here are some of the things I really like about my job:

- I think about math every day. It can be frustrating when I am stuck, but I find learning and investigating mathematics very satisfying. Over time, you learn how to deal with being stuck.

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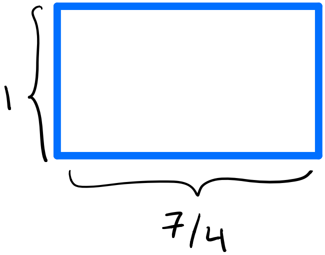
- I get to discuss mathematical topics I find really interesting with other mathematicians and with my students.
- There are mathematicians all over the world. I have traveled to places as far and wide as Montreal, Paris, Tokyo, and Rio de Janiero to visit other mathematicians and attend conferences and workshops. You get to connect with people from all over, who may be much younger or older, with very different experiences.

What else?

My life outside of my job, such as my hobbies and relationships, is important to me too. I love rock climbing, and my husband and I have a very cute dog named Willow (who also loves rock climbing).

Onto the math!

Packing squares into a rectangle



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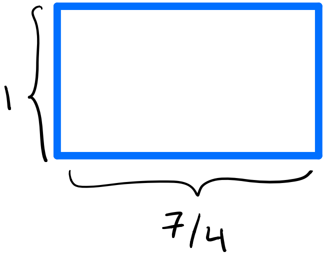

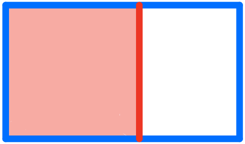

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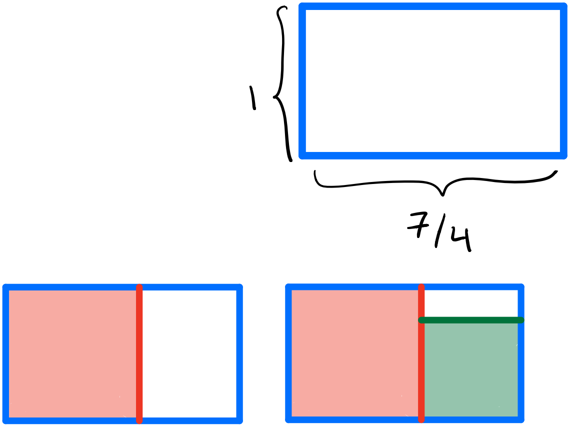

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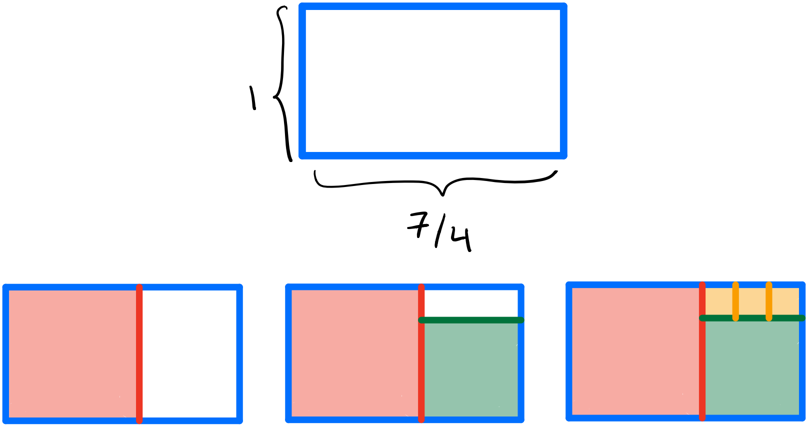
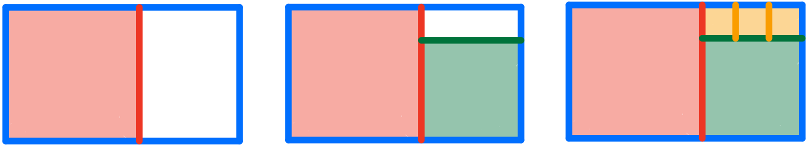


Questions

1. Find a square packing for the $1 \times \frac{4}{7}$ rectangle. What do you notice?
2. Do you think this procedure will always end? Why or why not?

Continued Fractions

Each time you draw a set of squares of the same size, write down the *number* of squares you have drawn.



When $x = \frac{7}{4}$, we write down the list in the format

$$[1; 1, 3]$$

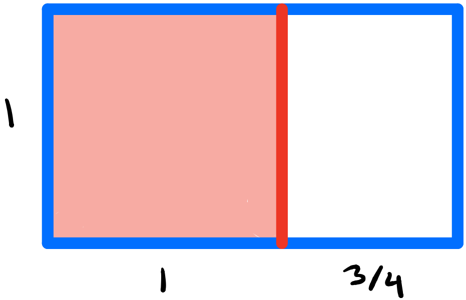
This is the **continued fraction** of the number $\frac{7}{4}$.

What does the continued fraction mean?

We know

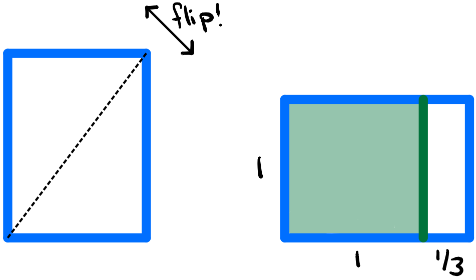
$$\frac{7}{4} = 1 + \frac{3}{4}$$

which we can see in the rectangle:



The meaning of continued fractions, continued

We also know that the shape of the square packing for $\frac{3}{4}$ should be the same as the shape of the square packing for $\frac{4}{3}$, just flipped.



We get $\frac{4}{3} = {\color{blue}1} + \frac{1}{3}$.

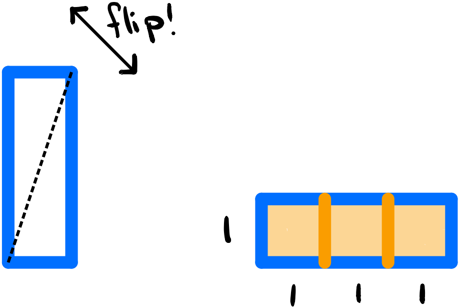
Plugging back in to our original calculation, we see

$$\frac{7}{4} = {\color{blue}1} + \frac{1}{\frac{4}{3}} = {\color{blue}1} + \cfrac{1}{{\color{blue}1} + \frac{1}{3}}$$

The meaning of continued fractions, Part 3

$$\frac{7}{4} = \color{blue}{1} + \frac{1}{\color{blue}{1} + \frac{1}{\color{blue}{3}}}$$

Once we see a fraction like $\frac{1}{n}$ where n is an integer, we know we can fill the rest of the rectangle with squares:



Working with continued fractions

You can compute a continued fraction without the pictures:

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You can also work backwards to get the number from the continued fraction.

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$$2 + \frac{1}{1 + \frac{1}{7}} = 2 + \frac{1}{\frac{7}{7} + \frac{1}{7}} = 2 + \frac{7}{8} = \frac{16}{8} + \frac{7}{8}$$

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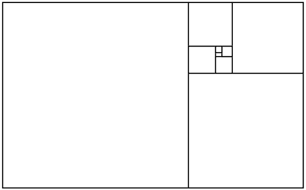
For negative numbers, the *first* number in the continued fraction is negative, and *all the rest* are positive:

$$-\frac{23}{8}=-3+\frac{1}{8}$$

so the continued fraction of $-\frac{23}{8}$ is $[-3;8]$.

Challenge 1: Find a formula for the continued fraction of $-x$ given the continued fraction of x. What change do you have to make when you apply your formula twice?

Infinite continued fractions



Some square packings don't end!

Question: What number has continued fraction $[1; 1, 1, 1, \dots]$?

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$$x = 1 + \frac{1}{x} \Leftrightarrow x^2 = x + 1 \Leftrightarrow x^2 - x - 1 = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{1^2 + 4}}{2}$$

Is $x = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$?

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Challenge 2: What is the number with continued fraction equal to n ones (i.e. $\underbrace{[1; 1, \ldots, 1]}_{n}$)?

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Challenge 3: Find the continued fraction of $\sqrt{3}$.

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Irrational numbers which aren't square roots! These are numbers which *can't* be written as $\frac{a}{b}$ where a and b are integers, or as $\frac{a}{b} + \sqrt{\frac{c}{d}}$.

For example:

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1...]$$

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But don't we already have decimals for that?

Best approximations

A rational number $\frac{a}{b}$ is a **best approximation** to x (rational or irrational!) if there's no other rational number $\frac{c}{d}$ where:

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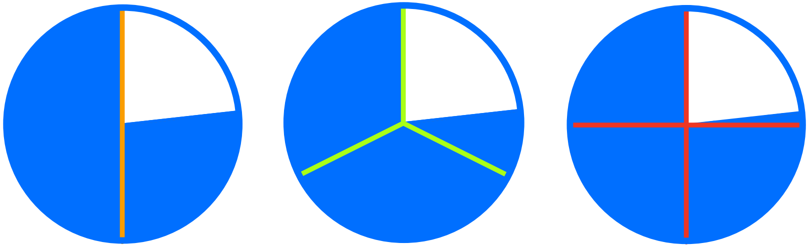

Figure: $\frac{77}{100}$ of the circle is blue. But $\frac{3}{4}$ is simpler than $\frac{77}{100}$.

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$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113} \approx 3.1415929$$

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Using decimals with denominator 1000 (so to 0.001 accuracy), we can only approximate $\pi \approx 3.142$, which is way less accurate!

Help with the Challenges

You can find help with Challenge 1 at http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html#section14.1

For Challenge 2, watch https://www.youtube.com/watch?v=RXPnMECdh6k, or read http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html#section9.1

For Challenge 3, there are similar examples at http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html#section6.1