An embedded genus-one helicoid

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There exists a properly embedded minimal surface of genus one with a single end asymptotic to the end of the helicoid. This genus-one helicoid is constructed as the limit of a continuous one-parameter family of screw-motion invariant minimal surfaces, also asymptotic to the helicoid, that have genus equal to one in the quotient.

cone metrics | global theory of minimal surfaces | flat structures

Complete Embedded Minimal Surfaces of Finite Topology and Infinite Total Curvature

We prove the existence of a properly embedded minimal surface in $\mathbf{R}^3$ with finite topology and infinite total curvature.[¶] In refs. 1 and 2, an immersed surface with these properties was proved to exist, and strong mathematical and graphical evidence was given to show that the constructed surface was embedded (i.e., free of self intersections). However, a proof of embeddedness was not found, even though this earlier existence result has a considerable influence of the subsequent development of the global theory of minimal surfaces. We give a full theoretical proof of the existence of an embedded example with finite topology and infinite total curvature. Such a surface has not been reported since 1776, when Meusnier (3) showed that the helicoid was a minimal surface. Our surface has genus one and is asymptotic to the helicoid.

We exhibit this minimal surface as a geometric limit of periodic embedded minimal surfaces. The periodic surfaces, $\mathcal{H}_k$, indexed by a real number $k \geq 1$, are invariant under a cyclic group of screw motions generated by σ_k: rotation by $2\pi k$ about the vertical axis, followed by a vertical translation by $2\pi k$. Thus, for fixed k, the quotient surface has two topological ends and genus one. The limit is taken as $k \to \infty$; a compact set in the limit surface $\mathcal{H}e_1$ is arbitrarily well-approximated (for k sufficiently large) by corresponding pieces of fundamental domains of $\mathcal{H}_k/\sigma_k$ (see Fig. 1).

The requirement to prove embeddedness was the main motivation of our work. We prove that embeddedness is inherited from the embeddedness of the approximating simpler (periodic) surfaces, using that in this particular minimal surface setting, the condition of being embedded is both open and closed on families. This method of proving embeddedness for surfaces defined using the Weierstrass representation contrasts with previous methods: here, the characteristic of being embedded follows naturally from the property holding for simpler surfaces, whereas previously one proved embeddedness by ad hoc methods, for instance by cutting the surface into graphs.

A second important feature is that we approximate a surface of finite topology (and finite symmetry group) by surfaces of infinite topology (and infinite symmetry group). We are unaware of this method being previously applied to produce a nonclassical example.

The third feature is that we construct the Weierstrass data of these approximating minimal surfaces in terms of flat singular structures on the tori corresponding to the quotients. The salient feature to note is that the defining flat structures have singularities corresponding to the two ends[‖] with cone angles of $\pm 2\pi k$. Thus, as the size of the twist tends to infinity, the cone angles also tend to infinity, with the limit surface, our genus-one helicoid, represented in terms of flat cone metrics with an infinite cone angle. The flat geometry of this end corresponds to the Weierstrass data for a helicoid, whose Gauss map has an essential singularity at the end. Establishing this correspondence required the development of a theory of singular flat structures that admits infinite cone angles.

Recently, Meeks and Rosenberg (4) showed that the helicoid is the unique simply connected, properly embedded (nonplanar) minimal surface $\mathbf{R}^3$ with one end. The method of proof uses in an essential manner the work of Colding and Minicozzi (5–8) concerning curvature estimates for embedded minimal disks and geometric limits of those disks. [Colding and Minicozzi recently showed that a complete and embedded minimal surface with finite topology in $\mathbf{R}^3$ must be proper (9).] These results together with the work we describe here (M. Weber, D.H., and M. Wolf, unpublished results) and numerical work of Traizet (M. Traizet, unpublished data) and Bobenko (10) suggest that there may be a substantial theory of complete embedded minimal surfaces with one end, infinite total curvature and finite topology (see also ref. 11). For complete embedded surfaces of finite total curvature, the theory is surveyed in ref. 1.

$\mathcal{H}e_1$, an Embedded Genus-One Helicoid

In 1993, Hoffman, Karcher, and Wei (12) constructed a surface, $\mathcal{H}e_1 \subset R^3$, which they called the genus-one helicoid.

It has the following properties:

(*i*) $\mathcal{H}e_1$ is a properly immersed minimal surface;
(*ii*) $\mathcal{H}e_1$ has genus one and one end asymptotic to the helicoid;
(*iii*) $\mathcal{H}e_1$ contains a single vertical line (the axis) and a single horizontal line. **Condition 1**

Note that *Condition 1.ii* implies that any $\mathcal{H}e_1$ has finite topology and infinite total curvature.[**]

We will refer to any surface with the properties in *Condition 1* as a genus-one helicoid and denote such a surface by $\mathcal{H}e_1$. It is known that any $\mathcal{H}e_1$ must be embedded outside of a compact set (12, 13). However, as mentioned above, even though computer simulations and computational estimates strongly suggested that this $\mathcal{H}e_1$ was embedded, an independent proof has been elusive.

We prove the following.

Theorem 1. *There exists an embedded* $\mathcal{H}e_1$.

We believe that the surface we have found is the same one constructed by Hoffman, Karcher, and Wei (12). In fact, we believe

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[¶]A surface is said to have "finite topology" if it is homeomorphic to a compact surface with a finite number of points removed.

[‖]There is an additional cone point (with cone angle 6π) in these structures at a vertical point.

[**]The helicoid, a surface swept out by a horizontal line rotating at a constant rate as it moves up a vertical axis at a constant rate, is clearly properly embedded and has finite topology (in fact it is simply connected). Because it is singly periodic and evidently not flat, it has infinite total curvature. Any periodic surface asymptotic to the helicoid must also have infinite total curvature.

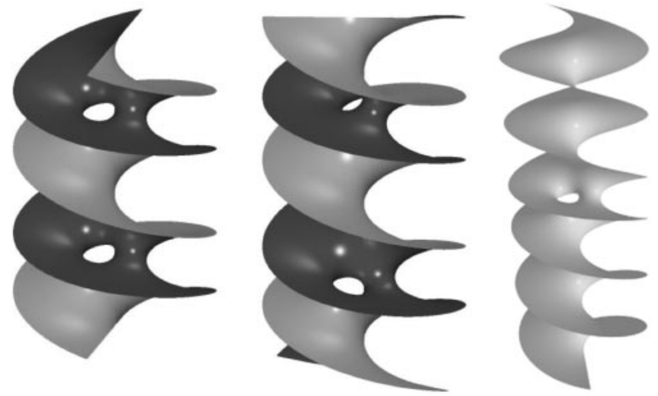

Fig. 1. Approximate surfaces $\mathcal{H}_k$ and the genus-one helicoid $\mathcal{H}e_1$. The two images at *Left* and *Center* are (a pair of) fundamental domains of elements in the sequence $\mathcal{H}_k$ of σ_k-invariant periodic surfaces whose geometric limit is the image (*Right*) of $\mathcal{H}e_1$, the genus-one helicoid. This minimal surface is embedded and has genus one with a single end asymptotic to the standard helicoid.

Conjecture 1. *There is a unique embedded* $\mathcal{H}e_1$.

Note that *Conjecture 1* does not assert that there is a unique $\mathcal{H}e_1$ and that it is an embedded surface.

$\mathcal{H}e_1$ As the Limit of a Family of Screw-Motion-Invariant, Embedded Minimal Surfaces

The starting point of our investigation is the embedded singly periodic genus-one helicoid.

Theorem 2 (2, 14). *There exists a unique*[††] *properly embedded, singly periodic minimal surface* $\mathcal{H}_1$, *whose quotient by vertical translations has the following conditions*:

(*i*) *has genus one and two ends;*
(*ii*) *has ends asymptotic to a full* 2π*-turn of a helicoid;*
(*iii*) *contains a vertical axis and two horizontal parallel lines.*

Condition 2

Hoffman and Wei (16) conducted a numerical investigation indicating that $\mathcal{H}_1$ could be also be deformed in a manner suggested by the symmetries of the helicoid.[‡‡]

The helicoid is not only invariant under a vertical translation by 2π; it is also invariant under vertical screw motions σ_k. For $k \geq 1$, imagine a periodic minimal surface, $\mathcal{H}_k$, invariant under a vertical screw motion σ_k and satisfying the following conditions.

The quotient of $\mathcal{H}_k$ by σ_k

(*i*) has genus one and two ends;
(*ii*) is asymptotic to a portion of the helicoid that has twisted through an angle of $2\pi k$;
(*iii*) contains a vertical axis and two parallel horizontal lines.

Condition 3

The geometric limit, as $k \to \infty$, of the singly periodic covers of these surfaces (once a vertical point is required to stay on the x-axis) should be an $\mathcal{H}e_1$. Moreover, if the family also depended continuously on k, then the embeddedness of $\mathcal{H}_1$ would be inherited by the $\mathcal{H}_k$. With some additional control on the limiting behavior, embeddedness could be shown to pass to the limit $\mathcal{H}e_1$.

We carry out this program completely.

[††]Uniqueness follows from an observation of Karcher, using the proof of ref. 14 and a uniqueness result in ref. 15.

[‡‡]An animation of the genus-one helicoid is available from the authors upon request.

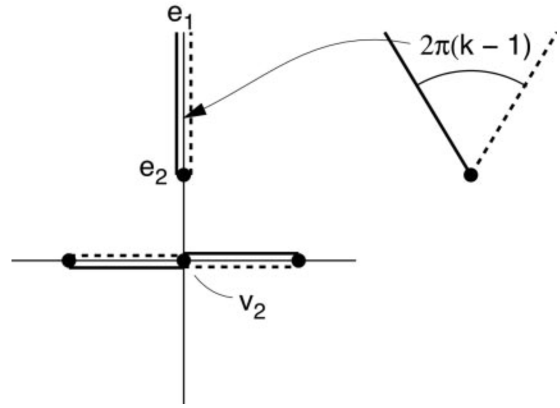

Fig. 2. The *gdh* structures for $\mathcal{H}_k$. The points e_1 and e_2 correspond to the ends of the surface and are cone points of cone angles $-2\pi k$ and $2\pi k$, respectively; the point V_2 is one of the two points where the tangent plane is horizontal and here has a cone angle of 6π. This development of the one-form *gdh* (and an analogous development of the form $\frac{1}{g}dh$) permits an easy verification of the horizontal period condition. [This last comment uses the elementary fact that if we develop a form α by the rule $w = \int^z \alpha$, then in the developed image, the form takes the form $\alpha = dw$ with periods $\int_\gamma \alpha = \int_{w(\gamma)} dw$. These periods are usually easy to compute because they are the difference of the endpoints of $w(\gamma)$.] The parameters of the development are k and the Euclidean distance d between v_2 and e_2.

Theorem 3. *For every* $k \geq 1$, *there exists a complete,* σ_k*-invariant, properly embedded minimal surface,* $\mathcal{H}_k$, *whose quotient by* σ_k *satisfies Condition 3. As* $k \to \infty$, *a limit surface exists and is an embedded* $\mathcal{H}e_1$, *i.e., a properly embedded minimal surface satisfying Condition 1.*

The Ideas Behind the Proof of *Theorem 3*

The procedure of constructing a minimal surface by prescribing Weierstrass data, a Riemann surface, $\mathcal{S}$, with a meromorphic function, g, and one form, dh, requires the demonstration that the two period conditions

$$\int_\alpha g dh = \overline{\int_\alpha \frac{1}{g} dh}$$

(the horizontal period condition) and $\mathrm{Re} \int_\alpha dh = 0$ (the vertical period condition) be satisfied for all cycles $\alpha \subset \mathcal{S}$. Weber (17) realized that the Weierstrass data for $\mathcal{H}_1$ of *Theorem 2* defined one-forms gdh and $(1/g)dh$ that differed by a scale factor and a translation, and this condition specified the underlying torus $\mathcal{S}$ and (up to one free variable) the one forms gdh and $(1/g)dh$. Moreover, the flat geometric structure defined by gdh can be realized by a simple geometric model. See Fig. 2 and note that for $\mathcal{H}_1$, i.e., $k = 1$, there is no inserted cone angle. This definition automatically solves the first period condition. The free parameter is used to satisfy the second period condition as follows. First, we realize that the flat structure of dh can be understood qualitatively (see Fig. 3 *Left*) as a planar domain. Then, for the parameter values at their limits (see Fig. 3 *Center* and *Right*), we are able to determine it explicitly, which allows us to apply the intermediate value theorem.

By sewing in a cone of angle $2\pi(k - 1)$, we are able to modify the flat geometric structure of gdh for $\mathcal{H}_1$ to produce candidate flat structures for $\mathcal{H}_k$ (see Fig. 2). The position of the vertex of the cone gives a real parameter $d > 0$. (For $k = 1$ we do not sew in a cone, but the choice of d is our free parameter, corresponding to the placement of a point corresponding to an end.)

For each $(k, d) \in [1, \infty) \times [0, \infty)$, the construction gives a candidate structure; on each of these sturctures, the one-form dh is determined up to a scale factor, and the horizontal period problem is solved. Thus, we are left to find, for each $k > 1$, a

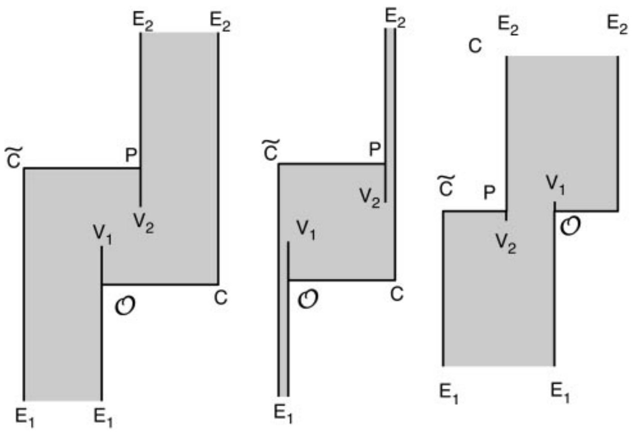

Fig. 3. The *dh* structure. The vertical period condition is satisfied when V_1 and V_2 lie in a single vertical line. The images in *Right* and *Center* depict the *dh* structure for *k* fixed, but *d* near zero (*Center*) and *d* very large (*Right*). Notice that in *Center*, V_1 lies to left of V_2, while in *Right*, V_1 lies to the right of V_2. The argument that for each fixed *k* there exists a $d = d(k)$ satisfying the vertical period condition is by the intermediate-value theorem, because as *d* increases, the point V_1 must pass from being on the left of V_2 to being on the right of V_2.

parameter $d \in (0, \infty)$ for which the vertical period problem is also solved; here this solution value d should depend continuously on the screw-motion parameter k.

To find such a family of $d = d(k)$, we show that the period of the height function over a given cycle naturally defines a real analytic function, $h = h(k, d)$, and the desired $d(k)$ would then satisfy $h(k, d(k)) = 0$. If $h = 0$ for some (k, d), then the corresponding (k, d)-flat geometric structure defines a properly immersed minimal surface $\mathcal{H}_k$ satisfying *Condition 3*. We show that the zero set of h contains a piecewise-smooth curve $\mathcal{C}$ that begins at the point $(1, d_1)$ corresponding to $\mathcal{H}_1$, and crosses each vertical line $\{k = const.\}$ in the (k, d) rectangle. Each point, (k, d), on this curve defines a properly immersed minimal surface satisfying the *Condition 3*.

To find this curve $\mathcal{C}$, we first compactify the "moduli space" of (k, d)-structures, adding degenerate structures for the loci $k = \infty$, $d = 0$, and $d = \infty$ in a manner compatible with the topology of $[1, \infty] \times [0, \infty]$. Then, we show that the height function h is continuous on the full compact rectangle $[1, \infty] \times [0, \infty]$ (i.e., h extends continuously to $d = 0, \infty$ and $k = \infty$). This method of compactifying has the crucial advantage that the signs of h on the degenerate surfaces $(k, 0)$ and (k, ∞) are evident and opposite, and so the intermediate-value theorem provides for a solution $(k_0, d) \in \mathcal{C}$ for each choice of $k_0 \in (1, \infty)$. More precisely, there must be a curve $\mathcal{C}$ on which $h = 0$ because this curve separates the neighborhoods of the boundary components $\{d = 0\}$ and $\{d = \infty\}$ on which h has opposite signs. Finally, a maximum-principle argument then shows that the embeddedness of $\mathcal{H}_1$ implies that all of the $\mathcal{H}_k$, $k > 1$, on this curve $\mathcal{C}$ are embedded.

For $k = \infty$, the (k, d)-structures are defined by sewing in a cone with an infinite cone angle (see below.) Thus, any structure (∞, d) corresponds to a potential $\mathcal{H}e_1$, which will exist provided $h(\infty, d) = 0$; the endpoint of $\mathcal{C}$ on the locus $\{k = \infty\}$ is then such a point. Thus, we obtain a limit flat structure that defines an $\mathcal{H}e_1$, and we argue that, as a limit of a family $\{\mathcal{H}_k\}$ of embedded surfaces, the surface $\mathcal{H}e_1$ is also embedded.

Existence and Uniqueness of Cone Metrics with Infinite Cone Angles

The approximating minimal surfaces $\mathcal{H}_k/\sigma_k$ each have meromorphic Weierstrass data, whereas the limit surface $\mathcal{H}e_1$ has Weierstrass data with an essential singularity at its end. This phenomenon of meromorphic data limiting on transcendental data corresponds on the level of the developed flat structures to taking limits of cone metrics whose cone angles grow without bound: the limit cone metric has a cone point with infinite cone angle. To accommodate the taking of limits of metrics with cone points of arbitrarily large cone angles, we introduce the definition of a cone point of simple exponential type and the idea of asymptotic isometry. In particular, an exponential cone of simple type is a cone isometric to the disk D with the metric

$$\left| e^{1/w} \frac{dw}{w^2} \right|$$

(the origin is the cone point), and two such cones with representations

$$\left| e^{1/w} \frac{dw}{w^2} \right| \quad \text{and} \quad \left| e^{1/z} \frac{dz}{z^2} \right|$$

are asymptotically isometric provided $\frac{dw}{dz}(0) = 1$.

We show that cone metrics are essentially determined by their cone points, with a natural Gauss–Bonnet-type restriction being the only obstruction to existence. These results extend the work of Troyanov (18) on cone metrics with positive and finite cone angles to the cases where the cone angles may be negative, zero, positive, or of simple exponential type.

We thank Hermann Karcher, Harold Rosenberg, Bill Meeks, and Fusheng Wei for useful conversations over the past years. Wei's ingenious ideas about how to represent the family $\mathcal{H}_k$ in a computationally tractable manner was key to establishing the belief that that the family not only existed but also converged to a genus-one helicoid. The idea that embeddedness was inherited from $\mathcal{H}_1$ by the family $\mathcal{H}_k$ came from discussions that one of us had with Meeks about the same phenomena for finite-total-curvature surfaces. Rosenberg's idea to sew a helicoid into a genus-one surface to create a genus-one helicoid, conveyed to Karcher in a conversation reported in ref. 12, was pivotal to the discovery of $\mathcal{H}e_1$. Karcher's deep understanding of the Weierstrass data of the surfaces $\mathcal{H}_1$ and $\mathcal{H}e_1$ was of crucial help at numerous instances, and his intelligence and spirit continue to be fundamental to our work. This work was partly supported by National Science Foundation Grants DMS-0139476 (to M. Weber), DMS-9971563 (to M. Wolf), and DMS-0139887 (to M. Wolf). D.H. was partly supported by U.S. Department of Energy Office of Energy Research Applied Mathematical Science Subprogram Research Grant DE-FG03-95ER25250/A007 and National Science Foundation, Division of Mathematical Sciences Research Grant DMS-0139410.

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