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POLYNOMIAL CUBIC DIFFERENTIALS AND CONVEX POLYGONS IN THE PROJECTIVE PLANE

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Abstract. We construct and study a natural homeomorphism between the moduli space of polynomial cubic differentials of degree d on the complex plane and the space of projective equivalence classes of oriented convex polygons with $d+3$ vertices. This map arises from the construction of a complete hyperbolic affine sphere with prescribed Pick differential, and can be seen as an analogue of the Labourie–Loftin parameterization of convex $\mathbb{RP}^2$ structures on a compact surface by the bundle of holomorphic cubic differentials over Teichmüller space.

1 Introduction

1.1 Motivating problem. Labourie [Lab07] and Loftin [Lof01] have independently shown that the moduli space of convex $\mathbb{RP}^2$ structures on a compact surface S of genus $g \geqslant 2$ can be identified with the vector bundle $\mathcal{C}(S)$ of holomorphic cubic differentials over the Teichmüller space $\mathcal{T}(S)$.

By definition, a convex $\mathbb{RP}^2$ structure on S is the quotient of a properly convex open set in $\mathbb{RP}^2$ by a free and cocompact action of a group of projective transformations. Identifying this convex domain with the projectivization of a convex cone in $\mathbb{R}^3$, one can consider smooth convex surfaces in $\mathbb{R}^3$ that are asymptotic to the boundary of the cone. A fundamental theorem of Cheng and Yau [CY75] (proving a conjecture of Calabi [Cal72]) shows that there is a unique such surface which is a *complete hyperbolic affine sphere*. Classical affine-differential constructions equip this affine sphere with a $\pi_1 S$-invariant Riemann surface structure and holomorphic cubic differential (the *Pick differential*), and the respective $\pi_1 S$-quotients of these give a point in $\mathcal{C}(S)$. The surjectivity of this map from $\mathbb{RP}^2$ structures to $\mathcal{C}(S)$ is established using a method of C. P. Wang, wherein the reconstruction of an affine sphere from the cubic differential data is reduced to the solution of a quasilinear PDE.

Since Wang's technique and the Cheng–Yau theorem apply to any properly convex domain, it would broadly generalize the Labourie–Loftin parameterization if one could characterize those pairs of simply-connected Riemann surfaces and holomorphic cubic differentials that arise from properly convex open sets in $\mathbb{RP}^2$. That is the basic motivating question we consider in this paper.

As stated, this question is probably too broad to have a satisfactorily complete answer. However, the question can be specialized in many ways by asking to characterize the cubic differentials corresponding to a special class of convex sets, or the convex sets arising from a special class of cubic differentials. Of course the Labourie–Loftin parameterization can be described this way, where one requires the convex domain to carry a cocompact group action of a given topological type. Benoist and Hulin have also considered aspects of this question, first for convex domains covering a noncompact surface of finite area [BH13], and later showing that domains with Gromov-hyperbolic Hilbert metrics correspond to the Banach space of L^∞ cubic differentials on the hyperbolic plane [BH14].

1.2 Main theorem. We consider another specialization of the motivating question that is essentially orthogonal to those of Labourie–Loftin and Benoist–Hulin, namely, identifying Riemann surfaces and holomorphic cubic differentials that correspond to *convex polygons* in $\mathbb{RP}^2$. Our main result is that the associated affine spheres give parabolic Riemann surfaces (biholomorphic to $\mathbb{C}$) and cubic differentials that are complex polynomials, with the degree of the polynomial determined by the number of vertices of the polygon.

For example, the affine sphere over the regular pentagon corresponds to the complex plane with the cubic differential $z^2 dz^3$. The fivefold rotational symmetry of the pentagon corresponds to the invariance of $z^2 dz^3$ under the automorphism $z \mapsto e^{2\pi i/5} z$.

In fact, as in the compact surface case, the affine sphere construction gives a homeomorphic identification between two moduli spaces. Consider the space of cubic differentials $p(z)dz^3$ on the complex plane where $p(z)$ is a polynomial of degree d, and let $\mathcal{MC}_d$ denote the quotient of this set by the action of the holomorphic automorphism group $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az+b\}$. Let $\mathcal{MP}_n$ denote the space of projective equivalence classes of convex polygons in $\mathbb{RP}^2$ with n vertices. We show:

Theorem A. *The affine sphere construction determines a homeomorphism*

$$\boldsymbol{\alpha} : \mathcal{MC}_d \to \mathcal{MP}_{d+3}.$$

That is, each polynomial cubic differential C is the Pick differential of a complete hyperbolic affine sphere $S \subset \mathbb{R}^3$, uniquely determined up to the action of $\mathrm{SL}_3\,\mathbb{R}$ and asymptotic to the cone over a convex polygon P. The map $\boldsymbol{\alpha}$ is defined by

$$\boldsymbol{\alpha}([C]) = [P],$$

where $[C]$ denotes the $\mathrm{Aut}(\mathbb{C})$-equivalence class of C and $[P]$ the $\mathrm{SL}_3\,\mathbb{R}$-equivalence class of P.

The spaces $\mathcal{MP}_n$ and $\mathcal{MC}_d$ and their properties are discussed in more detail in Sections 2 and 3, respectively. Since both of these are smooth orbifolds, it is natural to ask about the smoothness of the map $\boldsymbol{\alpha}$ itself. This issue and related conjectures are discussed in Section 9.

Our proof of the main theorem is direct: we construct mutually inverse maps $\mathcal{MC}_d \to \mathcal{MP}_{d+3}$ and $\mathcal{MP}_{d+3} \to \mathcal{MC}_d$ and show that each is continuous. (Most of the time we actually work with the lifted map between manifold covers of these orbifold moduli spaces, but we elide this distinction in the introduction).

1.3 Existence of polynomial affine spheres. The construction of the map $\boldsymbol{\alpha} : \mathcal{MC}_d \to \mathcal{MP}_{d+3}$ first requires an existence theorem, i.e. for any polynomial cubic differential on $\mathbb{C}$ there exists a corresponding complete hyperbolic affine sphere. As in the work of Labourie and Loftin, through the technique of Wang this becomes a problem about finding a conformal metric—the *Blaschke metric* of the affine sphere—that satisfies a quasilinear PDE involving the norm of the cubic differential. The cubic differential and the Blaschke metric together determine a flat $\mathfrak{sl}_3\mathbb{R}$-valued connection form, the trivialization of which gives the affine sphere (and a framing thereof).

The technique we apply to solve for the Blaschke metric (the method of super- and sub-solutions) naturally gives a result for a somewhat more general class of equations. In Section 5 we show:

Theorem B. *Let $\phi = \phi(z)dz^k$ be a holomorphic differential on $\mathbb{C}$ with $\phi(z)$ a polynomial. Then there exists a unique complete and nonpositively curved conformal metric σ on $\mathbb{C}$ whose Gaussian curvature function K satisfies*

$$K = (-1 + |\phi|^2_\sigma).$$

Here $|\phi|_\sigma$ denotes the pointwise norm of ϕ with respect to the Hermitian metric σ^{-k} on the bundle of order-k holomorphic differentials.

The existence part of this result appears in Theorem 5.1, and the uniqueness in Theorem 5.3; the details are modeled on those found in [Wan92, WA94, Han96, HTTW95]. We call the differential equation considered in this theorem the *coupled vortex equation*; the connection between this equation and the classical vortex equation from gauge theory is described at the beginning of Section 5.

Returning to the construction of $\boldsymbol{\alpha}$, the case $k = 3$ of Theorem B implies that there exists a complete affine sphere with any given polynomial Pick differential. The next step is to show that these affine spheres are asymptotic to cones over convex polygons with the right number of vertices.

1.4 Ţiţeica asymptotics. The simplest examples of complete affine spheres with nonzero Pick form are the *Ţiţeica surfaces*. These can be characterized as the affine spheres conformal to $\mathbb{C}$ and having constant nonzero Pick differential; all such surfaces are projectively equivalent, and each is asymptotic to the cone over a triangle.

A polynomial cubic differential on $\mathbb{C}$ can be identified, after removing a compact set containing the zeros, with the result of gluing of a finite collection of half-planes, each equipped with a constant differential cdz^3. Using this description, we show that the asymptotic geometry of an affine sphere with polynomial Pick differential (briefly, a *polynomial affine sphere*) can be asymptotically modeled by gluing together pieces of finitely many Țițeica surfaces. Corresponding pieces of the asymptotic triangular cones glue to give the cone over a convex polygon.

An important aspect of the picture that is missing from the sketch above represents most of the work we do in Section 6: In order to compare an affine sphere to a Țițeica surface, one must control not only the Pick differential, but also the Blaschke metric. Analyzing the structure equations defining an affine sphere, this becomes a question of comparing the *Blaschke error*, meaning the difference between the Blaschke metric of a polynomial affine sphere and that of a Țițeica surface, to the *frame size* of the Țițeica surface, meaning the spectral radius of its affine frame in the adjoint representation. If the product of frame size and Blaschke error decays to zero in some region, we find a unique Țițeica surface to which the polynomial affine sphere is asymptotic.

Using our estimates for solutions to the coupled vortex equation, we find that the Blaschke error decays exponentially, and more precisely that

$$\text{Blaschke error} = O\left(\frac{e^{-2\sqrt{3}\,r}}{\sqrt{r}}\right),$$

where r is the distance from the zeros of the polynomial (measured in a natural coordinate system). In the same coordinates, the affine frame of the Țițeica surface grows exponentially, but the rate depends on direction, i.e.

$$\text{Frame size} \approx Ce^{c(\theta)r}$$

along a ray of angle θ, where $c(\theta)$ is an explicit function satisfying

$$c(\theta) \leqslant 2\sqrt{3}$$

with equality for exactly $2(d+3)$ directions, for a polynomial of degree d. Away from these *unstable directions*, the error decay is therefore more rapid than the frame growth and we have a unique asymptotic Țițeica surface, giving $2(d+3)$ Țițeica surfaces in all.

1.5 Assembling a polygon. The final step in the construction of $\boldsymbol{\alpha}$ is to understand how the limiting Țițeica surface changes when we cross an unstable direction. Perhaps surprisingly, in this analysis the $1/\sqrt{r}$ factor in our bound for the error turns out to be crucial (and sharp).

By integrating the affine connection form over an arc of a circle of radius R joining two rays that lie on either side of an unstable direction, we determine the element of $\mathrm{SL}_3\,\mathbb{R}$ relating the limiting Țițeica surfaces along these rays. The interplay between the quadratic approximation to $c(\theta)$ near its $2\sqrt{3}$ maximum and the Blaschke error estimate show that this integral is essentially a Gaussian approximation to a delta function in θ multiplied by an off-diagonal elementary matrix. Letting $R \to \infty$ we find that the neighboring Țițeica surfaces are related by a particular type of unipotent projective transformation.

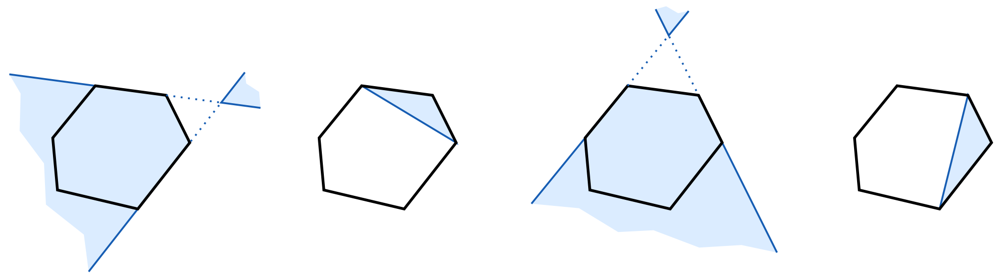

Figure 1: The vertex inscribed *triangles* and *edge circumscribed triangles* of a convex *polygon* in $\mathbb{RP}^2$ alternate in a natural cyclic order. Ţiţeica surfaces over these *triangles* assemble into an asymptotic model for the affine sphere over the *polygon*

These unipotent factors are the "glue" that let us move from an understanding of asymptotics in one direction, or in a sector, to the global picture. Using them, we find that each of the $2(d+3)$ transitions across an unstable direction reveals either a new edge or a new vertex, alternating to give a chain that closes up to form a convex polygon P with $(d+3)$ vertices. The polynomial affine sphere is asymptotic to the cone over P.

After the fact, the triangles associated to the individual Ţiţeica surfaces can also be described directly in terms of the polygon P: Each vertex of P forms a triangle with its two neighbors, giving $(d+3)$ *vertex inscribed triangles* of P. Each edge of P forms a triangle with the lines extending the two neighboring edges, giving $(d+3)$ *edge circumscribed triangles* of P. Each transition across an unstable direction flips from a vertex inscribed triangle to one of its neighboring edge circumscribed triangles, or vice versa (see Figure 1).

1.6 From polygons to polynomials. Constructing the inverse map $\boldsymbol{\alpha}^{-1} : \mathcal{MP}_{d+3} \to \mathcal{MC}_d$ amounts to showing that an affine sphere over a convex polygon has parabolic conformal type (i.e. is isomorphic to $\mathbb{C}$ as a Riemann surface) and that the Pick differential is a polynomial in the uniformizing coordinate. These properties are established in Section 7.

In this case there is no question of existence, as the Cheng–Yau theorem gives a complete affine sphere over any convex polygon (or indeed any properly convex set). To understand the conformal type and Pick differential of this affine sphere, we once again use the Ţiţeica surface as the key model and comparison object. The arguments are somewhat simpler than in the construction of $\boldsymbol{\alpha}$ described above.

Benoist and Hulin, using interior estimates of Cheng–Yau, showed that the k-jet of the affine sphere at a point depends continuously on the corresponding convex domain [BH13]. This implies that the Blaschke metric and Pick differential vary continuously in this sense (see Theorem 4.4 and Corollary 4.5 below). We use this continuity principle and projective naturality to compare the affine sphere over a polygon to the Ţiţeica surface over one of its vertex inscribed polygons.

We find that the Blaschke metric and the Pick differential of the polygonal affine sphere are comparable (with uniform multiplicative constants) to those of the Țițeica surface, at any point that is sufficiently close to the edges shared by the triangle and the polygon. Applying this construction to each of the vertex inscribed triangles, we find a "buffer" around the boundary of the polygon that contains no zeros of the Pick differential, and where the Blaschke metric is approximately Euclidean. It follows easily that the Pick form has finitely many zeros, that the conformal type of the Blaschke metric is $\mathbb{C}$, and finally that the Pick form is a polynomial.

1.7 Mapping of moduli spaces. Having established that polynomial affine spheres correspond to polygons and vice versa, the proof of the main theorem is completed by Theorem 8.1, where we use estimates and results from the preceding sections to show that this bijection between moduli spaces and its inverse are continuous. Also, since the normalization conventions of Sections 6 and 7 implicitly require working in manifold $\mathbb{Z}/(d+3)$-covers of the moduli spaces, we verify at this stage that our the constructions have the necessary equivariance properties to descend to the orbifolds themselves.

1.8 Related error estimates. The comparison between rates of error decay and frame growth which yields the finite set of asymptotic Țițeica surfaces for a polynomial affine sphere is an apparently novel element of our work on this class of surfaces. However, we build upon a substantial history of error estimates for Wang's equation and other geometric PDE. While both Loftin [Lof01] and Labourie [Lab07] focused on the Wang equation in their studies of affine spheres, Loftin [Lof04,Lof07,Lof15] began and developed a theory of error estimates for this equation. The structurally similar Bochner equation governing harmonic maps to hyperbolic surfaces had an analogous development of error estimates in [Min92,Wol91,Han96]. Crucial to refined error estimates are the use of sub- and super-solutions, first constructed in the present setting by Loftin [Lof04]; for the analogous Bochner equation and for open Riemann surfaces, this technique began with [Wan92] (see also [WA94]).

The passage from estimates for the solution of Wang's equation to an asymptotic description of the affine frame field requires an understanding the behavior of solutions to certain ordinary differential equations (i.e. the affine structure equations restricted to a curve). The technique we use here was introduced by Loftin in [Lof04]; there was no lower-dimensional Bochner equation analogue to this technique.

1.9 Other perspectives. To conclude the introduction we briefly comment on the potential relations between our main theorem and techniques from areas such as Higgs bundles and minimal surfaces. Some more concrete conjectures and possible extensions of our work are described after the proof of the main theorem, in Section 9.

In [Lab07], Labourie interprets the parameterization of convex $\mathbb{RP}^2$ structures by cubic differentials in terms of certain rank-3 Higgs bundles, thus identifying Wang's equation for the Blaschke metric of an affine sphere with Hitchin's self-duality equations for these bundles. The same construction applies in our setting: compactifying

$\mathbb{C}$ to $\mathbb{CP}^1$, our polynomial affine spheres correspond to *irregular Higgs bundles* on the projective line defined by a vector bundle and a meromorphic endomorphism-valued 1-form (which in this case has a single high-order pole).

Our existence theorem for Wang's equation (Theorem 5.1) is therefore equivalent to the existence of solutions to the self-duality equations for these bundles. The self-duality equations have been studied in the irregular case by Biquard–Boalch [BB04], extending Simpson's work for the case of a simple pole [Sim90]. However, there is a mismatch of hypotheses preventing us from deducing Theorem 5.1 directly from results in the Higgs bundle literature: The *irregular type* of the Higgs fields we consider, i.e. the Laurent expansion of the Higgs field at the pole, has nilpotent coefficient in the most singular term, while the type is usually assumed to be a regular semisimple element (for example in the results of [BB04]). Witten has proposed a way to generalize results from the semisimple case to arbitrary irregular types using branched coverings [Wit08], however a sketch is provided only in rank 2 and the details of a corresponding existence theorem have not appeared. We expect that pursuing these ideas further, one could give an alternate proof of Theorem 5.1 entirely by Higgs bundle methods.

The connection with irregular Higgs bundles also suggests that our main theorem could be related to the general phenomenon of equivalence between *Dolbeault moduli spaces* of Higgs bundles and *Betti moduli spaces* of representations of the fundamental group of a Riemann surface into a complex Lie group. It is known that the generalized Betti moduli space which corresponds to irregular Higgs bundles includes extra *Stokes data* at each pole of the Higgs field (see e.g. [Boa14]). This data consists of a collection of unipotent matrices cyclically ordered around each pole, and the possibility of a connection with the unipotent factors we find for a polygonal affine sphere in Lemma 6.5 is intriguing. We hope that by further developing this connection, one could interpret Theorem A as identifying the space of convex polygons in $\mathbb{RP}^2$ (stratified by the number of vertices) as a generalization of the Hitchin component of $\mathrm{SL}_3\,\mathbb{R}$ representations to the punctured Riemann surface $\mathbb{C}$ (with a stratification by the order of pole of the associated Higgs field at infinity).

Finally, we mention another interpretation of the affine sphere construction discussed by Labourie in [Lab07]: The combination of the Blaschke metric and the affine frame field of a convex surface in $\mathbb{R}^3$ induces a map to the symmetric space $\mathrm{SL}_3\,\mathbb{R}/\,\mathrm{SO}(3)$ which is a minimal immersion exactly when the original surface is an affine spherical immersion (the local version of being an affine sphere). By this construction, the flats of the symmetric space correspond to Ţiţeica affine spheres. Our main theorem can therefore be seen as identifying a moduli space of minimal planes in $\mathrm{SL}_3\,\mathbb{R}/\,\mathrm{SO}(3)$ that are asymptotic to finite collections of flats (in some sense that corresponds to Theorem 5.7) with a space of polynomial cubic differentials. It would be interesting to develop this picture more fully, for example by characterizing these minimal planes directly in terms of Riemannian geometry of the symmetric space, and possibly generalizing to the symmetric space of $\mathrm{SL}_n\,\mathbb{R}$ for $n > 3$.

2 Polygons

As in the introduction we will consider polygons in $\mathbb{RP}^2$ up to the action of the group $\mathrm{SL}_3\,\mathbb{R} \simeq \mathrm{PGL}_3\mathbb{R}$ of projective transformations. An elementary fact we will frequently use is:

Proposition 2.1. *The group* $\mathrm{SL}_3\,\mathbb{R}$ *acts simply transitively on* 4*-tuples of points in* $\mathbb{RP}^2$ *in general position.* □

Our convention is that a *convex polygon* in $\mathbb{RP}^2$ is a bounded open subset of an affine chart $\mathbb{R}^2 \subset \mathbb{RP}^2$ that is the intersection of finitely many half-planes. In particular a polygon is an open 2-manifold homeomorphic to the disk D^2. As usual, a polygon can also be specified by the 1-complex that forms its boundary, or by its set of vertices.

By *orientation* of the polygon we mean an orientation of its interior, as a 2-manifold, or equivalently an orientation of its boundary as a 1-complex. Whenever we list the vertices of an oriented polygon, it is understood that the list is ordered consistently with the orientation.

2.1 Spaces of polygons. Let $\mathcal{P}_k$ denote the set of oriented convex polygons in $\mathbb{RP}^2$ with k vertices (briefly, *convex k-gons*); this is an open subset of the symmetric product $\mathrm{Sym}^k(\mathbb{RP}^2)$. The group $\mathrm{SL}_3\,\mathbb{R}$ of projective automorphisms acts on $\mathcal{P}_k$ with quotient

$$\mathcal{MP}_k = \mathcal{P}_k / \mathrm{SL}_3\,\mathbb{R},$$

the *moduli space* of convex polygons. For $P \in \mathcal{P}_k$ we denote by $[P]$ its equivalence class in $\mathcal{MP}_k$.

By suitably normalizing a polygon, one can construct a natural "quasi-section" of the map $\mathcal{P}_k \to \mathcal{MP}_k$: Choose an oriented convex quadrilateral $Q_0 \subset \mathbb{RP}^2$ with vertices (q_1, q_2, q_3, q_4). This polygon Q_0 and the labeling of its vertices will be fixed throughout the paper. We say that $P \in \mathcal{P}_k$, $k \geqslant 4$, is *normalized* if it is obtained from Q_0 by attaching a convex $(k-2)$-gon to its (q_4, q_1) edge. Equivalently, an oriented convex polygon is normalized if its vertices are

$$(q_1, q_2, q_3, q_4, p_5, \ldots, p_k)$$

for some $p_i \in \mathbb{RP}^2$, $i = 5, \ldots, k$. In particular the vertices of a normalized polygon have a canonical labeling by $1, \ldots, k$.

Let $\mathcal{TP}_k \subset \mathcal{P}_k$ denote the set of normalized convex k-gons in $\mathbb{RP}^2$. Having fixed four vertices, the set $\mathcal{TP}_k$ is naturally is an open subset of $(\mathbb{RP}^2)^{k-4}$. In fact $\mathcal{TP}_k$ is contractible:

Proposition 2.2. *The space* $\mathcal{TP}_k$ *is diffeomorphic to* $\mathbb{R}^{2n-8}$.

Proof. Choose an affine chart of $\mathbb{RP}^2$ in which $q_1 = (0,1)$, $q_2 = (0,0)$, $q_3 = (1,0)$, and $q_4 = (1,1)$. By convexity, the remaining vertices of a normalized polygon must lie in the half-strip

$$\{(x,y) \mid 0 < x < 1,\ y > 1\}.$$

Setting $p_k = (x_k, y_k)$ we also have that x_k is monotonically decreasing with k, while the slopes m_k of the segments $\overline{p_{k-1}p_k}$ are monotonically increasing. Up to those two constraints, we may freely choose the pairs (x_j, m_j) for $j = 5, \dots, k$, which determine the polygon completely. Thus $\mathcal{TP}_k$ is parameterized by a product of open simplices:

$$\{(x_j, m_j) \mid 1 > x_5 > \cdots > x_k > 0,\ -\infty < m_5 < \cdots < m_k < \infty\} \simeq D^{2k-8}. \qquad \square$$

It is easy to see that $\mathcal{TP}_k$ intersects every $\mathrm{SL}_3\,\mathbb{R}$-orbit in $\mathcal{P}_k$: Given $P \in \mathcal{P}_k$, any four adjacent vertices of P are in general position and can therefore be mapped to $(q_1, \dots, q_4)$ by a unique element $A \in \mathrm{SL}_3\,\mathbb{R}$. Thus $A \cdot P \in \mathcal{TP}_k$ and we say that A *normalizes* P.

The only choice in the normalization construction is that of a vertex to map to q_1, and so there are exactly k ways to normalize an oriented convex k-gon P (though possibly some of them give the same normalized polygon). Equivalently, the set of intersection points of a $\mathrm{SL}_3\,\mathbb{R}$-orbit in $\mathcal{P}_k$ with $\mathcal{TP}_k$ has cardinality at most k, and the projection $\mathcal{TP}_k \to \mathcal{MP}_k$ is finite-to-one.

Furthermore, each fiber of this projection is the orbit of a natural $\mathbb{Z}/k$-action on $\mathcal{TP}_k$: Given $P \in \mathcal{TP}_k$ with vertices $(q_1, \dots, q_4, p_5, \dots, p_k)$, there is a projective transformation $A = A(P)$ uniquely determined by its action on the 4-tuple,

$$A : (q_2, q_3, q_4, p_5) \mapsto (q_1, q_2, q_3, q_4),$$

Then defining $\varrho(P) = A(P) \cdot P$ we have a map $\varrho : \mathcal{TP}_k \to \mathcal{TP}_k$. By construction P and $\varrho(P)$ lie in the same $\mathrm{SL}_3\,\mathbb{R}$-orbit and $\varrho^k = \mathrm{Id}$ follows since $\varrho^k(P) = B \cdot P$ where B is a projective transformation fixing the vertices of Q_0, hence $B = \mathrm{Id}$. It is straightforward to check that ϱ acts diffeomorphically on $\mathcal{TP}_k \subset (\mathbb{RP}^2)^{k-4}$, and in fact it acts by the restriction of a rational map defined over $\mathbb{Z}$.

Summarizing the discussion above, we find:

PROPOSITION 2.3. *The projection $\mathcal{TP}_k \to \mathcal{MP}_k$ can be identified with the quotient map of the $\mathbb{Z}/k$ action on $\mathcal{TP}_k$. Thus $\mathcal{MP}_k$ has the structure of an orbifold with universal cover $\mathcal{TP}_k \simeq \mathbb{R}^{2n-8}$.* $\square$

The fact that any convex polygon has a unique normalization once a vertex is chosen also shows that $\mathcal{TP}_k$ can be identified with a quotient space related to $\mathcal{MP}_k$: If we consider a space of pairs (P, v) where $P \in \mathcal{P}_k$ and v is a vertex of P, then the quotient of this space of *labeled polygons* by $\mathrm{SL}_3\,\mathbb{R}$ is in canonical bijection with $\mathcal{TP}_k$. In this description, the $\mathbb{Z}/k$ action cycles v around P while the map $(P, v) \mapsto P$ corresponds to $\mathcal{TP}_k \to \mathcal{MP}_k$.

The notations $\mathcal{TP}_k, \mathcal{MP}_k$ are meant to suggest *Teichmüller space* and *Moduli space*, respectively. In this analogy, the additional data of a normalization of a polygon is like the marking of a Riemann surface, and the $\mathbb{Z}/k$ action plays the role of the mapping class group. On both sides of the analogy, the Teichmüller space is contractible and smooth, while the quotient moduli space is only an orbifold. (Compare the "toy model" of the space of convex projective structures described in [FG07].)

So far we have considered our spaces of polygons to be topological spaces using what could be called the *vertex topology*, i.e. by considering the set of vertices of the polygon as a point in the symmetric product of $\mathbb{RP}^2$. Alternatively, one could introduce a topology on polygons using the Hausdorff metric on closed subsets of $\mathbb{RP}^2$; here (and throughout the paper) we take the closures of the convex domains whenever the Hausdorff topology is considered.

In fact these topologies are equivalent; it is immediate that vertex convergence implies Hausdorff convergence, and conversely we have:

PROPOSITION 2.4. *If a sequence P_n of convex polygons converges in the Hausdorff topology to a convex polygon P, and if P and P_n all have the same number of vertices, then P_n also converges to P in the vertex topology.*

Proof. First observe[1] that each vertex of v of P is a limit of a sequence v_n of vertices of P_n: Otherwise v would be a limit of interior points of edges, but not of their endpoints. By passing to a subsequence we could arrange for this sequence of edges to converge, giving a line segment in P of which v is an interior point, contradicting the assumption that v is a vertex.

Suppose P has k vertices. Applying the observation above to each vertex in turn we obtain k sequences, each having as nth element a vertex of P_n. While these sequences may overlap for some n, this can only happen for finitely many terms since these sequences have distinct limit points. Thus for all large n we have labeled the k vertices of P_n in such a way that vertices with a given label converge, as required. □

2.2 Example: pentagons. We have $\mathcal{TP}_5 \simeq \mathbb{R}^2$, and in fact the space is naturally an open triangle in $\mathbb{RP}^2$: The fifth vertex p_5 can be any point inside the triangle on the exterior of Q_0 formed by the $[q_4, q_1]$ edge and the lines extending its neighboring edges. (This model of $\mathcal{TP}_5$ is shown in Figure 5 in Section 9 below.)

Working in the affine coordinates of Proposition 2.2 the space becomes a half-strip,

$$\mathcal{TP}_5 = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ y > 1\}.$$

In the same coordinates, the generator of the $\mathbb{Z}/5$ action is the Cremona transformation

[1] This is an example of a general fact in convex geometry: In a Hausdorff-convergent sequence of compact convex subsets of a $\mathbb{R}^N$, each extreme point of the limit set is a limit of extreme points of the sequence.

$$\varrho(x,y) = \left(\frac{y(y-1)}{x^2 + x(y-1) + y(y-1)}, \frac{y(x+y-1)}{(x^2 + x(y-1) + y(y-1))} \right)$$

which can be expressed in a suitable homogeneous coordinate system as

$$[X : Y : Z] \mapsto [X(Z-Y) : Z(X-Y) : XZ].$$

This birational automorphism of $\mathbb{P}^2$ resolves to a biregular automorphism of a degree 5 del Pezzo surface by blowing up the four vertices of Q_0 (see e.g. [dF04, Sec. 4.6]). We are grateful to Stéphane Lamy for explaining this projective model. Returning to the affine half-strip model above, the unique fixed point of the $\mathbb{Z}/5$ action on $\mathcal{TP}_5$ is $\left(\frac{1}{2}, \frac{1}{4}(3+\sqrt{5})\right)$, which corresponds to the regular pentagon. The differential of ϱ at this fixed point is linearly conjugate to a rotation by 3/5 of a turn.

Topologically, the $\mathbb{Z}/5$ action rotates the interior of triangle $\mathcal{TP}_5$ about the fixed point. Blowing up the two vertices the triangle shares with Q_0 gives a pentagon in which the $\mathbb{Z}/5$ action can be seen as standard pentagonal rotational symmetry.

The quotient $\mathcal{MP}_5 = \mathcal{TP}_5/\langle\varrho\rangle$ is a topological open disk with an interior orbifold point (cone point) of order 5.

3 Cubic Differentials

We define a *polynomial cubic differential* to be a holomorphic differential on $\mathbb{C}$ of the form $C(z)dz^3$, where $C(z)$ is a polynomial function.

3.1 Spaces of cubic differentials. Let $\mathcal{C}_d \simeq \mathbb{C}^* \times \mathbb{C}^d$ denote the vector space of polynomial cubic differentials of degree d (with nonzero leading coefficient).

The group $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b\}$ acts on these differentials by pushforward. Let $\mathcal{MC}_d$ denote the quotient of $\mathcal{C}_d$ by this action. Given a cubic differential C, we denote its equivalence class by $[C]$.

As in the polygon case, the relationship between $\mathcal{C}_d$ and $\mathcal{MC}_d$ is clarified by considering an intermediate object space of "normalized" objects: If a polynomial cubic differential is written as

$$C = (c_d z^d + c_{d-1} z^{d-1} + \cdots + c_0) dz^3 \tag{1}$$

then we say C is *monic* if $c_d = 1$ and *centered* if $c_{d-1} = 0$. The latter condition means that the roots of C sum to zero. A cubic differential that is both monic and centered is *normalized*.

Let $\mathcal{TC}_d \subset \mathcal{C}_d$ denote the space of normalized polynomial cubic differentials. The set $\mathcal{TC}_d \simeq \mathbb{C}^{d-1}$ intersects every $\mathrm{Aut}(\mathbb{C})$-orbit in $\mathcal{C}_d$. Note that $z \mapsto (az + b)$ maps the differential (1) to

$$T^*C = (c_d a^{d+3} (z + b/a)^d + c_{d-1} a^{d+2} (z + b/a)^{d-1} + \cdots + c_0 a^3) dz^3.$$

Thus acting by $z \mapsto c_d^{-1/(d+3)} z + b$ makes an arbitrary differential monic, and a suitable translation factor b moves the root sum to zero. Moreover, if two normalized cubic differentials are related by $T(z) = az + b$, then the monic condition gives $a^{d+3} = 1$ and the centering implies $b = 0$, hence T is multiplication by a $(d+3)$-root of unity.

We therefore recover a description of $\mathcal{MC}_d$ as a space of orbits in $\mathcal{TC}_d$ of the group $\boldsymbol{\mu}_{d+3} \simeq \mathbb{Z}/(d+3)$ of roots of unity, where $\zeta \in \boldsymbol{\mu}_{d+3}$ acts on coefficients by

$$(c_{d-2}, c_{d-3}, \dots, c_0) \mapsto (\zeta^{d+1} c_{d-2}, \zeta^d c_{d-3}, \dots, \zeta^3 c_0). \tag{2}$$

The quotient by this action is a "weighted affine space", i.e. an affine chart of the weighted projective space $\mathbb{CP}(d+3, \widehat{d+2}, d+1, \dots, 3)$. To summarize,

Proposition 3.1. *The space $\mathcal{MC}_d$ is a complex orbifold (and a complex algebraic variety), the quotient of $\mathcal{TC}_d \simeq \mathbb{C}^{d+1}$ by the action of $\boldsymbol{\mu}_{d+3}$ described in (2).*

3.2 Example: Quadratic polynomials. We have $\mathcal{TC}_2 = \{(z^2 + c)dz^3\} \simeq \mathbb{C}$. The action of $\mathbb{Z}/5$ is generated by the rotation $z \mapsto \zeta z$, where $\zeta = \exp(2\pi i/5)$, acting by $c \mapsto \zeta^3 c$. The unique fixed point is $c = 0$ and the quotient $\mathcal{MC}_2 = \mathcal{TC}_2/\langle\zeta\rangle$ is a Euclidean cone with cone angle $2\pi/5$. Alternatively $\mathcal{MC}_2$ is the affine chart of $\mathbb{CP}(5, 3)$ in which the first homogeneous coordinate is nonzero.

3.3 Natural coordinates. A *natural coordinate* for a cubic differential C is a local coordinate w on an open subset of $\mathbb{C}$ in which $C = 2dw^3$. (The factor of 2 here is not standard, but it simplifies calculations later.) Such a coordinate always exists locally away from the zeros of C, because near such a point one can choose a holomorphic cube root of C and take

$$w(z) = \int_{z_0}^{z} \left(\tfrac{1}{2}C\right)^{1/3}. \tag{3}$$

Up to adding a constant, every natural coordinate for C has this form. Thus any two natural coordinates for C differ by multiplication by a power of $\omega = \exp(2\pi i/3)$ and adding a constant.

This local construction of a natural coordinate (and the integral expression above) analytically continues to any simply-connected set in the complement of the zeros of C to give a *developing map*, a holomorphic immersion that pulls back $2dw^3$ to C. The developing map need not be injective, but any injective restriction of it is a natural coordinate.

The metric $|C|^{2/3}$ defines a flat structure on $\mathbb{C}$ with singularities at the zeros of C. In a natural coordinate this is simply the Euclidean metric $2^{2/3}|dw|^2$. We call this the *flat structure* or *flat metric* associated to C. A zero of C of multiplicity k is a cone point with angle $\frac{2\pi}{3}(3+k)$. Straight lines in a natural coordinate are geodesics of this metric.

3.4 Half-planes and rays. We define a *C-right-half-plane* to be a pair (U, w) where $U \subset \mathbb{C}$ is open and w is a natural coordinate for C that maps U diffeomorphically to the right half-plane $\{\operatorname{Re} w > 0\}$. Note that U then determines w up to addition of a purely imaginary constant. Given a half-plane (U, w) there is an associated family of parallel right-half-planes $(U^{(t)}, w^{(t)})$, $t \in \mathbb{R}^+$, defined by $w^{(t)} = w - t$ and $U^{(t)} = w^{-1}(\{\operatorname{Re} w > t\}) \subset U$.

A path in $\mathbb{C}$ whose image in a natural coordinate for C is a Euclidean ray with angle θ will be called a *C-ray with angle θ*. Note that the angle is well-defined mod $2\pi/3$. In a suitable natural coordinate, a C-ray has the parametric form $t \mapsto b + e^{i\theta} t$.

Similarly, a *C-quasi-ray with angle θ* is a path that can be parameterized so that its image in a natural coordinate w has the form $t \mapsto e^{i\theta} t + \delta(t)$ where $|\delta(t)| = o(t)$.

Before discussing the geometry of an arbitrary polynomial cubic differential, we describe a configuration of C-rays and C-right-half-planes for the cubic differential $C = z^d dz^3$ that we intend to generalize: Consider the "star" formed by the $(d+3)$ Euclidean rays from the origin in $\mathbb{C}$,

$$\star_d := \{z \mid z^{d+3} \in \mathbb{R}^+\} = \{\arg z = 0 \mod 2\pi/(d+3)\}.$$

Since there is a natural coordinate for $z^d dz^3$ that is a real multiple of $z^{(d+3)/3}$, these are also $z^d dz^3$-rays with angle zero.

Now consider Euclidean sectors of angle $3\pi/(d+3)$ centered on each of the rays in $\star_d$; each such sector is naturally a $z^d dz^3$-right-half-plane in which the corresponding ray of $\star_d$ maps to $\mathbb{R}^+$. These sectors are pairwise disjoint except when surrounding neighboring rays in $\star_d$, in which case they overlap in a sector of angle $\pi/(d+3)$. In particular this overlap, when nonempty, maps by a natural coordinate to a sector in that coordinate of angle $\pi/3$. Finally, we observe that each of these Euclidean sectors of angle $3\pi/(d+3)$ is contained in the region between its neighboring rays from $\star_d$.

Thus we have constructed a system $\{(U_k, w_k)\}_{k=0,\dots,d+2}$ of $z^d dz^3$-right-half-planes that cover $\mathbb{C}^*$ and which are neighborhoods of the rays in $\star_d$, each neighborhood being disjoint from the other rays. Replacing these with the associated parallel half-planes $(U_k^{(t)}, w_k^{(t)})$, for some $t \in \mathbb{R}^+$, we have a collection of "eventual neighborhoods" of the rays of $\star_d$ that cover all but a compact set in $\mathbb{C}$.

In fact a collection of half-planes like this exists for any monic polynomial cubic differential, except that the rays of $\star_d$ will now only be *quasi-rays* in their respective half-plane neighborhoods. Specifically, we have:

Proposition 3.2. (Standard half-planes) *Let C be a monic polynomial cubic differential. Then there are $(d+3)$ C-right-half-planes $\{(U_k, w_k)\}_{k=0,\dots,d+2}$ with the following properties*:

(i) *The complement of $\bigcup_k U_k$ is compact.*
(ii) *The ray $\{\arg(z) = \frac{2\pi k}{d+3}\}$ is eventually contained in U_k.*
(iii) *The rays $\{\arg(z) = \frac{2\pi(k\pm1)}{d+3}\}$ are disjoint from U_k.*

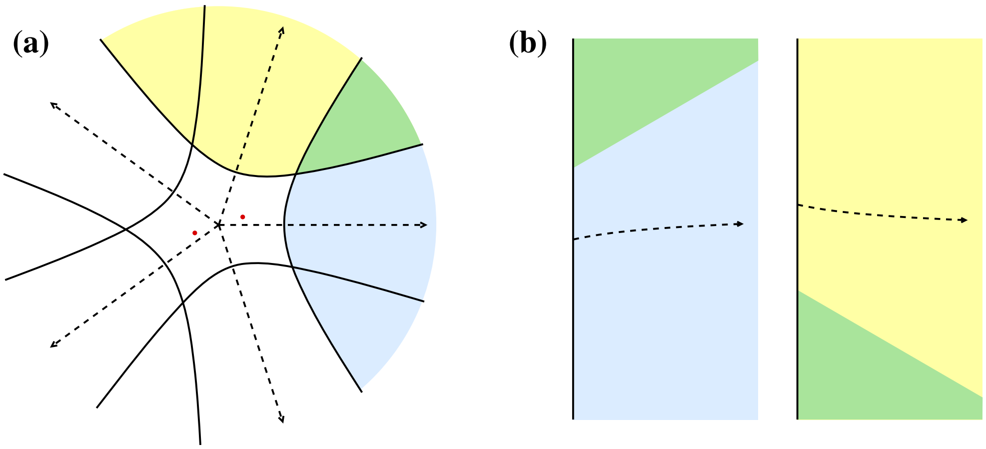

Figure 2: **a** Covering a neighborhood of infinity by five C-right-half-planes for the differential $C = (z^2 - (3+i)^2)dz^3$. **b** The edges of $\star_d$ (*dashed*) map to C-quasi-rays in the natural coordinates of these half-planes

(iv) *On* $U_k \cap U_{k+1}$ *we have* $w_{k+1} = \omega^{-1} w_k + c$ *for some constant* c, *and each of* w_k, w_{k+1} *maps this intersection onto a sector of angle* $\pi/3$ *based at a point on* $i\mathbb{R}$. (*Recall* $\omega = \exp(2\pi i/3)$.)

(v) *Each ray of* $\star_d$ *is a* C*-quasi-ray of angle zero in the associated half-plane* U_k. *More generally any Euclidean ray in* $\mathbb{C}$ *is a* C*-quasi-ray and is eventually contained in* U_k *for some* k.

Figure 2 shows an example of the configuration of half-planes given by this proposition.

Considering C as a meromorphic differential on $\mathbb{CP}^1$, this proposition describes the local structure of natural coordinates near a higher-order pole, and in this formulation it is well-known. For example, the corresponding description of half-planes for a meromorphic quadratic differential is given in [Str84, Sec. 10.4], and those arguments are easily adapted to cubic differentials. A proof of the proposition above is given in Appendix A.

4 Affine Spheres and Convex Sets

In this section, we briefly recall the definitions and results on affine spheres necessary to prove our main results. For more detailed background material we refer the reader to [NS94, LSZ93, Lof10].

4.1 Affine spheres. We consider locally strictly convex surfaces $M \subset \mathbb{R}^3$. A basic construction in affine differential geometry associates to such a surface a transverse vector field $\xi \pitchfork M$, the *affine normal* field, which is equivariant with respect to translations and the linear action of $\mathrm{SL}_3\,\mathbb{R}$. An *affine sphere* is a surface whose affine normal lines are concurrent at a point (the *center*). By applying a translation we can move the center of the affine sphere to the origin, in which case we have

$$\xi(p) = -Hp, \quad \text{for all } p \in M \subset \mathbb{R}^3$$

for some constant $H \in \mathbb{R}$, the *mean curvature*. We assume this normalization of the center from now on. We will consider only *hyperbolic* affine spheres, which are those with $H < 0$; by applying a dilation such a sphere can be further normalized so that $H \equiv -1$.

The second fundamental form of the convex surface M (relative to the transversal ξ) can be used to define an $\mathrm{SL}_3\,\mathbb{R}$-invariant Riemannian metric h, the *Blaschke metric*; specifically, this metric is seen in the Gauss equation which decomposes the flat connection of $\mathbb{R}^3$ into its tangential (TM) and normal ($\mathbb{R}\xi$) components:

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \quad X, Y \in \mathrm{Vect}(M)$$

We then have two connections on TM: The tangential component ∇ of the flat connection, and the Levi-Civita connection ∇^h of the Blaschke metric. The difference $(\nabla - \nabla^h)$ is a tensor of type $(2, 1)$, and using the isomorphism $TM \simeq T^*M$ induced by h we have an associated cubic form A on TM, the *Pick form* [Pic17].

We use the conformal class of the Blaschke metric to regard M as a Riemann surface. Blaschke showed that for an affine sphere, the Pick form A is the real part of a holomorphic cubic differential $C = C(z)dz^3$ [Bla23, p. 211]. We call C (which is uniquely determined by A) the *Pick differential* of the affine sphere.

All of the affine differential-geometric constructions above are local and can therefore be applied to immersed (rather than embedded) locally strictly convex surfaces in $\mathbb{R}^3$. This gives the notion of an *affine spherical immersion* $f : M \to \mathbb{R}^3$ of a Riemann surface M into $\mathbb{R}^3$ and associated Blaschke conformal metric h and Pick differential $C \in H^0(M, K_M^3)$.

We will refer to an affine spherical immersion as *complete* if the domain is complete with respect to the Blaschke metric. (This notion is sometimes called "affine complete".) Completeness in this sense has strong consequences:

Theorem 4.1 (Li [Li90, Li92]). *If an affine spherical immersion $f : M \to \mathbb{R}^3$ is complete, then it is a proper embedding, and its image is the boundary of an open convex set.* □

While stated here only in the 2-dimensional case, the result of Li applies to affine spheres of any dimension. The proof uses some of the estimates and the techniques developed by Cheng–Yau to prove a fundamental existence theorem for affine spheres (stated below as Theorem 4.3). More recently, Trudinger and Wang [TW02] have shown that completeness of the Blaschke metric gives the same conclusions as the theorem above (properness, etc.) under much weaker conditions—rather than requiring the immersion to be affine spherical immersion, local strict convexity alone is enough.

Completeness of an affine sphere also implies that the Blaschke metric is non-positively curved. This was first proved (as a statement about non-positive Ricci curvature for hyperbolic affine spheres of dimension at least two) by Calabi [Cal72], using a differential inequality on the norm of the Pick differential. A different proof was given by Benoist–Hulin [BH13] using Wang's equation (see (6) below). Li–Li–Simon [LLS04] show (by techniques in the spirit of those of Calabi) that if the curvature vanishes at a point, then it is identically zero and the surface is projectively equivalent to a specific example, the Țițeica surface, which discussed in more detail below. Summarizing, we have:

Theorem 4.2 [Cal72, BH13, LLS04]. *The Blaschke metric of a complete hyperbolic affine sphere M has nonpositive curvature. In fact, the curvature is either strictly negative or identically equal to zero, and in the latter case the affine sphere is $\mathrm{SL}_3\,\mathbb{R}$-equivalent to a surface of the form $x_1x_2x_3 = c$ for some nonzero constant c.* ☐

4.2 Frame fields. In much the same way that a surface immersed in Euclidean space can be locally reconstructed (up to an ambient Euclidean isometry) from its first and second fundamental forms, an affine spherical immersion can be recovered (up to the action of $\mathrm{SL}_3\,\mathbb{R}$) from the data of its Blaschke metric and Pick differential. In both cases one can consider this reconstruction as the integration of a connection 1-form with values in a Lie algebra to obtain a suitable frame field on the surface.

To describe the relevant integration process for an affine sphere, we introduce the complexified frame field F of an affine spherical immersion $f : M \to \mathbb{R}^3$,

$$F = (f\ f_z\ f_{\bar{z}}) \in \mathrm{GL}_3\,\mathbb{C}.$$

In fact, since f is real-valued, the frame F takes values in a fixed right coset of $\mathrm{GL}_3\,\mathbb{R}$ within $\mathrm{GL}_3\,\mathbb{C}$. Our standing assumption that f is normalized to have center at the origin means that f is proportional to its affine normal ξ, hence the components of this frame give both the affine normal direction and a complex basis for the tangent space. Here and throughout we use z to denote a local conformal coordinate for the Blaschke metric.

Following Wang [Wan91] and Simon–Wang [SW93], the Gauss and Weingarten structure equations for an affine spherical immersion can be stated in terms of the Darboux derivative $F^{-1}dF$ of the frame field. Writing $h = e^u|dz|^2$ and $C = C(z)dz^3$, the frame field of an affine sphere with mean curvature $H \equiv -1$ satisfies

$$F^{-1}dF = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^u \\ 1 & u_z & 0 \\ 0 & Ce^{-u} & 0 \end{pmatrix} dz + \begin{pmatrix} 0 & \frac{1}{2}e^u & 0 \\ 0 & 0 & \bar{C}e^{-u} \\ 1 & 0 & u_{\bar{z}} \end{pmatrix} d\bar{z}. \tag{4}$$

The integrability of this $\mathfrak{gl}_3\mathbb{C}$-valued form is equivalent to two additional (structure) equations on u and C:

$$C_{\bar{z}} = 0 \tag{5}$$

$$\Delta u = 2\exp(u) - 4|C|^2\exp(-2u). \tag{6}$$

The first equation simply requires the cubic differential C to be holomorphic. In the second equation (6), the (flat) Laplacian Δ is the operator $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. This nonlinear condition can be seen as imposing a relationship between the curvature of the Blaschke metric and the norm of the holomorphic cubic differential. More precisely (6) can be written as

$$K(h) = -1 + 2|C|_h^2, \tag{7}$$

where $K(h)$ denotes the Gaussian curvature function of the conformal metric $h = e^u|dz|^2$, and $|C|_h = |C|e^{-\frac{3}{2}u}$ is the pointwise h-norm of the cubic differential C.

Condition (6) (or its equivalent formulation (7)) is referred to in the affine sphere literature as *Wang's equation*. In the paper [Wan91], Wang studied its solutions to develop an intrinsic theory of affine spheres invariant under a cocompact group of automorphisms. It is also a variant of the *vortex equation* appearing in the gauge theory literature (see e.g. [JT80]), as explained in the next section. Labourie has interpreted (in [Lab07, Sec. 9]) this equation as an instance of Hitchin's self-duality equations for a rank-3 real Higgs bundle (E, Φ) over M with trivial determinant. In this perspective the Higgs field Φ is determined by the cubic differential C, a unitary connection A on E comes from the metric h, and integration of (4) corresponds to finding a local horizontal trivialization of the associated flat connection $A + \Phi + \Phi^*$.

Section 5 below is devoted to a study of solutions to equation (7) for polynomial cubic differentials on $\mathbb{C}$, and to a generalization to polynomial holomorphic differentials of any degree. These PDE results are applied in subsequent sections to construct the mapping of moduli spaces that is the subject of the main theorem.

4.3 Monge–Ampere, the Cheng–Yau Theorem and estimates. As we noted in the discussion of the completeness of affine spheres in Section 4.1, the seminal existence result for hyperbolic affine spheres is due to Cheng–Yau [CY75], with some clarifications on the notions of completeness (see the last sentence of the statement below) due to Gigena [Gig81], Sasaki [Sas80], and Li [Li90, Li92]. (The book [LSZ93] gives a comprehensive and coherent account of this theory.) The Cheng–Yau result says that hyperbolic affine spheres of a given mean curvature in $\mathbb{R}^3$ correspond to properly convex sets in $\mathbb{RP}^2$.

Theorem 4.3 (Cheng–Yau [CY86]). *Let $\mathcal{K} \subset \mathbb{R}^3$ denote an open convex cone which contains no lines. Then there is a unique complete hyperbolic affine sphere $M \subset \mathbb{R}^3$ of mean curvature $H = -1$ which is asymptotic to $\partial\mathcal{K}$.*

On the other hand, any complete affine sphere $M \subset \mathbb{R}^3$ with center 0 is asymptotic to the boundary of such a convex cone; this cone can be described as the convex hull of $M \cup \{0\}$. □

The Cheng–Yau theorem emerges from an approach to affine differential geometry through the analysis of nonlinear PDE of Monge–Ampere type. This approach is quite different from the frame field integration methods described above, and some of the estimates on affine invariants that come from the Monge–Ampere theory will be used in subsequent sections. Therefore, we will now briefly review the basics of this approach before formulating the estimates we need.

Cones $\mathcal{K}$ of the type considered in the Cheng–Yau theorem above correspond to properly convex open sets $\Omega \subset \mathbb{RP}^2$. (*Properly convex* means that the set can be realized as a bounded convex subset of an affine chart.) Given $\mathcal{K}$, we define $\Omega \subset \mathbb{RP}^2$ to be the set of lines through the origin in $\mathbb{R}^3$ that intersect $\mathcal{K}$ nontrivially (and hence in a ray). Conversely, the union of lines corresponding to points of a properly convex set Ω gives a "double cone" $\cup\Omega$ in which the origin is a cut point. Removing one of the two sides of the origin gives a convex cone containing no line, to which the Cheng–Yau theorem applies.

In fact, applying the Cheng–Yau theorem to either side of the double cone $\cup\Omega$ gives essentially the same affine sphere; like the cones themselves, the spheres are related by the antipodal map $p \mapsto -p$. Up to this ambiguity, one can therefore think of complete affine spheres in $\mathbb{R}^3$ as being parameterized by properly convex open sets in $\mathbb{RP}^2$.

Using this correspondence, we can consider any complete affine sphere in $\mathbb{R}^3$ as being parameterized as a "radial graph" over its corresponding projection to $\mathbb{RP}^2$. More precisely, if we consider Ω as a subset of an affine chart which we identify with the plane $\{(x, y, 1) \in \mathbb{R}^3\}$, then the point of the affine sphere that lies on the ray through $(x, y, 1)$ has the form

$$-\frac{1}{u(x,y)} \cdot (x, y, 1)$$

where $u = u(x, y)$ is a certain real, negative function on Ω. We call $u = u_\Omega$ the *support function* that defines the affine sphere. Since the surface is properly embedded and asymptotic to the boundary of the cone, we have $u \to 0$ on the boundary of Ω. Moreover, the condition that the surface is an affine sphere becomes a Monge–Ampere equation that the support function must satisfy:

$$\det(\mathrm{Hess}(u)) = u^{-4}.$$

The Cheng–Yau theorem is established by studying the Dirichlet problem for this equation (and its generalization to higher dimensions).

Following this approach, Benoist and Hulin use the maximum principle and interior estimates of Cheng–Yau to show that the support function and its derivatives depend continuously on the convex domain, in a sense which we now describe.

Let $\mathfrak{C}_*$ denote the set of pointed properly convex open sets in the real projective plane $\mathbb{RP}^2$, i.e.

$$\mathfrak{C}_* = \{(\Omega, x) \mid \Omega \subset \mathbb{R}^2 \subset \mathbb{RP}^2 \text{ open, bounded, and convex}, x \in \Omega\}.$$

We equip $\mathfrak{C}_*$ with the product of the Hausdorff topology on closed sets $\bar{\Omega}$ and the $\mathbb{RP}^2$ topology. Similarly let $\mathfrak{C}$ denote the set of open properly convex sets with the Hausdorff topology. Then we have:

Theorem 4.4 (Benoist and Hulin [BH13, Cor. 3.3]). *For any $k \in \mathbb{N}$, the k-jet of the support function u_Ω at $p \in \Omega$ is continuous as a function of $(\Omega, p) \in \mathfrak{C}_*$.*

More generally, the restriction of the support function and its derivatives to a fixed compact set depend continuously on the domain. That is, consider a properly convex open set $\Omega \subset \mathbb{RP}^2$, a compact set $K \subset \Omega$, and a neighborhood U of Ω in $\mathfrak{C}$ small enough so that $K \subset \Omega'$ for all $\Omega' \in U$. Then the restriction of the support function to K varies continuously in the C^k topology as a function of $\Omega \in U$.

Remark. The statement of Corollary 3.3 in [BH13] involves only the k-jet at a point. However, their proof also establishes the uniform C^k continuity on a compact subset that we have included in the theorem above. In fact, they derive the pointwise k-jet continuity from the C^k continuity on compacta.

As pointwise differential invariants, the Blaschke metric and the Pick form can be computed from derivatives of the support function. In fact, the Blaschke metric at a point is determined by the 2-jet of the support function at that point, and the Pick differential by the 3-jet. Therefore, the continuous variation statement above also yields:

Corollary 4.5. *The Blaschke metric and the Pick differential of the affine sphere over a properly convex domain Ω depend continuously on the domain, in the same sense considered in Theorem 4.4 (i.e. pointwise or on a fixed compact subset).* $\square$

4.4 Fundamental examples. Either connected component of a two-sheeted hyperboloid in $\mathbb{R}^3$ is a hyperbolic affine sphere. The center is the vertex of the cone to which the hyperboloid is asymptotic, and the corresponding convex domain in $\mathbb{RP}^2$ is bounded by a conic. The Blaschke metric is the hyperbolic metric, considering the hyperboloid as the Minkowski model of $\mathbb{H}^2$, and the Pick differential vanishes identically. This affine sphere is homogeneous, in that it carries a transitive action of a subgroup of $\mathrm{SL}_3\,\mathbb{R}$, in this case conjugate to $\mathrm{SO}(2,1)$.

Another homogeneous affine sphere, having nonzero Pick differential, will play an essential role in our proof of the main theorem. For any nonzero real constant c, each connected component of the surface $x_1x_2x_3 = c$ in $\mathbb{R}^3$ is a hyperbolic affine sphere centered at the origin [Tzi08]; the mean curvature of this surface is constant, depending on c. We call any surface that is equivalent to one of these under the action of $\mathrm{SL}_3\,\mathbb{R}$ (and hence also an affine sphere) a *Ţiţeica surface*. Each such surface carries a simply transitive action of a maximal torus of $\mathrm{SL}_3\,\mathbb{R}$; in the case of $x_1x_2x_3 = c$, it is the diagonal subgroup.

We now introduce a parameterization of a Ţiţeica surface that will be used extensively in Section 6. Let $\mathfrak{h} : \mathbb{C} \to \mathfrak{sl}_3\mathbb{R}$ be the map defined by

$$\mathfrak{h}(z) = \begin{pmatrix} 2\,\mathrm{Re}(z) & 0 & 0 \\ 0 & 2\,\mathrm{Re}(z/\omega) & 0 \\ 0 & 0 & 2\,\mathrm{Re}(z/\omega^2) \end{pmatrix}, \tag{8}$$

where $\omega = e^{2\pi i/3}$. Let

$$H(z) = \exp(\mathfrak{h}(z)) \tag{9}$$

denote the associated map to the diagonal subgroup of $\mathrm{SL}_3\,\mathbb{R}$. Then we obtain a parameterization of the component of $x_1x_2x_3 = c$ in the positive octant by the orbit map

$$T(z) = H(z) \cdot c^{1/3}(1, 1, 1) = c^{1/3}(e^{2\,\mathrm{Re}(z)}, e^{2\,\mathrm{Re}(z/\omega)}, e^{2\,\mathrm{Re}(z/\omega^2)}). \tag{10}$$

Taking $c = \frac{1}{3\sqrt{3}}$ gives a surface with mean curvature $H = -1$. In terms of the parameterization above, the Blaschke metric of this affine sphere is $e^u|dz|^2 = 2|dz|^2$ and the Pick differential is $C = 2dz^3$. We call this parameterized affine sphere T the *normalized Țițeica surface*.

Let F_T denote the frame field of the normalized Țițeica surface T. By homogeneity under the action of the diagonal group we have

$$F_T(z) = H(z) \cdot F_T(0).$$

Because they correspond to complete, flat Blaschke metrics and constant, nonzero Pick differentials, the Țițeica surfaces are natural comparison objects for any class of affine spheres with small (or decaying) Blaschke curvature. Later we will see that affine spheres corresponding to polynomial cubic differentials on $\mathbb{C}$ have this behavior at infinity.

Because it is asymptotic to the boundary of the positive octant in $\mathbb{R}^3$, projecting the normalized Țițeica surface to $\mathbb{RP}^2$ gives an open set $\mathbb{P}(T)$ that is the interior of a triangle whose vertices correspond to the coordinate axes; we denote this triangle by $\triangle_0$, and its vertices by v_{100}, v_{010}, and v_{001} according to their homogeneous coordinates. We use analogous notation for the three edges of the triangle, calling them $e_{011}, e_{101}, e_{110}$ according to the homogeneous coordinates of a point in the interior of the edge.

4.5 Affine spheres from planar data. Anticipating the construction in Section 6 of an affine sphere with prescribed polynomial Pick differential, we finish this section by observing that the results on affine spheres described thus far reduce this problem to one of constructing suitable planar data (i.e. a conformal metric on $\mathbb{C}$ satisfying Wang's equation).

PROPOSITION 4.6. *Let $C(z)dz^3$ be a holomorphic cubic differential on the complex plane $\mathbb{C}$. Let $h = e^u|dz|^2$ solve the Wang equation (6) and suppose that $e^u|dz|^2$ is a complete metric on $\mathbb{C}$. Then integration of the form (4) gives the complexified frame field $F : \mathbb{C} \to \mathrm{GL}_3\,\mathbb{C}$ of an affine sphere $f : \mathbb{C} \to \mathbb{R}^3$ with Blaschke metric h and Pick differential C. The map f is a proper embedding, and its image is asymptotic to the boundary of the cone over a convex domain in $\mathbb{RP}^2$.*

Proof. Integrability of the structure equations (4) follows because we assumed that C is holomorphic and that h, C satisfy (6). This guarantees an affine spherical immersion $\mathbb{C} \to \mathbb{R}^3$. By hypothesis the Blaschke metric of this immersion is complete, hence by Theorem 4.1 it is a proper embedding. The last sentence of the Cheng–Yau theorem (4.3) then completes the proof. □

Of course we have already seen one instance of this proposition: the Țițeica surface is the result of integrating $e^u = 2|dz|^2$ and $C = 2dz^3$; it is easy to check that this pair satisfies (6).

5 The Coupled Vortex Equation

5.1 Existence theorem. In this section we study the problem of prescribing a certain relationship between the curvature of a conformal metric on $\mathbb{C}$ and the norm of a holomorphic differential. The Wang equation (6) is one example of the class of equations we consider (and the only instance that is used in subsequent sections), but in this section we consider a more general class of equations to which our techniques naturally apply.

We begin with the following existence result:

Theorem 5.1. *Let $\phi = \phi(z)dz^k$ be a holomorphic differential of order k on $\mathbb{C}$, with $\phi(z)$ a polynomial that is not identically zero. Then there exists a complete, smooth, nonpositively curved conformal metric $\sigma = \sigma(z)|dz|^2$ on $\mathbb{C}$ satisfying*

$$K_\sigma = (-1 + |\phi|_\sigma^2) \tag{11}$$

where

- *$K_\sigma(z) = -(2\sigma(z))^{-1}\Delta(\log \sigma(z))$ is the Gaussian curvature, and*
- *$|\phi|_\sigma(z) = \sigma(z)^{-k/2}|\phi(z)|$ is the pointwise norm with respect to σ.*

Furthermore this metric can be chosen to satisfy $\sigma \geqslant |\phi|^{2/k}$, with equality at some point if and only if $\phi(z)$ is constant and $\sigma = |\phi|^{2/k}$.

Note that up to scaling of the holomorphic differential by a constant factor, the case $k = 3$ of (11) is Wang's equation.

Before proceeding with the proof, we briefly explain a connection (also noted in [Dun12, Sec. 3.1]) between (11) and the *vortex equations* from gauge theory. These equations were introduced in the Ginzburg–Landau model of superconductivity [GL50] and subsequently generalized and extensively studied in relation to Yang-Mills-Higgs theory (see e.g. [JT80, Bra91, GP94, Wit07]). In one formulation, the vortex equations on a Riemann surface reduce to a single equation for a Hermitian metric of a holomorphic line bundle; specifically, one fixes a holomorphic section ϕ of the bundle and asks for a Hermitian metric whose curvature differs from the pointwise norm of ϕ by a constant. Since the curvature of the Hermitian metric is an endomorphism-valued 2-form, it is first contracted with the Kähler form of fixed background metric on the surface to define a scalar equation.

In our situation we have a holomorphic section ϕ of the kth tensor power of the canonical bundle, and a Hermitian metric on this bundle is simply the tensor power of a conformal metric σ on the Riemann surface itself. If we use the Kähler form of σ instead of a fixed background metric, the associated vortex equation becomes (11). Thus, our equation involves an additional "coupling" between the curvature and norm functions that does not appear in the classical vortex equation setting, and we refer to (11) as the *coupled vortex equation*.

Proof of Theorem 5.1. Writing $\sigma(z) = \exp(u(z))$ the Equation (11) from the theorem becomes

$$\Delta u = 2e^u - 2|\phi|^2 e^{-(k-1)u} \tag{12}$$

and it is this form we use in the proof. We denote the right hand side of the equation above by $F(z, u)$.

We apply the method of sub-solutions and super-solutions for complete noncompact manifolds to (12): For equations of the form $\Delta u = F(z, u)$ where $\partial F/\partial u > 0$ it suffices to construct a pair of continuous functions on $\mathbb{C}$ which weakly satisfy

$$\begin{aligned} \Delta u_+ &\leqslant F(z, u_+), \\ \Delta u_- &\geqslant F(z, u_-). \end{aligned}$$

and where $u_- \leqslant u_+$. Then the method (cf. [Wan92, Thm. 9]) gives a smooth solution u on $\mathbb{C}$ satisfying $u_- \leqslant u \leqslant u_+$.

In our case, both u_- and u_+ will be slight modifications of the function

$$u_\phi := \frac{1}{k} \log |\phi|^2,$$

which corresponds to the conformal metric $|\phi|^{2/k}$. Define

$$\begin{aligned} u_+ &= \frac{1}{k} \log(a + |\phi|^2), \\ u_- &= \begin{cases} u_\phi & \text{if } |z| > d, \\ \max(u_\phi, h_{2d}) & \text{otherwise,} \end{cases} \end{aligned}$$

where a and d are positive constants whose values will be chosen later and

$$h_R(z) = 2 \log \left(\frac{2R}{R^2 - |z|^2} \right)$$

is the logarithm of the Poincaré metric density on the disk $\{|z| < R\}$ of constant curvature -1. In the exceptional case that ϕ is a constant function we modify the definition above and take $u_- = u_\phi$.

We must verify that for suitable choices of a and d these functions satisfy the required conditions. First, differentiating the expression for u_+ above we find that $\Delta u_+ \leqslant F(z, u_+)$ is equivalent to

$$2|\phi_z|^2 \leqslant k(a + |\phi|^2)^{\frac{1}{k}+1}.$$

Using that ϕ is a polynomial and comparing the rates of growth of the two sides, we find this inequality is always satisfied for $|z|$ sufficiently large. Thus, we may choose a large enough the inequality holds for all z.

Turning to the function u_-, if ϕ is nonconstant then we must first check the continuity of u_- on $\{|z| = d\}$ and at the zeros of ϕ. We assume that d is large enough so that all points z where $|\phi(z)| \leqslant 1$ lie in $\{|z| < d\}$, and also that $d > 4/3$ so $h_{2d}(z) < 0$ for $|z| \leqslant d$. This means that $u_- = u_\phi$ a neighborhood of $|z| = d$, and u_- is continuous there. Also, since h_{2d} is continuous on $|z| < d$ and bounded below, the function $\max(u_\phi, h_{2d})$ is continuous at the zeros of ϕ.

The function u_ϕ is subharmonic and $F(z, u_\phi) = 0$ on the complement of the zeros of ϕ, while h_R satisfies $\Delta h_R = 2e^{h_R} \geqslant F(z, h_R)$, thus, in a neighborhood of any point, the function u_- is either a subsolution of (12) or a supremum of two subsolutions. Hence u_- is itself a subsolution.

Finally we must compare u_+ and u_-. It is immediate from the definition that $u_+ \geqslant u_\phi$, and taking $a > 1$ we also have $u_+ \geqslant h_{2d}$ on $\{|z| < d\}$ because then

$$\inf u_+ \geqslant \frac{1}{k} \log a > 0 > \sup_{|z|<d} h_{2d}.$$

It follows that $u_+ \geqslant u_-$, and so the sub/supersolution method yields a C^∞ solution u and thus a corresponding metric $\sigma = e^u |dz|^2$.

By construction $u \geqslant u_- \geqslant u_\phi$, implying $\sigma = e^u \geqslant |\phi|^{2/k}$. Since the metric $|\phi|^{2/k}$ is complete, the metric σ is also complete. The condition $u \geqslant u_\phi$ also gives $e^u \geqslant 2|\phi|^2 e^{-(k-1)u}$ and thus by (12) we have $\Delta u \geqslant 0$, which implies that the metric σ is nonpositively curved.

Since $u \geqslant u_\phi$ with u a solution and u_ϕ a subsolution of the equation (12), the strong comparison principle (e.g. [Jos07, Thm. 2.3.1]) implies that on any domain where u_ϕ is continuous up to the boundary we have either $u > u_\phi$ or $u \equiv u_\phi$. Thus if $u(z_0) = u_\phi(z_0)$ for some z_0 (which therefore satisfies $\phi(z_0) \neq 0$), then u and u_ϕ agree in the complement of the zero set of ϕ. Since u_ϕ is unbounded near these zeros, while u extends smoothly over them, this means ϕ has no zeros at all, i.e. ϕ is constant, and thus $u = u_\phi$ everywhere. $\square$

By its construction from super- and sub-solutions, the proof above also gives the following basic bounds on the solution u:

Corollary 5.2 (Coarse bounds). *Let $e^u|dz|^2$ be the solution of (11) constructed in the proof of Theorem 5.1. Then there exist constants m, M depending continuously on the coefficients of the polynomial ϕ such that*

$$u(z) \geqslant \max(-m, u_\phi(z))$$

and

$$u(z) \leqslant u_\phi(z) + \frac{M}{|\phi(z)|^2}.$$

In particular we have $u(z) - u_\phi(z) \to 0$ *as* $z \to \infty$.

Proof. We assume ϕ is not constant, since otherwise $u = u_\phi$ and all of the bounds are trivial.

The subsolution u_- from Theorem 5.1 satisfies

$$u_-(z) \geqslant \max(\inf h_{2d}, u_\phi(z)),$$

since $\inf h_{2d}$ is achieved at $z = 0$ (and in particular within $|z| < d$). Taking $m = -\inf h_{2d} = 2\log d$ gives the lower bound.

On the other hand we have

$$u_+ = u_\phi + \frac{1}{k}\log\left(1 + \frac{a}{|\phi|^2}\right) \leqslant u_\phi + \frac{a}{k|\phi|^2}$$

where a is a (again positive) constant that depends on the coefficients of ϕ. Taking $M = a/k$ gives the desired upper bound.

Finally, as $z \to \infty$ we have $|\phi(z)| \to \infty$ and thus these bounds give $u(z) - u_\phi(z) \to 0$. □

We will improve these coarse bounds in Theorem 5.7.

5.2 Uniqueness. Complementing the existence theorem above, we have:

Theorem 5.3. *For any polynomial holomorphic differential* ϕ *of degree* k, *there is a unique complete and nonpositively curved solution of* (11).

Proof. Suppose that u and w are log-densities of solutions to (11), with u complete and nonpositively curved. Note that both metrics have curvature bounded below by -1. We will show that $w \leqslant u$, following the method of [Wan92, Sec. 5].

Let $\eta = w - u$. In terms of the Laplace–Beltrami operator $\Delta_u = e^{-u}\Delta$ of u and the pointwise norm $|\phi|_u = |\phi|e^{-ku/2}$, the fact that both w and u are solutions implies

$$\Delta_u \eta = 2e^\eta - 2|\phi|_u^2 e^{-(k-1)\eta} - 2 + 2|\phi|_u^2.$$

By (11) the nonpositive curvature of u implies that $|\phi|_u \leqslant 1$, giving

$$\Delta_u \eta \geqslant 2e^\eta - 2e^{-(k-1)\eta} - 2.$$

By a result [CY75] of Cheng and Yau, this differential inequality implies that η is bounded above: Since u is complete and has a lower curvature bound, applying Theorem 8 of [CY75] with $f(t) = 2e^t - 2e^{-(k-1)t} - 2$ and $g(t) = e^t$ gives

$$\sup \eta = \bar{\eta} < \infty.$$

Applying the generalized maximum principle [Omo67, Yau75] to η we find that there is a sequence $z_k \in \mathbb{C}$ such that

$$\lim_{k\to\infty} \eta(z_k) = \bar{\eta}$$
$$\limsup_{k\to\infty} \Delta_u \eta(z_k) \leqslant 0.$$

Using the bound $|\phi|_u \leqslant 1$ and passing to a subsequence we can assume $|\phi|_u(z_k)^2$ converges, say to $\lambda \in [0,1]$. Then substituting the expression for $\Delta_u \eta$ into the inequality above we find

$$(e^{\bar{\eta}} - 1) - \lambda(e^{-(k-1)\bar{\eta}} - 1) \leqslant 0.$$

This gives $\bar{\eta} \leqslant 0$, or equivalently, $w \leqslant u$.

If u and w are *both* complete and nonpositively curved we can apply this argument with their roles reversed to conclude $u = w$, hence there is at most one solution with these properties. By Theorem 5.1 there is at least one such solution. □

5.3 Continuity. For later applications it will be important to know that the metric satisfying (11) depends continuously on the holomorphic differential ϕ.

Let $\mathcal{D}_d^k \simeq \mathbb{C}^d$ denote the space of holomorphic differentials of the form $\phi(z)dz^k$, with ϕ a monic polynomial of degree d.

Let $\mathcal{CM}$ denote the set of smooth, strictly positive conformal metrics on $\mathbb{C}$, which we identify with $C^\infty(\mathbb{C})$ using log-density functions (i.e. the metric $e^u|dz|^2$ is represented by the function u).

Theorem 5.4 (Global C^0 and local C^1 continuity). *For any $k, d \in \mathbb{Z}^{\geqslant 0}$, considering the unique complete and nonpositively curved solution of* (11) *as a function of ϕ defines an embedding*

$$\mathcal{D}_d^k \hookrightarrow \mathcal{CM}$$

which is continuous in the uniform topology, i.e. as a map into $C^0(\mathbb{C})$. Furthermore, restricting to any compact set $K \subset \mathbb{C}$ defines a continuous map into $C^1(K)$.

Proof. A monic polynomial is determined by its absolute value, so it is immediate from (11) that the map is injective.

Consider a pair $\phi, \psi \in \mathcal{D}_d^k$. As monic polynomials of the same fixed degree, their difference is small in comparison to either one. Making this precise, for any $\epsilon > 0$ we can for example ensure that

$$\big||\psi(z)|^2 - |\phi(z)|^2\big| \leqslant \epsilon(1 + |\phi(z)|^2) \quad \text{for all } z \in \mathbb{C},$$

just by requiring the coefficients of ϕ and ψ to be sufficiently close.

Let u, v be the solutions to (12) corresponding to ϕ and ψ, respectively, and define $\eta = v - u$. To establish continuity of the map $\mathcal{D}_d^k \to \mathcal{CM}$ it suffices to bound the relevant norm of η in terms of ϵ.

Since $|\phi|_u \leqslant 1$ and u is bounded below (by Corollary 5.2), multiplying previous inequality above by e^{-ku} we find

$$||\phi|_u^2 - |\psi|_u^2| \leqslant C\epsilon, \tag{13}$$

for some constant C and for all ψ in some neighborhood of ϕ. In particular $|\psi|_u$ is bounded.

Calculating as in the proof of Theorem 5.3 we find that η satisfies

$$\begin{aligned} \Delta_u \eta &= 2e^\eta - 2|\psi|_u^2 e^{-(k-1)\eta} - 2 + 2|\phi|_u^2 \\ &= 2(e^\eta - 1) - 2|\psi|_u^2 (e^{-(k-1)\eta} - 1) + 2(|\phi|_u^2 - |\psi|_u^2). \end{aligned} \tag{14}$$

By [CY75, Thm. 8] the associated differential inequality

$$\Delta_u \eta \geqslant 2e^\eta - Me^{-(k-1)\eta} - 2,$$

where $M = 2 \sup |\psi|_u^2 < \infty$, implies as in Theorem 5.3 that η is bounded above. Applying the generalized maximum principle and passing to a suitable subsequence gives $\{z_k\}$ with

$$\lim_{k\to\infty} \eta(z_k) = \bar{\eta}, \quad \lim_{k\to\infty} |\phi|_u^2(z_k) = \lambda, \quad \lim_{k\to\infty} |\psi|_u^2(z_k) = \mu$$

and

$$(e^{\bar{\eta}} - 1) - \mu(e^{-(k-1)\bar{\eta}} - 1) + (\lambda - \mu) \leqslant 0.$$

Since (13) gives $|\lambda - \mu| \leqslant C\epsilon$, the inequality above implies

$$\bar{\eta} \leqslant \log(1 + C\epsilon) \leqslant C\epsilon.$$

Repeating this argument with the roles of ϕ and ψ reversed we find that

$$\sup_{\mathbb{C}} |\eta| \leqslant C'\epsilon,$$

and global C^0 continuity follows.

Substituting this bound on $|\eta|$ and the bound on $||\phi|_u^2 - |\psi|_u^2|$ from (13) into the right hand side of (14), we also find a uniform bound on the u-Laplacian,

$$\sup_{\mathbb{C}} |\Delta_u \eta| \leqslant C''\epsilon.$$

Local C^1 continuity now follows by standard estimates for the Laplace equation: Given a compact set $K \subset \mathbb{C}$, fix an open disk $\{|z| < R\}$ containing K. Corollary 5.2 provides an upper bound on $\sup_{|z|<R} u$ depending on R, so we get a bound on the flat Laplacian $\Delta = e^u \Delta_u$ of the form

$$\sup_{|z|<R} |\Delta\eta| = C(R)\,\epsilon.$$

Since $|\eta|$ and $|\Delta u|$ are each bounded by a fixed multiple of ϵ on $\{|z| < R\}$, standard interior gradient estimates (e.g. [GT83, Thm. 3.9]) give a proportional C^1 bound on η in the compact subset K. This establishes the local C^1 continuity of the map $\mathcal{D}_d^k \to \mathcal{CM}$. $\square$

Restricting attention to monic polynomials in the theorem above is a convenience that ensures injectivity of the map to $\mathcal{CM}$. However, continuity holds in general. Since (11) is invariant by holomorphic automorphisms, we can pull back by an automorphism of the form $z \to \lambda z$ to make an arbitrary polynomial differential monic, and we conclude:

COROLLARY 5.5. *The global C^0 and local C^1 continuity statements of Theorem 5.4 also apply to the space of polynomial differentials of fixed degree with arbitrary nonzero leading coefficient.*

5.4 Estimates. Our final goal in this section is to compare the solution $e^u|dz|^2$ of Equation (11) to the conformal metric $|\phi|^{2/k}$ by studying the difference of their logarithmic densities, the "error function"

$$u - u_\phi = u - \frac{1}{k}\log|\phi|^2.$$

We have already derived coarse bounds for this difference in Corollary 5.2. We begin by reinterpreting these in terms of $|\phi|^{2/k}$ metric geometry:

COROLLARY 5.6 (Coarse bound, intrinsic version). *Let ϕ and u be as above, and suppose $k > 1$. There exist constants A', R' and an exponent $\alpha > 1$ with the following property: If the $|\phi|^{2/k}$-distance from p to the zero set of ϕ is $r > R'$, then*

$$0 \leqslant u(p) - u_\phi(p) \leqslant A' r^{-\alpha}.$$

We note that for the purposes of the next theorem, a key feature of this coarse bound is that it yields an integrable function of r.

Proof. As before the constant case is trivial and we assume degree $d > 0$. Outside of a large open disk D containing the zeros of ϕ, the polynomial $\phi(z)$ is comparable to z^d by uniform multiplicative constants. Thus the $|\phi|^{2/k}$-distance $r = r(p)$ from a point p outside D to the zero set of ϕ is bounded above by a fixed multiple of the $|z|^{2d/k}|dz|^2$-distance from p to the origin. The latter distance can be explicitly calculated, giving

$$r < C|p|^{(d+k)/k}.$$

Since $|\phi|$ is also bounded below by a multiple of $|z|^d$, this implies

$$|\phi(p)| > C'|p|^d > C'' r^{dk/(d+k)}$$

and applying Corollary 5.2 we have

$$u(p) - u_\phi(p) < \frac{M}{|\phi(p)|^2} < \frac{A'}{r^{2dk/(d+k)}}$$

for p outside D, with A' determined by M and C''. Let $\alpha = 2dk/(d + k)$, and note that $\alpha > 1$ for d, k integers and $k > 1$. Fix R' large enough so that $r > R'$ implies that p is outside D. Then the statement follows for these A', R', α. $\square$

Our next goal is to show that the error $|u - u_\phi|$ is not just bounded at infinity, but exponentially small as a function of r.

Theorem 5.7 (Exponentially small error). *Let ϕ and u be as above. Then there exist constants A and R with the following property: If the $|\phi|^{2/k}$-distance from p to the zero set of ϕ is $r > R$, then*

$$u(p) - u_\phi(p) \leqslant A\frac{\exp(-\sqrt{2k}\, r)}{\sqrt{r}}.$$

In preparation for the proof, we introduce a convenient change of coordinates: let w be a local coordinate in which $|\phi| = |dw|^k$. We continue to use u to denote the log-density of σ, but now considered relative to $|dw|^2$, i.e.

$$\sigma = e^u|dw|^2,$$

so that u satisfies the equation

$$\Delta u = 2e^u - 2e^{-(k-1)u}, \tag{15}$$

with $\Delta = 4\frac{\partial^2}{\partial w \partial \bar{w}}$ now denoting (here and in the rest of this section) the flat Laplacian with respect to w.

In this coordinate system $u_\phi \equiv 0$, hence our goal is to show that u itself is exponentially small. More precisely, defining

$$\varepsilon(t) = \frac{\exp(-\sqrt{2k}\, t)}{\sqrt{t}}, \tag{16}$$

we must show that $u = O(\varepsilon(r))$.

Noting that the linearization of Equation (15) at $u = 0$ is $\Delta u = 2ku$, we first consider the Dirichlet problem for this linear equation in the upper half-plane $\mathbb{H} \subset \mathbb{C}$. Write $w = x + iy$ with $x, y \in \mathbb{R}$.

Lemma 5.8. *Suppose $g \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \geqslant 0$. Then there exists $h \in C^\infty(\mathbb{H})$ extending continuously to $\mathbb{R}$ that is a solution of the Dirichlet problem*

$$\Delta h = 2kh, \quad h|_{\mathbb{R}} = g,$$

and which satisfies

$$\begin{aligned} &0 \leqslant h \leqslant \sup g, \\ &h = O(\|g\|_1 \varepsilon(y)) \quad \text{as } y \to \infty, \end{aligned}$$

where the implicit constants in the second estimate are independent of g.

Proof. Throughout this proof we write $w = x + iy$, where $x, y \in \mathbb{R}$. A solution of the Dirichlet problem can be constructed by convolution

$$h(w) = \int_{\mathbb{R}} \frac{\partial G(w,\xi)}{\partial \operatorname{Im}(\xi)} g(\xi)\, d\xi, \tag{17}$$

where G is the Green's function G for the positive operator $-(\Delta - 2k)$, i.e. $(\Delta_\xi - 2k)G(w,\xi) = -\delta(w)$ and $G(w,\xi) = 0$ for $\xi \in \mathbb{R}$. Then G and its normal derivative along $\mathbb{R}$ are given [PZ02, Formula 7.3.2-3] in terms of the modified Bessel functions:

$$\begin{aligned} G(w,\xi) &= \frac{1}{2\pi}(K_0(\sqrt{2k}\,|w-\xi|) - K_0(\sqrt{2k}\,|w-\bar{\xi}|)) \\ \frac{\partial G(w,\xi)}{\partial \operatorname{Im}(\xi)} &= \frac{\sqrt{2k}\,y}{\pi|w-\xi|} K_1(\sqrt{2k}|w-\xi|) \quad \text{for } \xi \in \mathbb{R}. \end{aligned}$$

Since $|w - \xi| \geqslant y$ for $\xi \in \mathbb{R}$, and the Bessel function satisfies $K_1(t) = O(\varepsilon(t))$ as $t \to \infty$ [AS72, Formula 9.7.2], we therefore have

$$\sup_{\xi \in \mathbb{R}} \frac{\partial G(w,\xi)}{\partial \operatorname{Im}(\xi)} = O\left(\varepsilon(y)\right).$$

Substituting this into (17) gives $h = O(\|g\|_1 \varepsilon(y))$ as required.

The other bounds on h are immediate from (17): since $K_1 > 0$ and $g \geqslant 0$, we have $h \geqslant 0$. Since G is a fundamental solution we have, for any $w \in \mathbb{H}$,

$$\int_{\mathbb{R}} \frac{\partial G(w,\xi)}{\partial \operatorname{Im}(\xi)}\, d\xi = 1 - 2k \int_{\mathbb{H}} G(w,\xi)|d\xi|^2 < 1$$

and thus, again noting that $K_1 > 0$ and $g \geqslant 0$, we have that

$$h = \int_{\mathbb{R}} \frac{\partial G(w,\xi)}{\partial \operatorname{Im}(\xi)} g(\xi)\, d\xi \leqslant \sup g \int_{\mathbb{R}} \frac{\partial G(w,\xi)}{\partial \operatorname{Im}(\xi)}\, d\xi \leqslant \sup g. \qquad \square$$

Next we use the solution of the linearization constructed above to get a supersolution for the quadratic approximation of (15) on $\mathbb{H}$. Note that the right hand side of this equation is

$$f(u) := 2e^u - 2e^{-(k-1)u} = 2k\,u - k(k-2)\,u^2 + O(u^3).$$

Lemma 5.9. *Suppose $g \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ and that $0 \leqslant g \leqslant \frac{1}{(k-2)}$. Then there exists a function $v \in C^\infty(\mathbb{H})$ extending continuously to $\mathbb{R}$ such that*

$$\begin{aligned} &v|_{\mathbb{R}} \geqslant g, \\ &\Delta v \leqslant 2k\,v - k(k-2)\,v^2, \\ &\quad v = O(\|g\|_1 \varepsilon(y)) \quad \text{as } y \to \infty. \end{aligned}$$

Proof. First, consider an arbitrary function h satisfying $\Delta h = 2k\,h$. Then $v = h - \frac{(k-2)}{2}h^2$ satisfies

$$\Delta v - 2k\,v + k(k-2)\,v^2 = -k(k-2)^2\,h^3 + \frac{1}{4}k(k-2)^3\,h^4 - (k-2)\,|\nabla h|^2.$$

Since $|\nabla h| \geqslant 0$, we find that the right hand side is negative if $\sup h < \frac{4}{k-2}$.

Now let h be the solution of $\Delta h = 2k\,h$ given by Lemma 5.8 with boundary values $h|_{\mathbb{R}} = 2g$. Since $h \leqslant 2\sup g < \frac{2}{k-2}$, we find that $v = h - \frac{(k-2)}{2}h^2$ satisfies $0 < v < h$ and the calculation above shows

$$\Delta v \leqslant 2k\,v - k(k-2)\,v^2.$$

Since $0 < v < h$, it is immediate from Lemma 5.8 that $v = O(\|g\|_1 \varepsilon(t))$. Finally, we must verify that $v \geqslant g$ on $\mathbb{R}$. This follows because $h|_{\mathbb{R}} = 2g$ and $v = q(h)$ where the polynomial $q(t) = t - \frac{(k-2)}{2}t^2$ satisfies $q(t) > \frac{1}{2}t$ on the interval $[0, \frac{1}{k-2}]$ which contains the range of g. □

Proof of Theorem 5.7. First we find a suitable region and coordinate system in which to apply the previous lemmas. By Corollary 5.6 there is a compact set K in the plane outside of which $u - u_\phi < \frac{1}{2(k-2)}$. By Proposition A.1, any point p sufficiently far from K lies in a $|\phi|$-upper-half-plane (U, w) with $U \cap K = \emptyset$ and with $y(p) = \mathrm{Im}(w(p)) \geqslant r(p) - C$ for a constant C independent of p. For the rest of the proof we work in this coordinate w, identifying U with $\mathbb{H}$ and writing $\sigma = e^u|dw|^2$ and $|\phi|^{2/k} = e^{u_\phi}|dw|^2$. We therefore have $u_\phi(w) \equiv 0$ and $0 \leqslant u(w) < \frac{1}{2(k-2)}$.

Since Proposition A.1 gives $r(w) \geqslant c|w|$ for $w \in \mathbb{R}$ with $|w|$ sufficiently large, it follows from Corollary 5.6 that u is integrable on $\mathbb{R}$. Moreover u is everywhere less than $\frac{1}{2(k-2)}$, so we in fact have a bound on the L^1 norm of $u|_{\mathbb{R}}$ that depends on ϕ but which is independent of p.

Let v be the function on $\mathbb{H}$ given by Lemma 5.9 for $g = u|_{\mathbb{R}}$. Since $v = O(\epsilon(y))$ and $y(p) \geqslant r(p) - C$, the theorem will follow if we show $u \leqslant v$, or equivalently that the function $\eta = u - v$ is nowhere positive. Note that η is smooth on $\mathbb{H}$ and continuous on the closure $\bar{\mathbb{H}}$.

Suppose for contradiction that η is positive at some point, so the closed set $Q = \eta^{-1}([\epsilon, \infty)) \subset \bar{\mathbb{H}}$ is nonempty for some $\epsilon > 0$. Lemma 5.9 gives $\eta < 0$ on $\partial\mathbb{H}$, hence $Q \subset \mathbb{H}$. The same lemma and Corollary 5.2 respectively show $v \to 0$ and $u \to 0$ as $z \to \infty$, hence $\eta \to 0$ as $z \to \infty$, and Q is compact. Therefore η has a positive maximum at some point in Q.

Recall that we set $f(u) = 2e^u - 2e^{-(k-1)u}$, that u satisfies $\Delta u = f(u)$, and that v satisfies $v \geqslant 0$ and $\Delta v = 2kv - k(k-2)\,v^2 \leqslant f(v)$. Therefore

$$\begin{aligned}\Delta\eta &\geqslant 2e^u - 2e^{-(k-1)u} - 2e^v + 2e^{-(k-1)v}\\ &= 2e^v(e^\eta - 1) - 2e^{-(k-1)v}(e^{(-k-1)\eta} - 1).\end{aligned}$$

At a maximum we have $0 \geqslant \Delta\eta$, which in combination with the inequality above gives

$$(e^{\eta} - 1) \leqslant e^{-(k-2)v}(e^{-(k-1)\eta} - 1).$$

This shows $\eta \leqslant 0$ at any maximum, which is the desired contradiction. □

The C^0 bound of the previous theorem is easily improved to a C^1 bound:

Corollary 5.10. *Let ϕ and u be as above, and let $|\nabla f|_\phi$ denote the norm of the gradient of a function f with respect to the $|\phi|^{2/k}$-metric. Let r denote the $|\phi|^{2/k}$-distance from a point p to the zero set of ϕ. If $r > (R+1)$, where R is the constant from Theorem 5.7, then*

$$|\nabla(u - u_\phi)|_\phi(p) \leqslant C \exp(-\sqrt{2k}\, r)/\sqrt{r}).$$

Proof. Working as above in ϕ-natural coordinates, where $u_\phi = 0$ and the ϕ-gradient becomes the Euclidean one, we simply require a pointwise C^1 bound on the function u. Since u satisfies (15) and

$$e^{u} - e^{-(k-1)u} \leqslant C|u| \quad \text{for } |u| < 1,$$

the bound on u from the previous theorem shows that $|\Delta u|$ is also proportionally small throughout a disk of radius 1 centered at p. Applying the standard interior gradient estimate for Poisson's equation (see e.g. [GT83, Thm. 3.9]) to this disk then gives the desired bound for the derivative of u at its center. □

6 From Polynomials to Polygons

The results of the previous section give the following existence theorem for affine spheres with prescribed polynomial Pick differential:

Theorem 6.1. *For any polynomial cubic differential C on the complex plane, there exists a complete hyperbolic affine sphere in $\mathbb{R}^3$ that is conformally equivalent to $\mathbb{C}$ and which has Pick differential C with respect to some conformal parameterization. This affine sphere is uniquely determined by these properties, up to translation and the action of $\mathrm{SL}_3\,\mathbb{R}$.*

Proof. Let σ be the complete, nonpositively curved conformal metric satisfying (11) for $\phi = \sqrt{2}C$, given by Theorem 5.1. By Proposition 4.6 the pair σ, C can then be integrated to an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ which by Theorem 4.1 is properly embedded. Let M denote its image.

Suppose M' is another complete affine sphere conformally parameterized by $\mathbb{C}$ which has Pick differential C. Then the Blaschke metric σ' of M' is another solution of (11) for the same differential $\phi = \sqrt{2}C$ which is complete, and by Theorem 4.2, nonpositively curved. By Theorem 5.3 we have $\sigma' = \sigma$. Therefore, after translating M' so that it is centered at the origin, its complexified frame satisfies the same structure equations (4) as that of M. Hence the frames differ by a fixed element $A \in \mathrm{SL}_3\,\mathbb{R}$, and M' is the image of M by the composition of A and a translation. □

Though we will not use the following result in the sequel, we note in passing that uniqueness of the "polynomial affine sphere" considered above can be shown even under weaker hypotheses; we can drop the assumption of embeddedness and replace it with a curvature condition:

Theorem 6.2. *Suppose $f : \mathbb{C} \to \mathbb{R}^3$ is an affine spherical immersion with polynomial Pick differential and nonpositively curved Blaschke metric. Then the image of f is the complete affine sphere associated to C by Theorem 6.1.*

Proof. By Theorem 4.1, it suffices to show that the Blaschke metric of f is complete, for then the hypotheses of the uniqueness statement in Theorem 6.1 are again satisfied.

From the intrinsic formulation of Wang's equation (7) we find that nonpositive curvature of the Blaschke metric h implies $|C|_h \leqslant \frac{1}{2}$. Equivalently h is bounded below by a constant multiple of the conformal metric $|C(z)|^{2/3}|dz|^2$. Since C is a polynomial, it follows that both of these metrics are bounded below by a multiple of the Euclidean metric $|dz|^2$ on the complement of a compact set, and hence both are complete. $\square$

It would be interesting to know whether the curvature condition can also be dropped, i.e.

QUESTION. Does there exist an affine spherical immersion $f : \mathbb{C} \to \mathbb{R}^3$ with polynomial Pick differential whose Blaschke metric has positive curvature at some point?

Of course it follows from the developments above that such an immersion cannot be proper, and its Blaschke metric must be incomplete.

Having settled the basic existence and uniqueness results for polynomial affine spheres, the goal of the rest of this section is to show that the convex domains associated with these affine spheres are polygons.

Recall that $\star_d \subset \mathbb{C}$ is the union of $(d+3)$ evenly spaced rays from the origin that includes $\mathbb{R}^+$. We call a space homeomorphic to $\star_d$ an *open star*, and in such a space the homeomorphic images of the rays are the *edges*.

Theorem 6.3. *Let $f : \mathbb{C} \to M \subset \mathbb{R}^3$ be a conformal parameterization of a complete hyperbolic affine sphere whose Pick differential C is a polynomial of degree d. Then M is asymptotic to the cone over a convex polygon P with $d+3$ vertices.*

Moreover, if C is monic then the projectivization of $f(\star_d)$ gives an embedded open star in P whose edges tend to the vertices.

Before starting the proof we will introduce a key tool, the *osculation map*, and outline how the theorem will follow from an analysis of the asymptotic behavior of this map using estimates from the previous section.

Let z be a natural coordinate for the Pick differential defined in a region U. (In this section we will *not* be working with the global coordinate in which C is a polynomial, which in previous sections had also been denoted z.) Restricting the conformal parameterization f to U, we can consider it as a function of z, denoted $f(z)$. Let $F(z) = (f, f_z, f_{\bar{z}})$ be the associated complexified frame field. As before let $F_T(z)$ denote the frame field of the normalized Țițeica surface T. Define the *osculation map* $\widehat{F} : U \to \mathrm{GL}_3\,\mathbb{R}$ by

$$\widehat{F}(z) = F(z)F_T^{-1}(z).$$

Note that the value of this function lies in $\mathrm{GL}_3\,\mathbb{R}$ because both frame values $F(z)$ and $F_T(z)$ lie in the same right coset of $\mathrm{GL}_3\,\mathbb{R}$ within $\mathrm{GL}_3\,\mathbb{C}$ (as described in Section 4).

Evidently $\widehat{F}$ is constant if and only if f is itself a Țițeica surface, and more generally, left multiplication by $\widehat{F}(z_0)$ transforms the normalized Țițeica surface to one which has the same tangent plane and affine normal as f at the point $f(z_0)$. In this sense $\widehat{F}(z_0)$ represents the "osculating" Țițeica surface of f at z_0.

Recall the map $H(z) = \exp(\mathfrak{h}(z))$ we used in Section 4 (Equation (9)) to parameterize the Țițeica surface. Then a calculation with the affine structure equations (4) shows that the derivative $\widehat{F}^{-1}d\widehat{F} \in \Omega^1(U, \mathfrak{gl}_3\mathbb{R})$ of the osculation is given by

$$\begin{aligned} \widehat{F}^{-1}d\widehat{F} &= \mathrm{Ad}_{F_T}(F^{-1}dF - F_T^{-1}dF_T) \\ &= \mathrm{Ad}_{H(z)}\,\Theta(u(z)) \end{aligned} \tag{18}$$

where

$$\begin{aligned} \Theta(u(z)) = \mathrm{Ad}_{F_T(0)} \Bigg[&\begin{pmatrix} 0 & 0 & \frac{1}{2}e^u - 1 \\ 0 & u_z & 0 \\ 0 & 2e^{-u} - 1 & 0 \end{pmatrix} dz \\ &+ \begin{pmatrix} 0 & \frac{1}{2}e^u - 1 & 0 \\ 0 & 0 & 2e^{-u} - 1 \\ 0 & 0 & u_{\bar{z}} \end{pmatrix} d\bar{z} \Bigg] \end{aligned} \tag{19}$$

and $e^u|dz|^2$ is the Blaschke metric of M.

Notice that the estimates from the previous section show that $\Theta(u(z))$ is rapidly decaying toward zero as the distance from z to the zeros of C increases (since $F_T(0)$ is a constant matrix and in these coordinates the functions $\frac{1}{2}e^u - 1$, u_z, and $u_{\bar{z}}$ are all exponentially small). Ignoring the conjugation by the diagonal matrix $H(z)$ in (18) for a moment, this suggests that $\widehat{F}(z)$ should approach a constant as z goes to infinity—since its derivative is approaching zero—which would mean that the affine sphere f is asymptotic to a Țițeica surface.

However, the function $H(z)$ is itself exponentially growing as a function of z, with the precise rate of growth depending on the direction. Thus the actual asymptotic behavior of the osculation map depends on the competition between frame growth (i.e. H) and Blaschke error decay (i.e. Θ) as described in the introduction. In most directions, the exponential decay of Θ is faster than the growth of H, giving a well-defined limiting Țițeica surface and thus a portion of the projectivized image of M that is modeled on a triangle. In exactly $2(d + 3)$ *unstable directions* there is an exact balance, which allows the limit Țițeica surface to shift.

Thus, using the osculation map, the proof Theorem 6.3 splits into four steps:

1. *Finding stable limits*: By considering the osculation map restricted to a ray in a standard half-plane, use the exponential bounds from Theorem 5.7 and Corollary 5.10 to show the existence of a limit in any stable direction (and locally constant as a function of the ray).
2. *Finding unipotent factors*: By considering the osculation map restricted to an arc joining two rays on either side of an unstable direction, use the same bounds to show that the ray limits on either side differ by composition with a unipotent element of $\mathrm{SL}_3 \mathbb{R}$.
3. *Finding triangle pieces*: Show that there is a "vee" (two edges of a triangle) in the boundary of the projectivization of M corresponding to each interval of stable directions.
4. *Assembling the polygon*: Use the geometry of the unipotent factors to show that these triangle pieces glue up to form a polygon with $(d+3)$ vertices.

Proof of Theorem 6.3. Any polynomial affine sphere can be parameterized so that its Pick form is monic by composing an arbitrary parameterization with a suitable automorphism $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$. Therefore we can (and do) assume C is monic throughout the proof, since the only part of the theorem that involves a specific parameterization (i.e. the image $f(\star_d)$) includes monicity as a hypothesis.

Let U be one of the standard half-planes for C given by Proposition 3.2. Our first goal is to understand the part of $\partial\mathbb{P}(M)$ that arises from *rays* in U, i.e. paths of the form $\gamma(t) = b + e^{i\theta}t$ where $t \geqslant 0$ and b is arbitrary. Here we call $\theta \in [-\pi/2, \pi/2]$ the *direction* of the ray.

Step 1: Finding stable limits. We will say that such a ray is *stable* if $\theta \notin \{-\pi/2, -\pi/6, \pi/6, \pi/2\}$. Note the possible directions of stable rays form three intervals of length $\pi/3$, which we denote by

$$J_- = (-\pi/2, -\pi/6), \quad J_0 = (-\pi/6, \pi/6), \quad J_+ = (\pi/6, \pi/2).$$

Recall from Section 3 that a *quasi-ray* is a path that can be parameterized so that it eventually lies in a half-plane, in which it has distance $o(t)$ from a ray parameterized by arc length t. We say a quasi-ray is stable if the direction of some associated ray is stable.

The "stability" of rays and quasi-rays in these directions refers to convergence of the osculation map:

Lemma 6.4. *If γ is a stable ray or quasi-ray, then $\lim_{t\to\infty} \widehat{F}(\gamma(t))$ exists. Furthermore, among all such rays only three limits are seen: there exist $L_-, L_0, L_+ \in \mathrm{GL}_3 \mathbb{R}$ such that*

$$\lim_{t\to\infty} \widehat{F}(\gamma(t)) = \begin{cases} L_- & \text{if } \theta \in J_- \\ L_0 & \text{if } \theta \in J_0 \\ L_+ & \text{if } \theta \in J_+. \end{cases}$$

Proof. First we consider rays, and at the end of the proof we show that quasi-rays have the same behavior.

Let γ be a ray and for brevity write $G(t) = \widehat{F}(\gamma(t))$. By (18) we have

$$G(t)^{-1}G'(t) = \mathrm{Ad}_{H(\gamma(t))}\,\Theta(u)(\gamma'(t)).$$

Applying Theorem 5.7 and Corollary 5.10 to u and $\phi = \sqrt{2}C$, and using that $|\gamma'(t)| = 1$, we have

$$\Theta(u)(\gamma'(t)) = O(e^{-2\sqrt{3}t}/\sqrt{t}).$$

Note that the exponential decay rate of $2\sqrt{3}$ (rather than $\sqrt{6}$ seen in the theorems cited) reflects the fact that we are working in coordinates where $C = 2dz^3$ and $\phi = 2^{3/2}dz^3 = (\sqrt{2}dz)^3$, so $|\phi|^{2/3}$-distances are related to $|dz|^2$ distances by a factor of $\sqrt{2}$.

Conjugating Θ by the diagonal matrix $H(z)$ multiplies the (i,j) entry by

$$\lambda_{ij} := \exp(2\,\mathrm{Re}(z(\omega^{(1-i)} - \omega^{(1-j)}))). \tag{20}$$

In this case $z = \gamma(t) = b + e^{i\theta}t$, and taking the maximum over i and j we find

$$\lambda_{ij} = O(e^{c(\theta)t}) \tag{21}$$

where the optimal coefficient $c(\theta)$ has a simple geometric description: Inscribe an equilateral triangle in $|z| = 2$ with one vertex at $e^{i\theta}$. Project the triangle orthogonally to $\mathbb{R}$ and let $c(\theta)$ be the length of the resulting interval.

In particular, the coefficient $c(\theta)$ achieves its maximum $2\sqrt{3}$ exactly when one of the sides of the triangle is horizontal, or equivalently, if and only if the ray is *not* stable.

Combining these bounds for Θ and λ_{ij}, we find that for any stable ray, we have definite exponential decay in the equation satisfied by G, i.e.

$$G(t)^{-1}G'(t) = O(e^{-\alpha t}/\sqrt{t})$$

where $\alpha = 2\sqrt{3} - c(\theta) > 0$. Standard ODE techniques (see Lemma B.1(ii) in Appendix B) then show that $\lim_{t\to\infty} G(t)$ exists.

Now suppose that γ_1 and γ_2 are stable rays with respective angles θ_1, θ_2 that belong to the same interval (J_-, J_0, or J_+). We will show that $G_1(t)^{-1}G_2(t) \to I$ as $t \to \infty$, where $G_i(t) = \widehat{F}(\gamma_i(t))$. This means that $\widehat{F}$ has the same limit along these rays, giving L_-, L_0, and L_+ as in the statement of the lemma.

For any $t \geqslant 0$ let $\eta_t(s) = (1-s)\gamma_1(t) + s\gamma_2(t)$ be the constant-speed parameterization of the segment from $\gamma_1(t)$ to $\gamma_2(t)$. Let $g_t(s) = \widehat{F}(\eta_t(0))^{-1}\widehat{F}(\eta_t(s))$, which satisfies

$$\begin{aligned} g_t^{-1}(s)g_t'(s) &= \mathrm{Ad}_{H(\eta_t(s))}\,\Theta(u)(\eta_t'(s)) \\ g_t(0) &= I \\ g_t(1) &= G_1(t)^{-1}G_2(t). \end{aligned} \tag{22}$$

Since $|\eta_t'(s)| = O(t)$, the analysis above shows that $g_t^{-1}(s)g_t'(s) = O(\sqrt{t}e^{-\alpha t})$ where now $\alpha = \big(2\sqrt{3} - \sup_{\theta_1 \leqslant \theta \leqslant \theta_2} c(\theta)\big) > 0$ because $\pm\pi/2, \pm\pi/6 \notin [\theta_1, \theta_2]$. In particular by making t large enough we can arrange for $g_t^{-1}(s)g_t'(s)$ to be uniformly small for all $s \in [0,1]$. Once again standard ODE methods (Lemma B.1(i)) give the desired convergence,

$$G_1(t)^{-1}G_2(t) = g_t(1) \to I \quad \text{as } t \to \infty.$$

Finally, suppose that γ_1 is a stable quasi-ray, and γ_2 the ray that it approximates (with direction θ). We proceed as above to study $\eta_t(s) = (1-s)\gamma_1(t) + s\gamma_2(t)$ and the restriction of the frame field to this homotopy from γ_1 to γ_2. In this case we have the stronger bound on the derivative $|\eta_t'(s)| = o(t)$, and the previous bound on $g_t^{-1}(s)g_t'(s)$ applies again with exponent $\alpha = (2\sqrt{3} - c(\theta))$. Thus as before we find $g_t(1) \to I$ as $t \to \infty$, and that the frame field has the same limit on the stable quasi-ray γ_1 as on an associated stable ray γ_2.

Step 2: Finding unipotent factors. Next we will analyze the behavior of the osculation map near an unstable ray in order to understand the relationship between L_-, L_0, and L_+.

LEMMA 6.5. *Let L_-, L_0, L_+ be as in the previous lemma. Then there exist $a, b \in \mathbb{R}$ such that*

$$L_-^{-1}L_0 = \begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \textit{and} \quad L_0^{-1}L_+ = \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix}, \tag{23}$$

where the matrix entries not shown are zero.

Proof. We give a detailed proof for $L_0^{-1}L_+$ and then indicate what must be changed to handle the other case. We begin as in the last part of the previous proof, i.e. joining two rays by a path and studying the restriction of $\widehat{F}$ to the path.

Consider the rays $\gamma_0(t) = t$ and $\gamma_+(t) = e^{i\pi/3}t$. The restrictions $G_0 = \widehat{F} \circ \gamma_0$ and $G_+ = \widehat{F} \circ \gamma_+$ have respective limits L_0 and L_+. For any $t > 0$, join $\gamma_0(t)$ to $\gamma_+(t)$ by a circular arc

$$\eta_t(s) = e^{is}t, \quad \text{where } s \in [0, \pi/3]$$

and let $g_t(s) = \widehat{F}(\eta_t(0))^{-1}\widehat{F}(\eta_t(s))$. Then $g_t : [0, \pi/3] \to \mathrm{GL}_3\,\mathbb{R}$ satisfies the ordinary differential equation (22) with G_1, G_2 replaced by G_0, G_+.

Unlike the previous case, however, the coefficient

$$M_t(s) := \mathrm{Ad}_{H(\eta_t(s))}\,\Theta(u)(\eta_t'(s))$$

that appears in this equation is not exponentially small in t throughout the interval. At $s = \pi/6$, conjugation by $H(\eta_t)$ multiplies the $(1,3)$ entry of $\Theta(u)$ by a factor of $\exp(2\sqrt{3}t)$, exactly matching the exponential decay rate for Θ and giving

$$M_t(\pi/6) = O(|\eta_t'|/\sqrt{t}) = O(\sqrt{t}),$$

where in the second equality we used $|\eta_t'| = t$.

However, this potential growth in the coefficient matrix M_t is seen *only* in this $(1,3)$ entry, because by (20) the other entries are scaled by smaller exponential factors. (That is, the elementary matrix $E_{1,3}$ is the leading eigenvector of $\mathrm{Ad}_{H(\eta_t)}$.) Furthermore the effect rapidly decays as the angle moves away from $\pi/6$: For $\theta \in [0, \pi/3]$ and $c(\theta)$ as in (21) we have

$$c(\theta) = 2\sqrt{3}\cos\left(\frac{\pi}{6} - \theta\right) \leqslant 2\sqrt{3} - (\theta - \pi/6)^2.$$

Combining these two observations we can separate the unbounded entry in $M_t(s)$ and write

$$M_t(s) = M_t^0(s) + \mu_t(s)E_{13}$$

where $M_t^0(s) = O(\exp(-\alpha t))$ for some $\alpha > 0$, E_{13} is the elementary matrix, and

$$\mu_t(s) = O(|\eta_t'|\exp((2\sqrt{3} - c(s))t)/\sqrt{t}) = O(\sqrt{t}\exp(-(s - \pi/6)^2 t)).$$

This upper bound is a Gaussian function in s, normalized such that its integral over $\mathbb{R}$ is independent of t. (As $t \to \infty$ this Gaussian approximates a delta function at $s = \frac{\pi}{6}$.) Therefore the function $\mu_t(s)$ is uniformly absolutely integrable over $s \in [0, \pi/3]$ as $t \to \infty$.

With a coefficient of this form—an integrable component with values in a fixed 1-dimensional space, plus a small error—it follows from Lemma B.2 that the solution of the initial value problem (22) satisfies

$$\left\| g_t(\pi/3) - \exp\left(E_{13}\int_0^{\pi/3} \mu_t(s))\right)\right\| \to 0 \quad \text{as } t \to \infty.$$

Since $g_t(\pi/3) = \widehat{F}(t)^{-1}\widehat{F}(e^{i\pi/3}t) \to L_0^{-1}L_+$ as $t \to \infty$, this gives the desired unipotent form (23) for some $b \in \mathbb{R}$.

The value of $L_-^{-1}L_0$ is computed by a nearly identical argument applied to rays at angles $-\pi/3$ and 0. The only difference is that at $\theta = -\pi/6$, the leading eigenvector $\mathrm{Ad}_{H(e^{i\theta}t)}$ is the elementary matrix E_{12}, which becomes the dominant term in the coefficient $M_t(s)$. Exponentiating we find $L_-^{-1}L_0$ has the desired form (23). □

Step 3: Finding triangle pieces. We now turn to studying the shape of the projectivized image $\mathbb{P}(M) \subset \mathbb{RP}^2$. Let V denote the union of the edges e_{110} and e_{101} of the standard triangle $\triangle_0 \subset \mathbb{RP}^2$ that is the image of the normalized Ţiţeica surface T. (Recall the notation for vertices and edges of this triangle was described in Section 3.)

In the following proposition, we say that a ray has *height* y if it contains the point $z = 1 + iy$ in U, where $y \in \mathbb{R}$.

Lemma 6.6 (Projective limits in a half-plane). *Let $L_0 \in \mathrm{GL}_3\mathbb{R}$ be a limit of the osculation map of an affine sphere M as above. Then the following table describes the projective limits of f-images of stable* (quasi-)*rays in U:*

Type of path γ	***Direction*** θ	***Projective limit*** p_γ ***of*** $f(\gamma)$
Quasi-ray	$\theta \in (-\frac{\pi}{2}, -\frac{\pi}{3})$	$p_\gamma = L_0 \cdot v_{001}$
Ray (of height y)	$\theta = -\frac{\pi}{3}$	$p_\gamma \in L_0 \cdot e^\circ_{101}$
		($p_\gamma \to L_0 \cdot v_{001}$ *as* $y \to -\infty$)
Quasi-ray	$\theta \in (-\frac{\pi}{3}, \frac{\pi}{3})$, $\theta \neq \pm\frac{\pi}{6}$	$p_\gamma = L_0 \cdot v_{100}$
Ray (of height y)	$\theta = \frac{\pi}{3}$	$p_\gamma \in L_0 \cdot e^\circ_{110}$
		($p_\gamma \to L_0 \cdot v_{010}$ *as* $y \to \infty$)
Quasi-ray	$\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$	$p_\gamma = L_0 \cdot v_{010}$

And in particular:

- *The projectivization of any stable quasi-ray of angle zero in* U *tends to* $L_0 \cdot v_{100} \in L_0 \cdot V$ *(by the middle row of the table), and*
- *We have* $L_0 \cdot V \subset \partial\mathbb{P}(M)$ *(since* $V = v_{001} \cup e^\circ_{101} \cup v_{100} \cup e^\circ_{110} \cup v_{010}$*).*

Proof. First, using the explicit formula (10) for the normalized Ţiţeica surface T, it is easy to calculate the projective limit v_γ of the T-image of any ray or quasi-ray in the right half-plane. (At this point stability is not relevant.) The result is a table for T analogous to the one we seek for f (compare [Lof04, Tbl. 2]):

Type of path γ	**Direction** θ	**Projective limit** v_γ **of** $T(\gamma)$
Quasi-ray	$\theta < -\frac{\pi}{3}$	$v_\gamma = v_{001}$
Ray (of height y)	$\theta = -\frac{\pi}{3}$	$v_\gamma \in e^\circ_{101}$
		($v_\gamma \to v_{001}$ as $y \to -\infty$)
Quasi-ray	$\theta \in (-\frac{\pi}{3}, \frac{\pi}{3})$	$v_\gamma = v_{100}$
Ray (of height y)	$\theta = \frac{\pi}{3}$	$v_\gamma \in e^\circ_{110}$
		($v_\gamma \to v_{010}$ as $y \to \infty$)
Quasi-ray	$\frac{\pi}{3} < \theta$	$p_\gamma = v_{010}$

Now suppose γ is a *stable* ray or quasi-ray in U, and let $L_\gamma = \lim_{t\to\infty} \widehat{F}(\gamma(t))$. Since $f(z) = \widehat{F}(z)T(z)$, we find that the projective limits v_γ of $\mathbb{P}(T(\gamma))$ and p_γ of $\mathbb{P}(f(\gamma))$ are related by

$$p_\gamma = L_\gamma \cdot v_\gamma. \tag{24}$$

Note that since γ is a divergent path, each point p_γ obtained in this way lies on the boundary of $\mathbb{P}(M)$.

By Lemma 6.4 we have $L_\gamma \in \{L_-, L_0, L_+\}$ with the value depending only on θ. Hence the combination of formula (24) and the table of Ţiţeica limits gives the following characterization of f-limits:

Type of path γ	**Direction** θ	**Projective limit** p_γ **of** $f(\gamma)$
Quasi-ray	$\theta \in (-\frac{\pi}{2}, -\frac{\pi}{3})$	$p_\gamma = L_- \cdot v_{001}$
Ray (of height y)	$\theta = -\frac{\pi}{3}$	$p_\gamma \in L_- \cdot e_{101}^\circ$
		$(p_\gamma \to L_- \cdot v_{001}$ as $y \to -\infty)$
Quasi-ray	$\theta \in (-\frac{\pi}{3}, -\frac{\pi}{6})$	$p_\gamma = L_- \cdot v_{100}$
Quasi-ray	$\theta \in (-\frac{\pi}{6}, \frac{\pi}{6})$	$p_\gamma = L_0 \cdot v_{100}$
Quasi-ray	$\theta \in (\frac{\pi}{6}, \frac{\pi}{3})$	$p_\gamma = L_+ \cdot v_{100}$
Ray (of height y)	$\theta = \frac{\pi}{3}$	$p_\gamma \in L_+ \cdot e_{110}^\circ$
		$(p_\gamma \to L_+ \cdot v_{010}$ as $y \to \infty)$
Quasi-ray	$\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$	$p_\gamma = L_+ \cdot v_{010}$

This is nearly the characterization of projective limits we seek; if we replace all instances of L_- and L_+ with L_0 in the table above (and coalesce the middle three rows, where the limit becomes the same) we obtain exactly the statement of the lemma.

The proof is completed by using Lemma 6.5 to verify that in each place that L_- or L_+ appears in the previous table, the projective transformation is applied to a point in $\mathbb{RP}^2$ where it has the same action as L_0. Of the six affected rows, there are actually only two cases to consider:

- Since $L_0 = L_+U_+$ where U_+ is a unipotent that fixes the line in $\mathbb{RP}^2$ containing e_{110} pointwise, we have

$$L_0 \cdot e_{110} = L_+U_+ \cdot e_{110} = L_+ \cdot e_{110}.$$

  Since v_{100}, v_{010} are the endpoints of e_{110}, it follows that L_+ can be replaced by L_0 in the previous table.
- Since $L_0 = L_-U_+$ where U_- is a unipotent that fixes the line in $\mathbb{RP}^2$ containing e_{101} pointwise, we have

$$L_0 \cdot e_{101} = L_-U_- \cdot e_{101} = L_- \cdot e_{101}.$$

  Since v_{100}, v_{010} are the endpoints of e_{110}, it follows that L_- can be replaced by L_0 in the previous table. □

Step 4: Assembling the polygon. So far we have been working in a single fixed half-plane U for the Pick differential C. Now we consider how the picture changes as we move between the standard half-planes $U_0, \dots, U_{d+2}$ associated to C by Proposition 3.2. By the construction above, we obtain the following for each $0 \leqslant i \leqslant d+2$:

- A set of limits $L_-^{(i)}, L_0^{(i)}, L_+^{(i)} \in \mathrm{GL}_3\,\mathbb{R}$ of the osculation map in U_i restricted to stable rays
- Unipotent elements as in Lemma 6.5 that relate these limits, and
- The conclusion that $L_0^{(i)} \cdot V \subset \partial\mathbb{P}(M)$.

Thus each of the half-planes gives a piece of the boundary of $\partial\mathbb{P}(M)$ that is a "vee", i.e. the image of V by a projective transformation.

By studying the overlap between these edge pairs, we can finally establish:

LEMMA 6.7. *The projectivization $\mathbb{P}(M)$ of the affine sphere M is a convex polygon with $d+3$ vertices. The projectivization of $f(\star_d)$ is an embedded open star in $\mathbb{P}(M)$ whose edges tend to the vertices.*

Proof. Consider the f-images of rays in U_i with angle $\theta = \pi/3$, which by the previous lemma projectively limit on the edge $L_0^{(i)} \cdot e^\circ_{110}$.

By Proposition 3.2, the next half-plane U_{i+1} (with index understood mod $d+3$) intersects U_i in a sector that contains all but an initial segment from each of these rays. In the coordinate z_{i+1} of U_{i+1}, these rays have angle $\theta_{i+1} = -\pi/3$. Hence by applying the previous lemma in U_{i+1} we find the f-images of the same rays projectively limit on $L_0^{(i+1)} \cdot e^\circ_{101}$, and thus

$$L_0^{(i)} \cdot e^\circ_{110} = L_0^{(i+1)} \cdot e^\circ_{101}.$$

By continuity of projective transformations, we have the same equality for the associated closed edges. Furthermore, the previous lemma characterizes the behavior of the limit point as a function of the height of the ray, determining which pairs of endpoints are identified, namely:

$$\begin{aligned} L_0^{(i)} \cdot v_{100} &= L_0^{(i+1)} \cdot v_{001}, \\ L_0^{(i)} \cdot v_{010} &= L_0^{(i+1)} \cdot v_{100}. \end{aligned}$$

Thus if we orient the edge pair V from v_{001} to v_{010}, we have found that the union of $L_0^{(i)} \cdot V$ and $L_0^{(i+1)} \cdot V$ is an oriented chain of three edges in $\partial\mathbb{P}(M)$.

Allowing i to vary we find that the overlapping edge pairs $\{L_0^{(i)} \cdot V\}_{i=0\ldots d+2}$ assemble into a map

$$\Gamma : P_{d+3} \to \partial\mathbb{P}(M) \simeq S^1$$

where P_{d+3} is an abstract $(d+3)$-gon, considered as a simplicial 1-complex. By construction Γ is linear on each edge, its restriction to any pair of adjacent edges is an embedding (with image $L_0^{(i)} \cdot V$, for some i), and the image of any vertex of P_{d+3} is a corner of the convex curve $\partial\mathbb{P}(M)$ which can be described as $L_0^{(i)} \cdot v_{100}$ for some i.

Because adjacent pairs of edges embed, the map Γ is a local homeomorphism of compact, connected Hausdorff spaces. Thus Γ is a covering map, and in particular surjective. The image $\partial\mathbb{P}(M)$ is therefore a polygon and, considering that polygonal curve as a 1-complex, the covering is simplicial.

To identify the image as a $(d+3)$-gon, it remains to show that Γ is injective, which follows if it is injective on vertices.

Recall from Proposition 3.2 that for each i there is an edge γ_i of the standard star $\star_d$ that is eventually contained in that half-plane U_i in which it is a quasi-ray with direction $\theta_i = 0$. (This is the point at which we use that C is monic in an essential way.) Applying Lemma 6.6 to these quasi-rays find that the projectivizations $\mathbb{P}(f(\gamma_i))$ tend to the points $L_0^{(i)} \cdot v_{100}$, that is, the images of vertices of P_{d+3} by Γ.

Suppose for contradiction that $\mathbb{P}(\gamma_i)$ and $\mathbb{P}(\gamma_j)$ have the same limit point $x \in \partial\mathbb{P}(M)$ and $i \neq j$. Note $i \neq j \pm 1 \mod (d+3)$ since neighboring vertices (and the edge they span) map to distinct points by Lemma 6.6.

The union $\beta = \gamma_i \cup \gamma_j$ of two edges of $\star_d$ separates $\mathbb{C}$ into two components, and since $\mathbb{P}(f)$ is a homeomorphism onto the convex domain $\mathbb{P}(M)$, the image curve $\mathbb{P}(f(\beta))$ separates $\mathbb{P}(M)$. All branches of $\star_d$ contain the origin, but except for this common point the paths γ_{j+1} and γ_{j-1} lie in different components of $\mathbb{C}\backslash\beta$. Neither of $\gamma_{j\pm1}$ has projective image converging to x since these are the neighbors of γ_j. Thus each component of $\mathbb{P}(M\backslash f(\beta))$ accumulates on at least one boundary point of $\mathbb{P}(M)$ that is distinct from x. This is a contradiction, however, because $\mathbb{P}(f(\beta))$ is a properly embedded path in the open disk $\mathbb{P}(M)$ that limits on a single boundary point $x \in \partial\mathbb{P}(M)$ in both directions, so one of its complementary disks has x as the only limit point on $\partial\mathbb{P}(M)$.

Thus we find that Γ is injective, and that the projectivized image $\mathbb{P}(f(\star_d))$ gives an embedded star in $\mathbb{P}(M)$ that limits on the vertices of the polygon, giving an adjacency-preserving bijection of them with the edges of $\star_d$.

This completes the proof of the lemma, and of Theorem 6.3. $\square$

7 From Polygons to Polynomials

The main goal of this section is to establish the converse of Theorem 6.3:

Theorem 7.1. *Let M be a complete hyperbolic affine sphere in $\mathbb{R}^3$ asymptotic to the cone over a convex polygon with n vertices. Then the Blaschke metric of M is conformally equivalent to the complex plane $\mathbb{C}$ and its Pick differential is a polynomial cubic differential of degree $(n-3)$.*

Proof. If $n = 3$ then M is a Țițeica surface, so the statement follows immediately. For the rest of the proof we assume $n \geqslant 4$.

Let $P = \mathbb{P}(M)$ be the convex polygon. For any vertex v of P let τ_v be the triangle formed by v and its two neighboring vertices. We say τ_v is the *vertex inscribed triangle* of P at v.

Considering the triangle τ_v as the projectivized image of a Țițeica affine sphere, we can choose a parameterization $T_v : \mathbb{C} \to \mathbb{R}^3$ so that the projective image of the positive real axis is asymptotic to v.

By convexity τ_v is contained in P, so for each $z \in \mathbb{C}$ there is a unique point $M_v(z) \in M$ collinear with $T_v(z)$ and the origin. This gives a smooth embedding $M_v : \mathbb{C} \to M$.

Lemma 7.2. *Equip* $\mathbb{C}$ *with the Euclidean metric* $|dz|^2$ *and* M *with either the Blaschke metric or the Pick differential metric* $|C|^{2/3}$*. Then the differential* $dM_v : T_z\mathbb{C} \to T_{M_v(z)}M$ *is bilipschitz when* $\mathrm{Re}(z)$ *is large. Moreover there are constants* R, Λ *such that for any* z *with* $\mathrm{Re}(z) \geqslant R$ *and any* $\xi \in T_z\mathbb{C}$ *we have*

$$\frac{1}{\Lambda}\|\xi\| \leqslant \|dM_v(\xi)\| \leqslant \Lambda\|\xi\|.$$

Proof. We will show that the differential of the map from the Ţiţeica surface over τ_v to M obtained by projecting along rays through $0 \in \mathbb{R}^3$ is bilipschitz in the region corresponding to $\mathrm{Re}(z) \geqslant R$. Since M_v is the composition of this projection with the parameterization T_v, and since both the Blaschke and Pick differential metrics of the Ţiţeica surface are multiples of $|dz|^2$ in that parameterization, the lemma will follow.

For the remainder of the proof we consider the images of both T_v and M_v to be parameterized by their common projectivization, which is the triangle τ_v. Composing this parameterization with the inverse of T_v, the coordinate z of $\mathbb{C}$ becomes a function on the triangle τ_v; we denote the image of $p \in \tau_v$ by $z(p)$. We must show that for any $p \in \tau_v$ with $\mathrm{Re}(z(p))$ large, the respective metrics of T_v and M are uniformly comparable at p.

The triangle τ_v is an orbit of a maximal torus in $\mathrm{SL}_3\mathbb{R}$ (as is the surface T_v itself). Fix a basepoint $p_0 \in \tau_v$ and for any other $p \in \tau_v$ let $A(p)$ be the element of this torus mapping p to p_0.

The key observation is that by taking $\mathrm{Re}(z(p))$ large enough, we can assure that the image $A(p)\cdot P$ of the polygon P is arbitrarily close to τ_v in the Hausdorff topology.

To see this, first normalize with a projective transformation so that τ_v is the standard triangle $\triangle_0$, the vertex v is v_{100}, and T_v is the normalized Ţiţeica surface. Then $A(p) = H(z(p_0))H(z(p))^{-1}$ is diagonal and taking $\mathrm{Re}(z(p))$ to be large makes the $(1,1)$ entry of $A(p)$ small. This means that the projective action of $A(p)$, while preserving the two shared edges of P and τ_v, maps the rest of P very close to the third edge of τ_v (which is e_{011} in this normalization): geometrically, the map $A(p)$ sends p to p_0 and fixes the vertices of the triangle τ_v, with v being a repelling fixed point, and the other vertices being hyperbolic fixed points. Thus for $\mathrm{Re}(z(p))$ large enough, the image $A(p) \cdot P$ lies in any chosen Hausdorff neighborhood of τ_v.

Now we use the projective naturality of the Blaschke metric and the Pick differential. Instead of comparing the metrics of the affine spheres over τ_v and P at an arbitrary point $p \in \tau_v$, it suffices to compare the metrics of the affine spheres over $A(p) \cdot \tau_v = \tau_v$ and $A(p) \cdot P$ at the fixed point p_0. By Corollary 4.5, both the Blaschke metric and the Pick differential metric at p_0 vary continuously in the Hausdorff topology on pointed convex sets. Taking R large enough, we can assume that $A(p) \cdot P$ lies in a neighborhood of τ_v such that the Blaschke metric on the tangent space at p_0 is Λ-bilipschitz to that of the Ţiţeica surface over τ_v, and similarly for the Pick differential metric, as required. $\square$

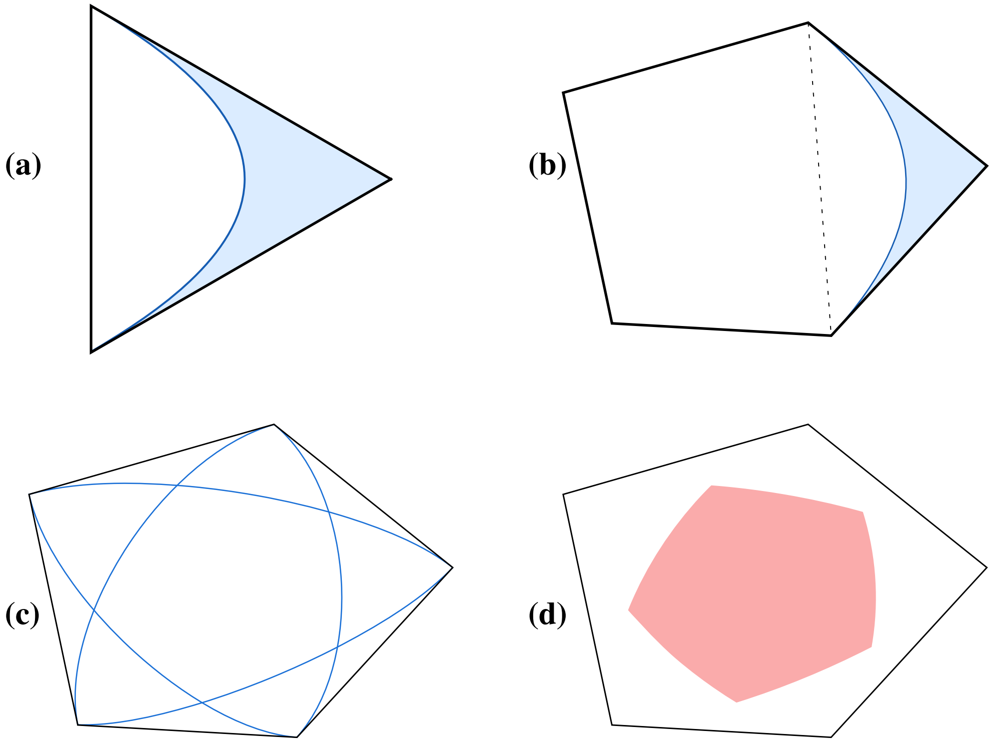

Figure 3: **a** The image of $\{\mathrm{Re}(z) > R\}$ in the *triangle* projectivization of a Ţiţeica surface; **b** the corresponding region in a vertex inscribed *triangle of a polygon*; **c** repeating this at each vertex, we obtain a set of *barrier curves*, and **d** the *core*, a compact set containing all zeros of the Pick differential

We remark that the proof above actually shows more: By taking R large enough, we can make the constant K as close to 1 as we like. However, we will not need this refined version of the estimate in what follows.

In the normalized Ţiţeica surface, the projectivized image of $\{\mathrm{Re}(z) > R\}$ is the intersection of a neighborhood of the union of v_{100} and the open edges e°_{110}, e°_{101} with the interior of the standard triangle (see Figure 3a). Correspondingly, the part of τ_v in which the estimate of the previous Lemma applies is a neighborhood of v and the adjacent open edges. It is bounded by a curve that joins the neighboring vertices of v, namely, the image of $\mathrm{Re}(z) = R$ in the Ţiţeica surface over the vertex inscribed triangle (see Figure 3b). We call this the *barrier curve at* v.

Applying Lemma 7.2 to each vertex of P in turn, we find that its conclusion applies in a set of n half-planes that cover all but a compact subset K of the interior of P (see Figure 3c, d); this set is a closed curvilinear polygon bounded by arcs from the barrier curves. We call K the *core* of P.

Using this construction, we can establish some key properties of the Pick differential and Blaschke metric:

LEMMA 7.3. (i) *The Pick differential has finitely many zeros.*

(ii) *The Pick differential metric* $|C|^{2/3}$ *of* M *is quasi-isometric to the Blaschke metric of* M, *and in particular, it is complete.*

Proof. (i) The Pick differential has no zeros in any of the half-planes given by Lemma 7.2, hence the zeros all lie in the core K, which is compact. The Pick differential is holomorphic and does not vanish identically, so its zeros have no accumulation point. The zero set is therefore compact and discrete, hence finite.

(ii) Outside the core, the Pick differential and Blaschke metrics of M are uniformly comparable. Because the core is compact, it has finite diameter for both metrics. Thus a geodesic for one metric can be split into a part of bounded diameter and a part in which the other metric is bounded above and below, giving quasi-isometry. The Blaschke metric is complete, so this shows $|C|^{2/3}$ is complete as well. □

By analyzing the continuity of the construction Lemma 7.2 as a function of the vertices of the polygon, we can also show that this compact set containing the zeros of the Pick differential for P has the same property for polygons sufficiently close to P. This observation will be used in Section 8.

LEMMA 7.4. *Let* $P \in \mathcal{P}_n$ *be a convex polygon in* $\mathbb{RP}^2$. *Then there exists a compact subset* $\widehat{K}$ *of the interior of* P *and a neighborhood* U *of* P *in* $\mathcal{P}_n$ *with the following property: if* $P' \in U$ *and if* M' *is the complete hyperbolic affine sphere asymptotic to the cone over* P', *then all of the zeros of the Pick differential of* M' *lie over* K.

Proof. While the construction of barrier curves in a polygon involves some choices, we will show that one can make the construction continuous in a small neighborhood of P, i.e. so that the barrier curve at a vertex varies continuously in the Hausdorff topology when a small deformation is applied to the vertices of P. Of course this will also imply that the core $K(P)$ varies continuously as well.

The lemma will then follow by taking $\widehat{K}$ to be a compact set containing a neighborhood of $K(P)$. For P' sufficiently close to P, the core $K(P')$ will be contained in this neighborhood of $K(P)$ and hence $\widehat{K}$ will contain the Pick zeros of P' as well.

To choose barrier curves continuously, first consider polygons P that have a fixed vertex inscribed triangle T at v (that is, we have fixed the location of v and its two neighbors). In the proof of Lemma 7.2 the barrier is constructed as the image of a vertical line $\{\mathrm{Re}(z) = R\}$ in the conformal parameterization of the Țițeica surface over T. The barrier curve is completely determined by the real number R, which must be large enough so that the associated subset of a maximal torus in $\mathrm{SL}_3\,\mathbb{R}$ maps the polygon P into a certain Hausdorff neighborhood of T. Choosing R large enough, we can ensure this not only holds for P, but also for the union of all polygons in a small neighborhood of P that share this vertex inscribed triangle. Hence a fixed barrier curve works for all of these polygons.

Now consider the general case, i.e. polygons P' near P with no restriction on the vertices. Working in a sufficiently small neighborhood of P gives a natural bijection from the vertices of P' to those of P. As in the normalization construction of Section 2, there is a unique projective transformation $A(P')$ that maps four chosen vertices of P' to the corresponding vertices of P, and this projective map varies continuously with P'. Selecting v, its two neighbors, and an arbitrary fourth vertex, we get normalizing projective transformations $A(P') \in \mathrm{SL}_3 \mathbb{R}$ so that $A(P') \cdot P'$ shares the vertex inscribed triangle at v with P.

Thus, after applying a projective transformation $A(P')$, we are reduced to the case considered before, where a fixed barrier curve could be used. We therefore define the barrier curve for P' by applying $A(P')^{-1}$ to this fixed curve. Since $A(P')$ is a continuous function of P', the curves constructed this way also vary continuously. □

Returning to consideration of a fixed affine sphere M over a polygon P, we can now identify the conformal type of the Blaschke metric (or the conformally equivalent Pick differential metric).

LEMMA 7.5. *The affine sphere M is conformally equivalent to $\mathbb{C}$.*

Proof. Since M is simply-connected and noncompact, we need only show that it is not conformally equivalent to the unit disk Δ.

Suppose for contradiction that $M \simeq \Delta$ and write $C = C(z)dz^3$ where $C(z)$ is a holomorphic function. By Lemma 7.3(i) we have $C(z) = p(z)H(z)$ where p is a polynomial and H has no zeros.

The unit disk does not admit a complete *flat* conformal metric, since the developing map of the Euclidean structure induced by such a metric would be a conformal isomorphism $\Delta \to \mathbb{C}$. The conformal metric $|H|^{2/3}|dz|^2$ is flat, because $\log |H|$ is harmonic, and therefore it is not complete.

But a divergent path of finite $|H|^{2/3}$-length also has finite $|C|^{2/3}$-length because the polynomial p is bounded on Δ. Thus the Pick differential metric is not complete, contradicting Lemma 7.3(ii). □

We remark that the finiteness of the zero set of C means that the *integral curvature* of the Pick differential metric is finite. Huber showed that any Riemann surface which admits a complete conformal metric of finite integral curvature is conformally parabolic [Hub57, Thm. 15], and the proof above is an adaptation of Huber's argument to this special case. For smooth conformal metrics and simply-connected surfaces, the same result was proved earlier by Blanc and Fiala [BF42].

LEMMA 7.6. *The function $C(z)$ is a polynomial.*

The follows from a lemma of Osserman [Oss86, Lem. 9.6], generalizing a result of Finn [Fin65, Thm. 17]. While these authors consider complete conformal metrics of the form $|f|^2|dz|^2$, where f is holomorphic, their arguments easily extend to $|C|^{2/3}$. For the reader's convenience we sketch the argument while incorporating the necessary changes for this case:

Proof. Write $C(z) = p(z)e^{G(z)}$ where p is a polynomial and G an entire function on $\mathbb{C}$. We show that completeness of $|C|^{2/3}$ implies that G is constant.

Taking an integer $N > \frac{1}{3}\deg(p)$ we have

$$|C(z)|^{1/3} = O(|z^N e^{G(z)}|) \quad \text{as } z \to \infty. \tag{25}$$

The function $F(z)$ with $F'(z) = z^N e^{G(z)}$ and $F(0) = 0$ has a zero of order exactly $N+1$ at 0, hence $\zeta = F^{1/(N+1)}$ is single-valued and has an inverse function $z(\zeta)$ in some neighborhood of 0.

In fact this inverse must exist globally: Otherwise there would be a radial path of the form $t \mapsto t\zeta_0$, $t \in [0,1)$, $|\zeta_0| = R$ on which $z(\zeta)$ is defined but cannot be extended. The image Γ of this path by $z(\zeta)$ satisfies

$$\int_\Gamma |z^N e^{G(z)}||dz| = \int_\Gamma |F'(z)||dz| = \int_\Gamma |d(\zeta^n)| = R^{N+1}.$$

The path Γ is not divergent, since by (25) this would contradict completeness. Thus along the path there is a sequence $z_n(\zeta_n) \to z_0$ with $\zeta_n \to \zeta_0$. But $F'(z_0) \neq 0$, allowing extension of $z(\zeta)$ over ζ_0, a contradiction.

Thus $F^{1/(N+1)}$ is entire and invertible, hence linear, making F a polynomial. Thus G is constant, and C is also a polynomial. □

By Theorem 6.3, the degree d of the polynomial $C(z)$ is $(n-3)$, completing the proof of Theorem 7.1. □

8 Mapping of Moduli Spaces

The two preceding sections show that a complete hyperbolic affine sphere is asymptotic to a polygon if and only if it has conformal type $\mathbb{C}$ and polynomial Pick differential, and that all polynomials arise in this way from polygons. In this section we combine and extend these results to prove the main theorem (also relying on the Cheng–Yau theorem and the results of Section 5.3 on continuous variation of solutions to the vortex equation). Precisely, we show:

Theorem 8.1. *For any integer $d \geqslant 0$, the construction of an affine sphere with polynomial Pick differential given by Theorem 6.1 induces a $\mathbb{Z}/(d+3)$-equivariant homeomorphism*

$$\alpha : \mathcal{TC}_d \to \mathcal{TP}_{d+3},$$

and thus also a quotient homeomorphism $\boldsymbol{\alpha} : \mathcal{MC}_d \to \mathcal{MP}_{d+3}$.

Proof. The proof will proceed in several steps.
Construction of the map. Let $C \in \mathcal{TC}_d$ be a normalized polynomial cubic differential of degree d. By Theorem 6.1 we have a complete conformally parameterized affine sphere $f_C^0 : \mathbb{C} \to \mathbb{R}^3$ with Pick differential C. By Theorem 6.3, the projectivized

image of this affine sphere is a polygon P_C^0 and the rays of $\star_d$ map to curves that limit projectively to the vertices of P_C^0. Thus the counterclockwise order of the edges of $\star_d$, starting from $\mathbb{R}^+$, induces a labeling of the vertices of P_C^0 by $(p_1, \ldots, p_{d+3})$. Let A be a projective transformation that normalizes this polygon at p_1, i.e. mapping (p_1, p_2, p_3, p_4) to (q_1, q_2, q_3, q_4). Then $P_C := A \cdot P_C^0 \in \mathcal{TP}_{d+3}$ is the projectivized image of the conformally parameterized affine sphere $f_C := A \cdot f_C^0$ (which still has Pick differential C). Define

$$\alpha(C) := P_C.$$

To summarize, $\alpha(C)$ is the polygon obtained by solving the vortex equation with cubic differential C, integrating to obtain an affine sphere with vertices naturally labeled by the $(d+3)$-roots of unity, and then adjusting by a projective transformation to normalize the polygon at the first vertex.

Equivariance. Let $\zeta = \exp(2\pi i/(d+3))$ be the generator of the group $\boldsymbol{\mu}_{d+3}$ of $(d+3)$-roots of unity and denote its action on $\mathcal{TC}_d$ by pushforward through $z \mapsto \zeta z$ by $C \mapsto \zeta \cdot C$.

Since the Pick differential of $f_C(\zeta z)$ is the pullback $\zeta^{-1} \cdot C$, we find that $f_{\zeta \cdot C}(\zeta z)$ has Pick differential $\zeta^{-1}\zeta \cdot C = C$, so by the uniqueness part of Theorem 6.1 there exists $A \in \mathrm{SL}_3 \mathbb{R}$ such that

$$f_{\zeta \cdot C}(\zeta z) = A \cdot f(z).$$

Note that $z \mapsto \zeta z$ permutes the rays of $\star_d$, acting as a $(d+3)$-cycle. Thus up to projective transformations the normalized polygons $\alpha(C)$ and $\alpha(\zeta \cdot C)$ are the same, but under this isomorphism the labeling of their vertices by $1, \ldots, (d+3)$ is shifted by one. This is the definition of the action of ϱ, the generator of the $\mathbb{Z}/(d+3)$ action on $\mathcal{TP}_d$, and so α is equivariant.

It follows that α induces a map $\boldsymbol{\alpha} : \mathcal{MC}_d \to \mathcal{MP}_{d+3}$, and that for any $C \in \mathcal{C}_d$ the image $\alpha([C])$ is simply $\mathrm{SL}_3 \mathbb{R}$ equivalence class of the projectivized image of f_C or f_C^0 (i.e. in describing the quotient map, no normalization is required).

Injectivity. If $\alpha(C) = \alpha(C') = P$ then the uniqueness part of the Cheng–Yau theorem (4.3) shows that the associated affine spheres coincide, and so the two conformal parameterizations f_C, $f_{C'}$ are related by an automorphism of $\mathbb{C}$. Both C and C' are normalized, so this automorphism must be multiplication by a $(d+3)$-root of unity, which permutes the rays of $\star_d$. Since $\alpha(C) = \alpha(C')$, we also have that f_C and $f_{C'}$ induce the same map from the rays of $\star_d$ to vertices of P. Hence this permutation must be trivial and the automorphism is the identity, i.e. $C = C'$.

Surjectivity. By equivariance it is enough to check that $\boldsymbol{\alpha}$ is surjective. Let $[P] \in \mathcal{MP}_{d+3}$. We can choose a representative oriented polygon P that lies in a fixed affine chart $\mathbb{C} \simeq \mathbb{R}^2 \simeq \{(x_1, x_2, x_3) | x_3 = 1\} \subset \mathbb{R}^3$ of $\mathbb{RP}^2$ in such a way that the orientation of P agrees with that of $\mathbb{C}$.

By the Cheng–Yau existence theorem (4.3) we have a complete hyperbolic affine sphere M_P asymptotic to the cone over P. By Theorem 7.1, there is a conformal parameterization of M_P by $\mathbb{C}$ such that Pick differential is a polynomial cubic differential C. Using Theorem 6.1 we obtain another parameterized affine sphere M_C with Pick differential C, which by definition has projectivization representing $\boldsymbol{\alpha}([C])$.

By the uniqueness part of Theorem 6.1 there is an element of $\mathrm{SL}_3\,\mathbb{R}$ mapping M_C to M_P, thus identifying their oriented projectivizations and giving $\boldsymbol{\alpha}([C]) = [P]$. Hence $\boldsymbol{\alpha}$ is surjective.

Continuity. For $d \leqslant 1$ all of the moduli spaces in question are finite sets with the discrete topology, and there is nothing to prove. We assume for the rest of the proof that $d > 1$.

First we consider a map related to α in which the polygon is normalized in a different way. For any $C \in \mathcal{TC}_d$ we can compose f_C with an element of $\mathrm{SL}_3\,\mathbb{R}$ so that its complexified frame at $0 \in \mathbb{C}$ agrees with that of the normalized Ţiţeica surface. Let $f_C^\circ : \mathbb{C} \to \mathbb{R}^3$ denote the resulting map; note that the projectivized image of f_C° is a polygon, but not necessarily a normalized one. We have the associated map

$$\begin{aligned} \alpha^\circ : \mathcal{TC}_d &\to \mathcal{P}_{d+3}, \\ C &\mapsto \mathbb{P}(f_C^\circ(\mathbb{C})). \end{aligned} \tag{26}$$

The advantage of working with α° is that the shared frame at the origin means that developing maps $\{f_C^\circ\}_{C \in \mathcal{TC}_d}$ are solutions of a fixed initial value problem for the system of ODEs (4) where only the coefficients of the system are varying. In contrast, the maps $\{f_C\}$ have a shared normalization only "at infinity".

Using Theorem 5.4 we will now show that α° is continuous with respect to the Hausdorff topology.

Fix $C \in \mathcal{TC}_d$ and $\epsilon > 0$. We must find a neighborhood V of C in $\mathcal{TP}_{d+3}$ such that for all $C' \in V$ we have $\alpha^\circ(C) \subset N_\epsilon(\alpha^\circ(C'))$ and $\alpha^\circ(C') \subset N_\epsilon(\alpha^\circ(C))$.

Let $P = \alpha^\circ(C)$. First select a radius R large enough so that so that $P \subset N_{\epsilon/2}(f_C^\circ(B_R))$ where $\bar{B}_R = \{|z| \leqslant R\}$. Now, by Theorem 5.4 we can ensure that the Blaschke metric densities of C and C' are arbitrarily close in $C^1(\bar{B}_R)$ by making the coefficients of C and C' sufficiently close. By (4) this shows that the coefficients of the respective connection forms can be made uniformly close (C^0), and applying continuous dependence of solutions to ODE initial value problems gives the same conclusion for $f_C^\circ, f_{C'}^\circ$ (and moreover, for their respective frame fields F_C°, $F_{C'}^\circ$). In particular we can choose a neighborhood of C so that $\mathbb{P}(f_{C'}^\circ)$ is $\epsilon/2$-close to $\mathbb{P}(f_C^\circ)$ in $\bar{B}_R$, giving

$$\alpha^\circ(C) = P \subset N_\epsilon(f'_C(\bar{B}_R)) \subset N_\epsilon(\alpha^\circ(C'))$$

for all C' in this neighborhood.

The "outer" continuity follows similarly by considering tangent planes to the affine sphere. Recall that the image of f_C° is strictly convex and asymptotic to the boundary of the cone over P, which is a polyhedral cone with $d+3$ planar faces (corresponding to the edges of P). Thus, we may approximate each of the planes in $\mathbb{R}^3$ containing one of these faces as closely as we wish by the tangent plane to an appropriately chosen point on f°_C. Selecting one such point for each face—call these "sample points"—the tangent planes become lines in $\mathbb{RP}^2$ which determine a $(d+3)$-gon $\widehat{P}$ that approximates P. Moreover, by convexity f°_C lies above its tangent planes, which means that P lies inside $\widehat{P}$. We call $\widehat{P}$ the *outer polygon* determined by the sample points.

Choose $R > 0$ so that $\bar{B}_R$ contains a set of $(d+3)$ sample points for which the outer polygon approximates P well enough that $\widehat{P} \subset N_{\epsilon/2}(P)$. As above we conclude from Theorem 5.4 that any C' close to C determines a frame field $F^\circ_{C'}$ that is uniformly close to $F^\circ_{C'}$ on $\bar{B}_R$, and in particular the tangent planes to $f^\circ_{C'}$ approximate those of f°_C. Thus there is a neighborhood of C in which the outer polygon for C' lies in a $\epsilon/2$-neighborhood of that for C, giving

$$\alpha^\circ(C') \subset N_{\epsilon/2}(\widehat{P}) \subset N_\epsilon(P) = N_\epsilon(\alpha^\circ(C)),$$

for all C' in this neighborhood.

Thus α° is continuous with respect to the Hausdorff topology. By Proposition 2.4 this implies that α° is also continuous with respect to the usual vertex topology on $\mathcal{P}_{d+3}$.

Finally, we return to the original map α: Since the polygon $\alpha(C)$ is simply the normalization of $\alpha^\circ(C)$, the continuous variation of the vertices of $\alpha^\circ(C)$ implies that the projective transformations that accomplish this normalization also vary continuously. Hence the map α is the composition of α° with a continuous family of projective transformations, and hence also continuous.

Continuity of the inverse. At this point we have shown that α is a continuous bijection between spaces homeomorphic to $\mathbb{R}^N$. To establish continuity of α^{-1} we need to show that α is closed. Since proper continuous maps on locally compact spaces are closed, it suffices to show that α is a proper map.

Suppose for contradiction that α is not proper, i.e. that there exists a sequence $C_n \to \infty$ in $\mathcal{TC}_d$ so that $\alpha(C_n) \to P$. Write $P_n = \alpha(C_n)$.

Let $Z_n \subset \mathbb{C}$ denote the set of roots of the polynomial C_n. Note that the cardinality of Z_n is at most d. Since C_n is monic and centered, a bound on the diameter of the set Z_n (with respect to the Euclidean metric of $\mathbb{C}$) would give a bound on all coefficients of C_n. Since $C_n \to \infty$, no such bound applies, and we find that the Euclidean diameter of Z_n is unbounded as $n \to \infty$. Replacing C_n with a subsequence, we assume from now on that the diameter of Z_n actually tends to infinity.

Let r_n denote the diameter of the set Z_n with respect to the flat metric $|C_n|^{2/3}$ of the cubic differential. We claim that r_n tends to infinity. To see this, consider paths in $\mathbb{C}$ that connect all of the zeros of C_n, i.e. maps $[0,1] \to \mathbb{C}$ such that Z_n is a subset of the image. We call these *spanning paths*. Ordering the elements of Z_n and connecting them in order by $|C_n|^{2/3}$-geodesic segments gives a spanning path of length at most $(d-1)r_n$. We will show that the minimum length of a spanning path tends to infinity, and hence that $r_n \to \infty$ as well.

Suppose that there exists a spanning path whose Euclidean distance from Z_n is never greater than k. Then the Euclidean k-neighborhood of Z_n is connected and hence the set Z_n has Euclidean diameter at most $2kd$. Since this diameter tends to infinity, we find that for large n, any spanning path contains a point that is very far from Z_n in the Euclidean sense. Since $|C_n(z)| > 1$ whenever $d(z, Z_n) > 1$ (by monicity), this also shows that the minimum $|C_n|^{2/3}$-length such a path diverges as $n \to \infty$, as required.

On the other hand, Lemma 7.4 gives a compact set $\widehat{K}$ in the interior of P that contains the zeros of the Pick differential for all polygons in a neighborhood U of P. Since $\alpha(C_n) \to P$, for large n we have $\alpha(C_n) \in U$. Since the restrictions of the Pick differential metrics to $\widehat{K}$ vary continuously in the Hausdorff topology (by Corollary 4.5), the diameter of $\widehat{K}$ in the Pick differential metric of $\alpha(C_n)$ is bounded as $n \to \infty$, as is the sequence r_n. This is the desired contradiction.

We conclude that the continuous bijection α is proper, and so α^{-1} is continuous. □

This completes the proof of Theorem A from the introduction.

9 Complements and Conjectures

In this final section we discuss alternative approaches to some of the results proved above and a few directions for further work related to the homeomorphism $\boldsymbol{\alpha} : \mathcal{MC}_d \to \mathcal{MP}_{d+3}$.

9.1 Continuity method. By the Invariance of Domain Theorem, a continuous, locally injective, and proper map between manifolds of the same dimension is a homeomorphism. Using this to establish that a map is homeomorphic is sometimes called the "continuity method".

Since we establish continuity, injectivity, and properness of the map α, the continuity method could be applied to show that it is a homeomorphism. Such an approach would obviate the construction of the inverse map α^{-1} and the need to establish surjectivity of α, giving a slightly shorter proof of the main theorem.

We prefer the argument given in Section 8 because it highlights the way in which existence theorems for the vortex equation and the Monge–Ampere equation (i.e. the Cheng–Yau theorem) give rise to mutually inverse maps between moduli spaces. Also, while the continuity method typically gives only indirect information about the properties of the inverse map, we hope that the explicit constructions of both α and α^{-1} will be helpful toward further study of the local or differential properties of this isomorphism of moduli spaces.

9.2 Hilbert metric geometry. There is a classical projectively invariant Finsler metric on convex domains in $\mathbb{RP}^n$, the *Hilbert metric*, which is defined using projective cross-ratios, generalizing the Beltrami–Klein model of the hyperbolic plane (see e.g. [BK53]). This metric is Finsler and is not Riemannian unless the domain is bounded by a conic [Kay67]. Thus, in general the Hilbert metric is quite different from the Blaschke and Pick differential (Riemannian) metrics considered above.

However, Benoist and Hulin used the Benzécri cocompactness theorem in projective geometry to show that the Hilbert metric of a properly convex domain in $\mathbb{RP}^2$ is uniformly comparable to the Blaschke metric, in the sense that the ratio of their norm functions is bounded above and below by universal constants [BH13, Prop. 3.4]. The same arguments show that some multiple of the Hilbert metric gives an upper bound on the Pick differential metric.

Therefore, in any instance where coarse geometric properties of the Blaschke metric are considered, this comparison principle would allow one to work instead with the Hilbert metric. Since the Hilbert metrics of polyhedra have been extensively studied (e.g. in [dlH93, FK05, Ber09, CV11, CVV11]), one might ask whether such results could be brought to bear on the study of polygonal affine spheres. We mention here only one result in this direction, an alternative proof of a weaker form of Theorem 6.3:

Theorem 9.1. *Suppose $M \subset \mathbb{R}^3$ is an complete affine sphere conformally equivalent to $\mathbb{C}$ and having polynomial Pick differential C. Then M is asymptotic to the cone over a convex polygon.*

Proof. Colbois and Verovic showed that a convex domain in $\mathbb{RP}^n$ whose Hilbert metric is quasi-isometric to a normed vector space (or even which quasi-isometrically embeds in such a space) is a convex polyhedron [CV11]. Applying the $n = 2$ case of this theorem, we can then conclude that the projectivization of M is a convex polygon if we show that its Hilbert metric is quasi-isometric to the Euclidean plane.

For any polynomial cubic differential C, the singular flat metric $|C|^{2/3}$ is quasi-isometric to the Euclidean plane. Hence the Pick differential metric of M has this property.

Theorem 6.2 shows that the Blaschke metric of M comes from a solution of the vortex equation for the polynomial Pick differential, whereupon Corollary 5.2 implies that the Pick differential and Blaschke metrics of M are quasi-isometric. Hence the Blaschke metric of M is quasi-isometric to the plane.

Since the Hilbert metric is bilipschitz to the Blaschke metric, it too is quasi-isometric to the plane, as required. □

Note that this argument does not relate the number of vertices of the polygon to the degree of the polynomial. It would be interesting to know if these Hilbert-geometric techniques could be pushed further to give a complete proof of Theorem 6.3.

9.3 Pick zeros and the Fence conjecture. By Theorem 7.1, each convex polygon P in $\mathbb{RP}^2$ with n vertices is associated to an affine sphere whose Pick differential is a polynomial of degree $(n - 3)$. The Pick zeros therefore give a projective

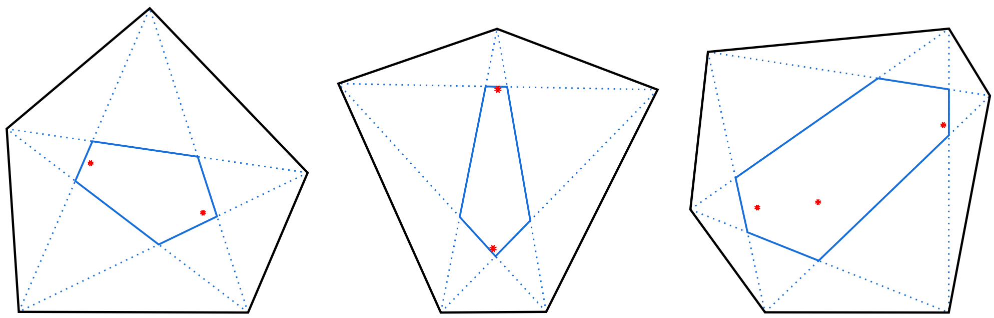

Figure 4: Pick zeros and fences for some convex polygons, computed by numerical solution of the vortex equation and integration of the affine frame field

invariant of P whose value is a set of $(n-3)$ interior points counted with multiplicity. While constructed through transcendental and analytic methods, it would be interesting to understand whether any properties of the Pick zeros can be related directly to the projective or algebraic geometry of the polygon P.

We will state one conjecture in this direction about bounding the Pick zeros in terms of diagonals of P. To formulate it, we first recall the compact region constructed in Section 7 which contains all of the Pick zeros. Corresponding to each vertex v of P there is a smooth arc inside P (a barrier curve) which joins the neighbors v_-, v_+ of v and which lies inside the triangle $\triangle v v_- v_+$. We cut P along these arcs, each time discarding the region on the same side of the barrier as v. What remains is the *core*.

Some simple computer experiments, in which the Pick zeros of some families of convex n-gons were computed for $n \leqslant 7$, suggest that it might be possible to replace the barrier curve at v with the line segment $[v_-, v_+]$ and still bound the Pick zeros (see Figure 4). That is, cutting away from P each of the triangles formed by a consecutive triple of vertices, we obtain a smaller convex n-gon, which we call the *fence*, and we conjecture:

Conjecture 9.2 (Fence conjecture). *For any convex polygon in $\mathbb{RP}^2$ with $n \geqslant 5$ vertices, the Pick zeros lie inside the fence. Equivalently, the Pick differential of the affine sphere over a convex polygon has no zeros over the vertex inscribed triangles.*

The restriction to $n \geqslant 5$ only excludes cases in which the conjecture is vacuous or trivial: For triangles there are no Pick zeros. For quadrilaterals, the fence always reduces to a point, which is the intersection of the two diagonals. All convex quadrilaterals are projectively equivalent, and symmetry considerations show the intersection of diagonals is also the unique zero of the Pick differential (for which we could choose $C = z dz^3$ as a representative).

Finally we note that the fence conjecture can be seen as a limiting case of the construction of the barrier curves in Section 7: that construction involves the choice of a real constant R so that the barrier curves correspond to $\{\mathrm{Re}(z) = R\}$ in the

Țițeica affine spheres over the inscribed triangles. Lemma 7.2 implies that any sufficiently large R can be used. On the other hand, the fence is obtained as the limit of these curves when R tends to $-\infty$. Thus the fence conjecture would follow if the hypothesis $\{\mathrm{Re}(z) > R\}$ could be dropped in Lemma 7.2.

9.4 Differentiability, Poisson structures, and flows. The spaces $\mathcal{TC}_d$ and $\mathcal{TP}_{d+3}$ are smooth manifolds, and we have shown that the map α is a homeomorphism between them. We expect that α has additional regularity:

CONJECTURE 9.3. *The map $\alpha : \mathcal{TC}_d \to \mathcal{TP}_{d+3}$ is a diffeomorphism.*

Of course, since α is $\mathbb{Z}/(d+3)$-equivariant, the conjecture is equivalent to the statement that the quotient map $\boldsymbol{\alpha} : \mathcal{MC}_d \to \mathcal{MP}_{d+3}$ is a diffeomorphism of orbifolds.

The differentiability of α itself would follow from a sufficiently strong estimate concerning the smooth dependence of the solution to Wang's equation on the holomorphic cubic differential. While estimates of this type are routine when considering a fixed compact subset of the domain, the global nature of the map α would seem to require more control. For example, the constructions of Section 6 show that the vertices of the polygon $\alpha(C)$ are determined by fine limiting behavior of the Blaschke metric at infinity, through the unipotent factors constructed in Lemma 6.5.

Similarly, the differentiability of α^{-1} might be established by studying the dependence of the k-jet of the Blaschke metric at a point of a convex polygon as a function of the vertices, generalizing Theorem 4.4. However, one would also need to control the variation of the uniformizing coordinate z in which the Pick differential becomes a polynomial.

Assuming for the moment that α is a diffeomorphism, several questions arise about its possible compatibility with additional differential-geometric structures of its domain and range.

For example, in addition to its complex structure, the space $\mathcal{MC}_d$ carries a holomorphic action of $\mathbb{C}^*$, which is the quotient of the action on polynomials by scalar multiplication. Restricting to the subgroup $S^1 = \{e^{i\theta}\} \subset \mathbb{C}^*$ gives a flow on $\mathcal{MC}_d$ with closed leaves, which we call the *circle flow*. Since the Wang equation involves the cubic differential only through its norm, the Blaschke metric (as a function on $\mathbb{C}$) is constant on these orbits, and the associated affine spheres are intrinsically isometric. The extrinsic geometry is necessarily changing, however, since the image of a (nontrivial) S^1-orbit by $\boldsymbol{\alpha}$ is a circle in $\mathcal{MP}_{d+3}$.

It is natural to ask whether the images of circle orbits in $\mathcal{MP}_{d+3}$ could be recognized in terms of intrinsic features of that space, or in terms of projective geometry of polygons, without direct reference to the map $\boldsymbol{\alpha}$.

For example, there is a natural Poisson structure on a space closely related to $\mathcal{MP}_k$: Define a *twisted polygon* with k vertices to be a map $P : \mathbb{Z} \to \mathbb{RP}^2$ that conjugates the translation $i \mapsto (i+k)$ of $\mathbb{Z}$ with a projective transformation $M \in \mathrm{SL}_3\,\mathbb{R}$. Here we say M is the *monodromy* of the twisted polygon. The space $\tilde{\mathcal{MP}}_k$ of $\mathrm{SL}_3\,\mathbb{R}$-equivalence classes of twisted k-gons is a real algebraic variety which is

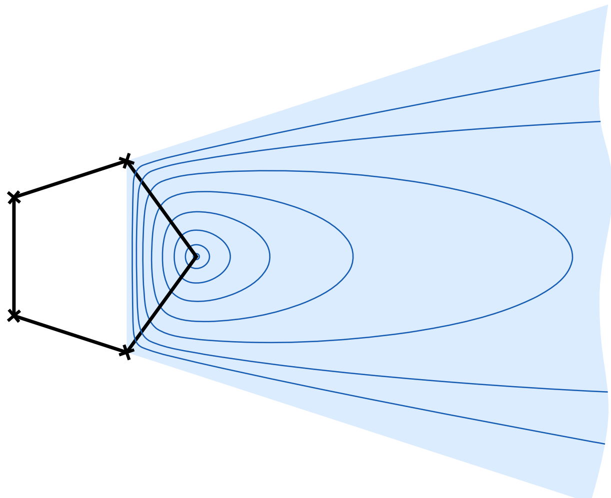

Figure 5: Stratification of $\mathcal{TP}_5$ by level sets of the product of corner invariants. Here $\mathcal{TP}_5$ is identified with the set of possible locations for a fifth vertex when the first four are fixed (*the marked points*). Thus $\mathcal{TP}_5$ is a triangle in $\mathbb{RP}^2$ (*the shaded region*, which extends beyond this affine chart)

stratified by conjugacy classes of the monodromy, and which contains $\mathcal{MP}_k$ as the stratum with trivial monodromy. The variety $\tilde{\mathcal{MP}}_k$ is smooth and of dimension $2k$ in a neighborhood of $\mathcal{MP}_k$.

Ovsienko, Schwartz, and Tabachnikov introduced in [OST10] a natural Poisson structure on $\tilde{\mathcal{MP}}_k$ as part of their study of the *pentagram map*, a dynamical system on polygons and twisted polygons which, in our terminology, maps a convex polygon to its fence. We wonder if this same Poisson structure could be related to the circle flow considered above, and in particular if the circle flow is defined by a Hamiltonian. More precisely, we ask:

Question. Conjugating the circle flow on $\mathcal{MC}_d$ by $\boldsymbol{\alpha}$ we obtain a flow on $\mathcal{MP}_{d+3}$. Is it the restriction of Hamiltonian flow on $\tilde{\mathcal{MP}}_{d+3}$?

We remark that recent work of Bonsante–Mondello–Schlenker give an affirmative answer to a seemingly analogous question for quadratic differentials on compact surfaces: In [BMS13] they introduce a circle flow on $\mathcal{T}(S) \times \mathcal{T}(S)$ that is induced by the $e^{i\theta}$ multiplication of holomorphic quadratic differentials and a harmonic maps construction that involves the $k = 2$ case of the vortex equation (11) from Section 5. In [BMS12] it is shown that this *landslide flow* is Hamiltonian for the product of Weil–Petersson symplectic structures.

We close with a final conjecture about the circle flow which was suggested by computer experiments in the pentagon case ($d = 2$).

Associated to each vertex v of a polygon P in $\mathbb{RP}^2$ there is a projective invariant $x_v = x_v(P) \in \mathbb{R}$ known as the *corner invariant*. It is defined as follows: Consider the chain of five consecutive vertices of the polygon in which v is the middle. Join v to each other vertex in the chain by lines, obtaining four lines that are concurrent at v. The cross ratio of these lines is x_v.

Schwartz observed in [Sch92] that a pentagon in $\mathbb{RP}^2$ is uniquely determined up to projective transformations by its corner invariants, and more generally that the product $X = \prod_v x_v$ of the corner invariants is a "special" function on the space of pentagons. For example, this function is invariant under the pentagram map and it has a unique minimum at the regular pentagon. The non-minimal level sets of the function X foliate the rest of $\mathcal{MP}_5$ by real-algebraic curves homeomorphic to S^1.

Our computational experiments suggest that these curves are images of orbits of the circle flow on $\mathcal{MC}_2$:

Conjecture 9.4. *The map $\boldsymbol{\alpha} : \mathcal{MC}_2 \to \mathcal{MP}_5$ sends each orbit of the circle flow to a level set of the product of the corner invariants.*

The corner invariants and stratification by level sets of X can also be defined on the manifold cover $\mathcal{TP}_5$, and we recall from Section 2 that this space is naturally a triangle in $\mathbb{RP}^2$. The corresponding stratification of this triangle is shown in Figure 5.

Generalizing the previous conjecture, it would be interesting to know whether there exist any nontrivial circle orbits in $\mathcal{MC}_d$ that map to real-algebraic curves in $\mathcal{MP}_{d+3}$ for $d > 2$, or more generally whether these circle orbits are contained in real-algebraic subvarieties of positive codimension. Positive answers would evince a compatibility between $\boldsymbol{\alpha}$ and the algebraic structure of $\mathcal{MP}_{d+3}$.

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Appendix A: Existence of Standard Half-Planes

In this appendix, we construct the half-plane subdomains we need for our analysis of the large scale geometry of the affine spheres in Section 7 and of the basic decay estimates for the general vortex equation in Section 5.4. We begin with the case of finding half-planes for cubic differentials and then generalize the argument to holomorphic differentials of any order.

A.1 Half-planes for Cubic Differentials. Recall the precise statement from Section 3:

PROPOSITION 10.1 (Standard half-planes). *Let C be a monic polynomial cubic differential. Then there are $(d+3)$ C-right-half-planes $\{(U_k, w_k)\}_{k=0,\dots,d+2}$ with the following properties:*

(i) *The complement of $\bigcup_k U_k$ is compact.*
(ii) *The ray $\{\arg(z) = \frac{2\pi k}{d+3}\}$ is eventually contained in U_k.*
(iii) *The rays $\{\arg(z) = \frac{2\pi(k\pm 1)}{d+3}\}$ are disjoint from U_k.*
(iv) *On $U_k \cap U_{k+1}$ we have $w_{k+1} = \omega^{-1} w_k + c$ for some constant c, and each of w_k, w_{k+1} maps this intersection onto a sector of angle $\pi/3$ based at a point on $i\mathbb{R}$. (Recall $\omega = \exp(2\pi i/3)$.)*
(v) *Each ray of $\star_d$ is a C-quasi-ray of angle zero in the associated half-plane U_k. More generally any Euclidean ray in $\mathbb{C}$ is a C-quasi-ray and is eventually contained in U_k for some k.*

Proof. The point of the proof is to treat C as a small deformation of z^d, and to construct half-planes for C as small deformations of the ones described above for z^d.

We construct U_k and then verify its properties. Define

$$\zeta_k = \tfrac{3}{d+3} z^{\frac{d+3}{3}} \exp\left(\tfrac{2\pi i k}{d+3}\right). \tag{27}$$

Here we use the principal branch of the logarithm to define this fractional power of z, so $\{\zeta_k \in \mathbb{R}^+\}$ corresponds to $\{\arg z = 2\pi k/(d+3)\}$, and so that ζ_k is a conformal coordinate on a sector centered at $\{\arg z = 2\pi k/(d+3)\}$ mapping it to $\mathbb{C}\backslash\mathbb{R}^-$.

For the moment we fix k and for brevity write $\zeta = \zeta_k$. Observe that $z^d dz^3 = d\zeta^3$. For $C = C(z)dz^3$ where $C(z)$ is a general monic polynomial of degree d, we instead have

$$C = (1 + O(|\zeta|^{-\frac{3}{d+3}}))d\zeta^3$$

where the implicit constant depends on C but can be made uniform if an upper bound is imposed on the coefficients of the polynomial.

Restricting attention to $|\zeta|$ large enough so that C is nonzero, we find that C has holomorphic cube root of the form

$$\sqrt[3]{C} = (1 + O(|\zeta|^{-\frac{3}{d+3}}))d\zeta. \tag{28}$$

Fix a small $\epsilon > 0$. For any $s \gg 0$, consider the region

$$\Omega_{\epsilon,s} = \left\{\zeta \mid \arg(\zeta - s) \in \left(-\tfrac{\pi}{2} - \epsilon, \tfrac{\pi}{2} + \epsilon\right)\right\}$$

which is a "slightly enlarged C-right half-plane" with s on its boundary.

This domain has the following properties:

- It is nearly convex, i.e. any pair of points $x, y \in \Omega_{\epsilon,s}$ are joined by a path $\gamma \subset \Omega_{\epsilon,s}$ of length $|\gamma| \leqslant L|x-y|$, for some $L = L(\epsilon) > 1$, and
- The real part of ζ approaches $-\infty$ at a linear rate on the boundary, i.e. if $\zeta \in \partial\Omega_{\epsilon,s}$ and $|\zeta|$ is large enough, then $-\operatorname{Re}(\zeta) \geqslant c|\zeta|$ for some $c > 0$.
- It is far from the origin, i.e. the minimum of $|\zeta|$ on $\Omega_{\epsilon,s}$ is $s\cos(\epsilon) \gg 0$.

Since $\frac{d}{d+3} < 1$, the estimate (28) and the last property above show that for any $\delta > 0$ we can choose s large enough so that

$$\left| \frac{\sqrt[3]{C}}{d\zeta} - 1 \right| < \delta \tag{29}$$

throughout $\Omega_{\epsilon,s}$.

Now integrate $\sqrt[3]{C}$ as in (3) to get a natural coordinate w for C. The estimate above shows that w is approximately a constant multiple of ζ. For example, it follows easily from this bound and the near-convexity of $\Omega_{\epsilon,s}$ that w is injective on $\Omega_{\epsilon,s}$ as long as $L\delta < 1$. Fix s large enough so that this holds.

The linear growth of $-\operatorname{Re}(\zeta)$ on $\partial\Omega_{\epsilon,s}$ and the sublinear bound on $(\sqrt[3]{C}/d\zeta - 1)$ from (28) also show that $\operatorname{Re}(w)$ is bounded from above on $\partial w(\Omega_{\epsilon,s})$, and therefore that $w(\Omega_{\epsilon,s})$ contains a half-plane $\{\operatorname{Re}(w) > t\}$. Let $U_k^{(t)}$ denote the region in the z-plane corresponding to $\{\operatorname{Re}(w) > t\}$, and let $w_k = w - t$ be the adjusted natural coordinate making $(U_k^{(t)}, w_k)$ into a C-right-half-plane.

Applying (28) again we can estimate the shape of $U_k^{(t)}$ in the ζ coordinate: It is a perturbation of a right half-plane $\{\operatorname{Re}(\zeta) > c\}$ by $o(|\zeta|)$, and thus for any $\epsilon' > 0$ it contains all but a compact subset of the sector of angle $\pi - \epsilon'$ centered on $\{\zeta \in \mathbb{R}^+\}$. It also follows that for t large enough, the set $U_k^{(t)}$ is disjoint from the rays $\arg(\zeta) = \pm 2\pi/3$ (which lie in the left half-plane $\{\operatorname{Re}(\zeta) < 0\}$). Fixing such t, let $U_k = U_k^{(t)}$.

Using $z = \left(\frac{d+3}{3}\zeta\right)^{\frac{3}{d+3}}$, we have corresponding estimates for the shape of U_k in the z coordinate, which show that it is asymptotic to a sector of angle $\frac{3\pi}{d+3}$. More precisely, for any $\delta' > 0$ there exists $R > 0$ so that U_k contains the part of a sector of angle $\frac{3\pi}{(d+3)} - \delta'$ outside the R-disk, i.e.

$$\left\{ |z| > R \text{ and } \left| \arg(z) - \tfrac{2\pi k}{d+3} \right| < \tfrac{3\pi}{2(d+3)} - \tfrac{\delta'}{2} \right\} \subset U_k. \tag{30}$$

Here R depends on k for the moment, and we also note that U_k is disjoint from $\{\arg(z) = \frac{2\pi(k\pm1)}{d+3}\}$. The latter condition means that U_k is contained in the sector

$$U_k \subset \left\{ \left| \arg(z) - \tfrac{2\pi k}{d+3} \right| < \tfrac{2\pi}{(d+3)} \right\}. \tag{31}$$

Repeating the construction above for each k we obtain $(d+3)$ such C-right-half-planes, and by taking a maximum over radii of excluded balls, we assume (30) holds for a uniform constant R.

To complete the proof we must verify (i)–(v).

Property (ii) is an immediate consequence of (30) and (iii) is immediate from (31). Taking the union of (30) over $0 \leqslant k \leqslant (d+2)$ also shows $\{|z| > R\} \subset \bigcup_k U_k$, giving (i).

Now we consider the relation between natural coordinates on $U_k \cap U_{k+1}$. Since any two natural coordinates are related by an additive constant and a power of ω, the ratio dw_{k+1}/dw_k is constant. To establish (iv) we need only show this constant is equal to ω. It is immediate from (27) that the coordinates ζ_k, ζ_{k+1} satisfy $d\zeta_{k+1}/d\zeta_k = \omega$ on their common domain. Since the natural coordinate w_k for U_k satisfies $dw_k = (1 + o(|\zeta_k|))d\zeta_k$, we find that dw_{k+1}/dw_k approaches ω at infinity, and is therefore equal to ω everywhere.

Finally, the C-right-half-plane U_k is constructed so that the ray $\arg(z) = 2\pi k/(d+3)$ of corresponds to $\{\zeta \in \mathbb{R}^+\}$, and integrating (28) along this path shows that it is a C-quasi-ray of angle zero. Similarly, any ray in $\mathbb{C}$ eventually lies in one of the sectors (30), and therefore in some U_k, where (28) shows it is a C-quasi-ray. Thus (v) follows. $\square$

10.1 Half-planes for k-differentials. We now extend and adapt some of the previous discussion of half-planes to k-differentials $\phi = \phi(z)dz^k$, where $\phi(z)$ is a polynomial of degree d. These results are used in Section 5.4.

Define a $|\phi|$-*upper-half-plane* to be a pair (U, w) where $U \subset \mathbb{C}$ is an open set and $w : U \to \mathbb{H}$ a conformal map to the upper half-plane $\mathbb{H}$ such that $|\phi| = |dw|^k$ on U. We will show that every point in $\mathbb{C}$ that is far enough from the zeros of ϕ lies in such a half-plane.

Note that unlike the discussion for cubic differentials above, the phase of ϕ is ignored here; a $|\phi|$-upper-half-plane is also a $|e^{i\theta}\phi|$-upper-half-plane. We are also constructing *upper* half-planes for the absolute value of a k-differential, rather than the *right* half-planes for a cubic differential that we did previously. These different conventions are convenient for the respective applications of the constructions in the main text.

Define $r : \mathbb{C} \to \mathbb{R}^{\geqslant 0}$ by

$$r(p) = \mathrm{d}_{|\phi|}(p, \phi^{-1}(0))$$

where $\mathrm{d}_{|\phi|}(\cdot, \cdot)$ denotes the distance function associated to the singular flat metric $|\phi|^{2/k}$. Thus r is the $|\phi|^{2/k}$-distance to the zeros of ϕ, or equivalently the maximal radius of a flat disk that embeds in $(\mathbb{C}, |\phi|^{2/k})$ with center at p.

Proposition A.1. *Let $\phi = \phi(z)dz^k$ be a k-differential on $\mathbb{C}$ with $\phi(z)$ a monic polynomial of degree k. Let K be a compact set in the plane containing the zeroes of ϕ. Then there are constants C, c, R_0 with $c > 0$ so that for any point $p \in \mathbb{C}$ with $r(p) > R_0$, there exists a $|\phi|$-upper-half-plane (U, w) with $U \cap K = \emptyset$ such that $\mathrm{Im}(w(p)) \geqslant r(p) - C$. In addition, on the boundary of this half-plane we have $r(x) \geqslant c|\mathrm{Re}(w(x))|$, for x large.*

Proof. Pulling back by $z \mapsto e^{i\theta} z$ we can reduce to the case where $p \in \mathbb{R}$, at the cost of replacing the monic polynomial with one having leading coefficient of unit modulus. We assume this from now on.

The basic existence argument is very similar to Proposition 3.2, so we will simply explain what must be changed. (The direct translation of that argument will of course give a ϕ-right half-plane; at the last step we will rotate by $\frac{\pi}{2}$.) Define

$$\zeta = \frac{k}{d+k} z^{\frac{d+k}{k}}$$

using the principal branch of the logarithm, so that $z^d dz^k = (d\zeta)^k$. Expressing ϕ in this coordinate and estimating as in the proof of Proposition 3.2 we find

$$\phi^{\frac{1}{k}} = e^{i\eta}(1 + O(|\zeta|^{-\frac{k}{d+k}}))d\zeta,$$

for some $\eta \in \mathbb{R}$. Now fix a small positive constant ϵ. For any $s > 0$ define

$$\Omega_{s,\epsilon} = \left\{\zeta \mid \arg(\zeta - s) \in \left(-\tfrac{\pi}{2} - \epsilon, \tfrac{\pi}{2} + \epsilon\right)\right\}.$$

For s large enough this region is disjoint from K (hence it contains no zeros of ϕ) and as before we find that integration of $\phi^{\frac{1}{k}}$ gives a conformal mapping of $\Omega_{s,\epsilon}$. More precisely, defining

$$w_0(\zeta) = \int_s^\zeta e^{-i\eta}\phi^{\frac{1}{k}}$$

we have $|dw_0|^k = |\phi|$, $w_0(s) = 0$, and the map w_0 is a small perturbation of a translation, i.e.

$$w_0(\zeta) = (\zeta - s) + O(|\zeta - s|^{\frac{d}{d+k}}). \tag{32}$$

We fix such s, noting that this constant can be taken to depend only on the coefficients of the polynomial $\phi(z)$ (and not on the point p under consideration).

The estimate above shows that the boundary of $w_0(\Omega_s)$ is approximated by the union of two rays $\{\arg w_0 = \pm(\frac{\pi}{2} + \epsilon)\}$, with the actual boundary being a displacement of this by $o(|w_0|)$. In particular $\mathrm{Re}(w_0) \to -\infty$ linearly along this boundary curve, and the same holds for a sufficiently small rotation of this region about the origin, e.g. the set $e^{i\theta} w_0(\Omega_s)$ for $\theta < \frac{\epsilon}{2}$. We conclude that $e^{i\theta} w_0(\Omega_s)$ contains a right half-plane $\{\mathrm{Re}(w_0) > t\}$ for a constant t depending only on the coefficients of $\phi(z)$, and for all sufficiently small θ.

Estimate (32) also shows that $\arg(w_0(p))$ is small for large $p \in \mathbb{R}^+$. Assuming p is large enough so that $\arg w_0(p) < \frac{\epsilon}{2}$, and defining $w_1 = e^{-i \arg w_0(p)} w_0$ we have $w_1(p) \in \mathbb{R}^+$ and $w_1(\Omega_{s,\epsilon})$ contains $\{\mathrm{Re}(w_1) > t\}$. Finally, taking $w = i(w_1 - t)$ we get a conformal map onto the upper half-plane $\mathbb{H}$ taking p to a point on $i\mathbb{R}$.

We claim (U, w) is the desired $|\phi|$-upper-half-plane, where U is the region in the z-plane corresponding to $w^{-1}(\mathbb{H}) \subset \Omega_{s,\epsilon}$. First, we have constructed this set under the assumption that $|p|$ is larger than some constant depending on the coefficients of $\phi(z)$; since the function r is continuous, we can choose R_0 so that $r(p) > R_0$ implies that $|p|$ is sufficiently large. We have $|dw| = |dw_0|$ and thus $|dw|^k = |\phi|$ and this region is a $|\phi|$-upper-half-plane. The function r grows linearly on the boundary of U because the boundary of $w(\Omega_{s,\epsilon})$ approximates (with sublinear error) a ray whose argument differs from that of the boundary of U by at least $\frac{\epsilon}{2}$; this ensures a zero-free disk centered at each boundary point of U with $|\phi|$-radius growing linearly with $|w|$, giving the desired constant c.

Finally, we must consider the relation between $\mathrm{Im}(w(p))$ and $r(p)$. Let p_0 denote the point that corresponds to the origin in the w-plane. The segment on $i\mathbb{R}$ in the w-plane from p to p_0 is a $|\phi|$-geodesic of length $\mathrm{Im}(w(p))$, hence $r(p) \leqslant \mathrm{Im}(w(p)) + r(p_0)$. But from the definition of the map w we see that, in the z-plane, the point p_0 has modulus bounded in terms of the constants s and t chosen above, which in turn depend only on the coefficients of ϕ. Thus $r(p_0)$ is bounded by the supremum of r on a fixed closed disk in the z-plane, and taking C to be this supremum we conclude $\mathrm{Im}(w(p)) \geqslant r(p) - C$. □

Appendix B: ODE Asymptotics

In this section we collect some results on asymptotics of solutions to initial value problems for ODE that are used in Section 6. These techniques and results are certainly well-known; our goal here is simply to collect precise statements and corresponding references to standard texts.

We consider the equation

$$F'(t) = F(t)A(t) \tag{33}$$

on intervals $J \subset \mathbb{R}$, where the coefficient $A : J \to \mathfrak{gl}_n\mathbb{R}$ is a continuous function and the solution is a matrix-valued function $F : J \to \mathrm{GL}_n\,\mathbb{R}$. This equation is equivalent to the statement that $A(t)dt = F(t)^{-1}dF(t)$ is the pullback of the Maurer–Cartan form on $\mathrm{GL}_n\,\mathbb{R}$ by the map F.

In the small coefficient case (A near zero), one expects the solution to (33) to be approximately constant. To quantify this, fix a norm $\|\cdot\|$ on the space of $n \times n$ matrices. Considering bounded and unbounded intervals separately, we have:

Lemma B.1. (i) *There exist $C, \delta_0 > 0$ such that if $\|A(t)\| < \delta < \delta_0$ for all $t \in [a, b]$, then the solution F of (33) with $F(a) = I$ satisfies $|F(t) - I| < C\delta$ for all $t \in [a, b]$. The constants C and δ_0 can be taken to depend only on an upper bound for $|b - a|$.*

(ii) *If $\int_a^\infty \|A(t)\| dt < \infty$ then any solution of (33) on $[a, \infty)$ satisfies $F(t) \to F_0$ as $t \to \infty$, for some $F_0 \in \mathrm{GL}_n\,\mathbb{R}$.* □

While we have stated these results only for the $\mathrm{GL}_n\,\mathbb{R}$ case, they are standard facts about linear ODE that can be found, for example, in [Har02]: Part (i) is an application of [Har02, Lemma IV.4.1] to the equation satisfied by $F(t) - I$, while (ii) follows from [Har02, Theorem X.1.1].

Next we consider the case when the coefficient $A(t)$ is not pointwise bounded, but instead is nearly concentrated in a 1-dimensional subspace of $\mathfrak{gl}_n\mathbb{R}$ and has bounded mass.

Lemma B.2. *There exist $M, C, \delta_1 > 0$ with the following property: Let $A(t) = s(t) \cdot X + B(t)$ where $X \in \mathfrak{gl}_n\mathbb{R}$ and $s : [a,b] \to \mathbb{R}$ and $B : [a,b] \to \mathfrak{gl}_n\mathbb{R}$ are continuous functions. If $\int_a^b |s(t)|\, dt < M$ and $\|B(t)\| < \delta < \delta_1$ for all $t \in [a,b]$, then the solution of (33) with $F(a) = I$ satisfies*

$$\left\| F(t) - \exp\left(\left(\int_a^t s(t)dt \right) \cdot X \right) \right\| \leqslant C\delta,$$

for all $t \in [a,b]$.

Proof. Define $G(t) = \exp\left(\left(\int_a^t s(t)\, dt\right) \cdot X\right)$. This function satisfies $G'(t) = G(t)s(t)X$ and $G(a) = I$. The integral bound on $s(t)$ and continuity of the exponential map give a uniform upper bound on $\|G(t)\|$ in terms of M and $\|X\|$.

Let $H(t) = F(t)G(t)^{-1}$. Then we have

$$H'(t) = H(t) \left(G(t)B(t)G(t)^{-1} \right).$$

Choosing δ_1 small enough and using the uniform bound on $\|G(t)\|$ we can assume that $\|G(t)B(t)G(t)^{-1}\| < C'\delta < \delta_0$ for some C', where δ_0 is the constant from Lemma B.1. Applying part (i) of that lemma to the equation above we obtain

$$\|F(t)G(t)^{-1} - I\| = \|H(t) - I\| < C''\delta.$$

Since $\|G(t)\|$ is bounded this gives $\|F(t) - G(t)\| < C\delta$ as desired. □

References

[AS72] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications, New York (1972).

[BB04] O. Biquard and P. Boalch. Wild non-abelian Hodge theory on curves. *Compositio Mathematica*, (1)**140** (2006), 179–204.

[Ber09] A. Bernig. Hilbert geometry of polytopes. *Archiv der Mathematik (Basel)*, (4)**92** (2009), 314–324.

[BF42] Ch. Blanc and F. Fiala. Le type d'une surface et sa courbure totale. *Commentarii Mathematici Helvetici*, **14** (1942), 230–233.

[BH13] Y. Benoist and D. Hulin. Cubic differentials and finite volume convex projective surfaces. *Geometry and Topology*, (1)**17** (2013), 595–620.

[BH14] Y. Benoist and D. Hulin. Cubic differentials and hyperbolic convex sets. *Journal of Differential Geometry*, (1)**98** (2014), 1–19.

[BK53] H. Busemann and P. Kelly. *Projective Geometry and Projective Metrics*. Academic Press Inc, New York (1953).

[Bla23] W. Blaschke. *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. II. Affine Differentialgeometrie, bearbeitet von K. Reidemeister. Erste und zweite Auflage*. Springer, Berlin, (1923).

[BMS12] F. Bonsante, G. Mondello, and J.-M. Schlenker. A cyclic extension of the earthquake flow II (2012) (preprint). arXiv:1208.1738

[BMS13] F. Bonsante, G. Mondello, and J.-M. Schlenker. A cyclic extension of the earthquake flow I. *Geometry and Topology*, (1)**17**(2013), 157–234.

[Boa14] P. Boalch. Geometry and braiding of Stokes data; fission and wild character varieties. *Annals of Mathematics (2)*, (1)**179** (2014), 301–365.

[Bra91] S.B. Bradlow. Special metrics and stability for holomorphic bundles with global sections. *Journal of Differential Geometry*, (1)**33** (1991), 169–213.

[Cal72] E. Calabi. Complete affine hyperspheres. I. In: *Symposia Mathematica*, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971). Academic Press, London (1972), pp. 19–38.

[CV11] B. Colbois and P. Verovic. Hilbert domains that admit a quasi-isometric embedding into Euclidean space. *Advances in Geometry*, (3)**11** (2011), 465–470.

[CVV11] B. Colbois, C. Vernicos, and P. Verovic. Hilbert geometry for convex polygonal domains. *Journal of Geometry*, (1–2)**100** (2011), 37–64.

[CY75] S.Y. Cheng and S.-T. Yau. Differential equations on Riemannian manifolds and their geometric applications. *Communications on Pure and Applied Mathematics*, (3)**28** (1975), 333–354.

[CY86] S.Y. Cheng and S.-T. Yau. Complete affine hypersurfaces. I. The completeness of affine metrics. *Communications on Pure and Applied Mathematics*, (6)**39** (1986), 839–866.

[dF04] T. de Fernex. On planar Cremona maps of prime order. *Nagoya Mathematical Journal*, **174** (2004), 1–28.

[dlH93] P. de la Harpe. On Hilbert's metric for simplices. In: *Geometric Group Theory, Vol. 1 (Sussex, 1991). London Mathematical Society Lecture Note Series*, Vol. 181. Cambridge University Press, Cambridge (1993), pp. 97–119.

[Dun12] M. Dunajski. Abelian vortices from sinh-Gordon and Tzitzeica equations. *Physics Letters B*, (1)**710** (2012), 236–239.

[FG07] V.V. Fock and A.B. Goncharov. Moduli spaces of convex projective structures on surfaces. *Advances in Mathematics*, (1)**208** (2007), 249–273.

[Fin65] R. Finn. On a class of conformal metrics, with application to differential geometry in the large. *Commentarii Mathematici Helvetici*, **40** (1965), 1–30.

[FK05] T. Foertsch and A. Karlsson. Hilbert metrics and Minkowski norms. *Journal of Geometry*, (1–2)**83** (2005), 22–31.

[Gig81] S. Gigena. On a conjecture by E. Calabi. *Geometriae Dedicata*, (4)**11** (1981), 387–396.

[GL50] V.L. Ginburg and L.D. Landau. *Zh. Eksp. Teor. Fiz.*, **20** (1950), 1064. English translation in Collected Papers of L.D. Landau, pp. 546–568, Pergamon Press (1965).

[GP94] O. García-Prada. A direct existence proof for the vortex equations over a compact Riemann surface. *Bulletin of the London Mathematical Society*, (1)**26** (1994), 88–96.

[GT83] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order. Grundlehren der Mathematischen Wissenschaften*, Vol. 224, 2nd edn. Springer, Berlin (1983).

[Han96] Z.-C. Han. Remarks on the geometric behavior of harmonic maps between surfaces. In: *Elliptic and Parabolic Methods in Geometry (Minneapolis, MN, 1994)*. A K Peters, Wellesley (1996), pp. 57–66.

[Har02] P. Hartman. *Ordinary Differential Equations. Classics in Applied Mathematics*, Vol. 38. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2002)

[HTTW95] Z.-C. Han, L.-F. Tam, A. Treibergs, and T. Wan. Harmonic maps from the complex plane into surfaces with nonpositive curvature. *Communications in Analysis and Geometry*, (1–2)**3** (1995), 85–114.

[Hub57] A. Huber. On subharmonic functions and differential geometry in the large. *Commentarii Mathematici Helvetici*, **32**(1957), 13–72.

[Jos07] J. Jost. *Partial Differential Equations. Graduate Texts in Mathematics*, Vol. 214, 2nd edn. Springer, New York (2007).

[JT80] A. Jaffe and C. Taubes. *Vortices and Monopoles. Progress in Physics*, Vol. 2. Birkhäuser, Boston (1980). Structure of static gauge theories.

[Kay67] D. Kay. The ptolemaic inequality in Hilbert geometries. *Pacific Journal of Mathematics*, **21** (1967), 293–301.

[Lab07] F. Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure and Applied Mathematics Quarterly* (4, part 1)**3** (2007), 1057–1099.

[Li90] A.M. Li. Calabi conjecture on hyperbolic affine hyperspheres. *Mathematische Zeitschrift*, (3)**203** (1990), 483–491.

[Li92] A.M. Li. Calabi conjecture on hyperbolic affine hyperspheres. II. *Mathematische Annalen*, (3)**293** (1992), 485–493.

[LLS04] A.-M. Li, H. Li, and U. Simon. Centroaffine Bernstein problems. *Differential Geometry and its Applications*, (3)**20** (2004), 331–356.

[Lof01] J. Loftin. Affine spheres and convex $\mathbb{RP}^n$-manifolds. *American Journal of Mathematics*, (2)**123** (2001), 255–274.

[Lof04] J. Loftin. The compactification of the moduli space of convex $\mathbb{RP}^2$ surfaces. I. *Journal of Differential Geometry*, (2)**68** (2004), 223–276.

[Lof07] J. Loftin. Flat metrics, cubic differentials and limits of projective holonomies. *Geometriae Dedicata*, **128** (2007), 97–106.

[Lof10] J. Loftin. Survey on affine spheres. In: *Handbook of Geometric Analysis, No. 2. Advanced Lectures in Mathematics*, Vol. 13. International Press, Somerville (2010), pp. 161–191.

[Lof15] J. Loftin. Convex $\mathbb{RP}^2$ structures and cubic differentials under neck separation (2015) (preprint). arXiv:1506.03895

[LSZ93] A.M. Li, U. Simon, and G.S. Zhao. *Global Affine Differential Geometry of Hypersurfaces. de Gruyter Expositions in Mathematics*, Vol. 11. Walter de Gruyter & Co, Berlin (1993).

[Min92] Y. Minsky. Harmonic maps, length, and energy in Teichmüller space. *Journal of Differential Geometry*, (1)**35** (1992), 151–217.

[NS94] K. Nomizu and T. Sasaki. *Affine Differential Geometry. Cambridge Tracts in Mathematics*, Vol. 111. Cambridge University Press, Cambridge (1994).

[Omo67] H. Omori. Isometric immersions of Riemannian manifolds. *Journal of the Math-*

ematical Society of Japan, **19** (1967), 205–214.

[Oss86] R. Osserman. *A Survey of Minimal Surfaces*, 2nd edn. Dover Publications Inc, New York (1986).

[OST10] V. Ovsienko, R. Schwartz, and S. Tabachnikov. The pentagram map: a discrete integrable system. *Communications in Mathematical Physics*, (2)**299** (2010), 409–446.

[Pic17] G. Pick. Über affine Geometrie iv: Differentialinvarianten der Flächen gegenüber affinen Transformationen. *Leipziger Berichte*, **69** (1917), 107–136.

[PZ02] A.D. Polyanin and V.F. Zaitsev. *Handbook of Linear Partial Differential Equations for Engineers and Scientists*. CRC Press, Boca Raton (2002).

[Sas80] T. Sasaki. Hyperbolic affine hyperspheres. *Nagoya Mathematical Journal*, **77** (1980), 107–123.

[Sch92] R. Schwartz. The pentagram map. *Experimental Mathematics*, (1)**1** (1992), 71–81.

[Sim90] C. Simpson. Harmonic bundles on noncompact curves. *Journal of the American Mathematical Society*, (3)**3** (1990), 713–770.

[Str84] K. Strebel. *Quadratic Differentials. Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Vol. 5. Springer, Berlin (1984).

[SW93] U. Simon and C.P. Wang. Local theory of affine 2-spheres. In: *Differential Geometry: Riemannian Geometry (Los Angeles, CA, 1990). Proceedings of Symposia in Pure Mathematics*, Vol. 54. American Mathematical Society, Providence (1993), pp. 585–598.

[TW02] N.S. Trudinger and X.-J. Wang. Affine complete locally convex hypersurfaces. *Inventiones Mathematicae*, (1)**150** (2002), 45–60.

[Tzi08] G. Tzitzéica. Sur une nouvelle classe de surfaces. *Rendiconti del Circolo Matematico di Palermo*, (1)**25** (1908), 180–187.

[WA94] T.Y. Wan and T.K. Au. Parabolic constant mean curvature spacelike surfaces. *Proceedings of the American Mathematical Society*, (2)**120** (1994), 559–564.

[Wan91] C.P. Wang. Some examples of complete hyperbolic affine 2-spheres in $\mathbf{R}^3$. In: *Global Differential Geometry and Global Analysis (Berlin, 1990). Lecture Notes in Mathematics*, Vol. 1481. Springer, Berlin, (1991), pp. 271–280.

[Wan92] T.Y. Wan. Constant mean curvature surface, harmonic maps, and universal Teichmüller space. *Journal of Differential Geometry*, (3)**35** (1992), 643–657.

[Wit07] E. Witten. From superconductors and four-manifolds to weak interactions. *Bulletin of the American Mathematical Society (N.S.)*, (3)**44** (2007), 361–391 (electronic).

[Wit08] E. Witten. Gauge theory and wild ramification. *Analysis and Applications (Singapore)*, (4)**6** (2008), 429–501.

[Wol91] M. Wolf. High energy degeneration of harmonic maps between surfaces and rays in Teichmüller space. *Topology*, (4)**30** (1991), 517–540.

[Yau75] S.T. Yau. Harmonic functions on complete Riemannian manifolds. *Communications on Pure and Applied Mathematics*, **28** (1975), 201–228.

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