PROJECTIVE STRUCTURES, GRAFTING, AND MEASURED LAMINATIONS

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Abstract. We show that grafting any fixed hyperbolic surface defines a homeomorphism from the space of measured laminations to Teichmüller space, complementing a result of Scannell-Wolf on grafting by a fixed lamination. This result is used to study the relation between the complex-analytic and geometric coordinate systems for the space of complex projective ($\mathbb{CP}^1$) structures on a surface.

We also study the rays in Teichmüller space associated to the grafting coordinates, obtaining estimates for extremal and hyperbolic length functions and their derivatives along these grafting rays.

1. Introduction

In this paper we compare two perspectives on the theory of complex projective structures on surfaces by studying the grafting map of a hyperbolic surface.

A complex projective (or $\mathbb{CP}^1$) structure on a compact surface S is an atlas of charts with values in $\mathbb{CP}^1$ and Möbius transition functions. Let $\mathcal{P}(S)$ denote the space of (isotopy classes of) marked complex projective structures on S, and let $\mathcal{T}(S)$ be the Teichmüller space of (isotopy classes of) marked complex structures on S. Because Möbius maps are holomorphic, there is a forgetful projection $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$.

An analytic tradition, having much in common with univalent function theory, parameterizes the fiber $\pi^{-1}(X)$ using the Schwarzian derivative, identifying $\mathcal{P}(S)$ with the total space of the bundle $\mathcal{Q}(S) \to \mathcal{T}(S)$ of holomorphic quadratic differentials.

A second, more synthetic geometric description of $\mathcal{P}(S)$ is due to Thurston, and proceeds through the operation of *grafting* – a construction which traces its roots back at least to Klein [Kle, §50, p. 230], with a modern history developed by many authors ([Mas1], [Hej], [ST], [Gol], [GKM],[Tan], [McM], [SW]). The simplest example of grafting may be described as follows.

Start with a hyperbolic surface $X \in \mathcal{T}(S)$ and a simple closed geodesic γ on X; then construct a new surface by removing γ from X and replacing it

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Date: April 18, 2007.

2000 *Mathematics Subject Classification.* Primary 30F60.

The first author was partially supported by an NSF postdoctoral research fellowship.

The second author was partially supported by NSF grants DMS-0139877 and DMS-0505603.

with the Euclidean cylinder $\gamma \times [0,t]$. The result is $\mathrm{Gr}_{t\gamma} X$, the *grafting of X by $t\gamma$*, which is a surface with a ($C^{1,1}$ Riemannian) metric composed of alternately flat or hyperbolic pieces. Furthermore, $\mathrm{Gr}_{t\gamma} X$ has a canonical projective structure that combines the Fuchsian uniformization of X and the Euclidean structure of the cylinder $\gamma \times [0,t]$ (for details, see [SW, §1] [Tan, §2] [KT]).

Thurston showed that grafting extends naturally from weighted simple closed geodesics to the space $\mathcal{ML}(S)$ of measured geodesic laminations, and thus defines a map

$$\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S). \tag{1.1}$$

Moreover, this map is a homeomorphism; for a proof of this result, see [KT].

A natural problem is to relate the analytic and geometric perspectives on the space of projective structures, for example by comparing the product structure of $\mathcal{ML}(S) \times \mathcal{T}(S) \simeq \mathcal{P}(S)$ to the bundle structure induced by the projection $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$.

Results on grafting. We compare these two perspectives by studying the *conformal grafting map* $\mathrm{gr} = \pi \circ \mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$, i.e. $\mathrm{gr}_\lambda X$ is the conformal structure which underlies the projective structure $\mathrm{Gr}_\lambda X$. Fixing either of the two coordinates we have the X-grafting map $\mathrm{gr}_{\bullet} X : \mathcal{ML}(S) \to \mathcal{T}(S)$ and the λ-grafting map $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$. These maps reflect how the base coordinate of the complex-analytic fibration $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ is related to the geometric product structure $\mathcal{ML}(S) \times \mathcal{T}(S) \simeq \mathcal{P}(S)$. Our main result is

Theorem 1.1. *For each $X \in \mathcal{T}(S)$, the X-grafting map $\mathrm{gr}_{\bullet} X : \mathcal{ML}(S) \to \mathcal{T}(S)$ is a tangentiable diffeomorphism (and hence a homeomorphism).*

Momentarily deferring a brief discussion of the term *tangentiable diffeomorphism*, we note that this theorem is a natural complement to the result of Scannell-Wolf on the λ-grafting map.

Theorem 1.2 (Scannell-Wolf [SW, Thm. A])**.** *For each $\lambda \in \mathcal{ML}(S)$, the map $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ is a real-analytic diffeomorphism.*

The discrepancy between diffeomorphism and tangentiable diffeomorphism in Theorems 1.1 and 1.2 is related to the lack of a natural differentiable structure on $\mathcal{ML}(S)$. Bonahon showed that grafting is differentiable in the weak sense of being *tangentiable*; see §2 below or [Bon5] for details.

Returning to the original problem of comparing different coordinate systems for $\mathcal{P}(S)$, Theorems 1.1 and 1.2 can be used to study the fiber $P(X) = \pi^{-1}(X)$ and its relation to the grafting coordinates. Let us denote the two factors of the map $\mathrm{Gr}^{-1} : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S)$ by

$$Gr^{-1}(Z) = (p_{\mathcal{ML}}(Z), p_{\mathcal{T}}(Z)).$$

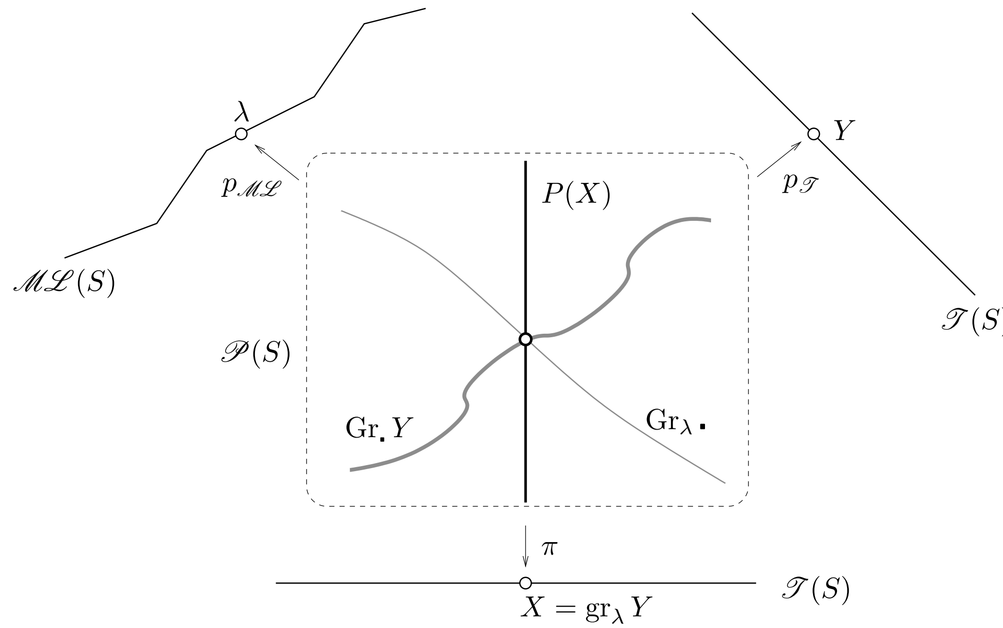

FIGURE 1. The bundle $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ of $\mathbb{CP}^1$ structures over Teichmüller space and the product structure $\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \simeq \mathcal{P}(S)$ induced by grafting.

Thus the maps $p_{\mathcal{ML}} : \mathcal{P}(S) \to \mathcal{ML}(S)$ and $p_{\mathcal{T}} : \mathcal{P}(S) \to \mathcal{T}(S)$ send a projective structure to one of its two grafting coordinates, and we think of them as projections. We prove:

Corollary 1.3. *For each $X \in \mathcal{T}(S)$ the restriction $p_{\mathcal{ML}}|_{P(X)} : P(X) \to \mathcal{ML}(S)$ is a tangentiable diffeomorphism, and $p_{\mathcal{T}}|_{P(X)} : P(X) \to \mathcal{T}(S)$ is a C^1 diffeomorphism.*

This corollary improves the existing regularity results for these projection maps, from which it was known that that $p_{\mathcal{ML}}|_{P(X)}$ is a homeomorphism (a corollary of Theorem 1.2, see [D2, §4]) and that $p_{\mathcal{T}}|_{P(X)}$ is a proper C^1 map of degree 1 (see [Bon5, Thm. 3] and [D1, Lem. 7.6, Thm. 1.1]). The relationship between the maps $p_{\mathcal{T}}$, $p_{\mathcal{ML}}$, and π is represented schematically in Figure 1.

One can also apply Theorem 1.1 and Corollary 1.3 to study the *pruning map* (the inverse of grafting) and the parameterization of quasi-Fuchsian manifolds by their convex hull geometry. We explore these directions in §5.

Methods. From a general perspective, the proof of the main theorem relies on relating two established techniques in hyperbolic geometry. The first is the analytic study of the prescribed curvature (Liouville) equation (see e.g. [Wol4] [ZT], [Tro]), and the second is the complex duality between bending and twisting (see e.g. [Wol1], [Wol2], [Wol3], [Pla], [Ser2], [Ser3]) and its generalization to complex earthquakes and duality between grafting and shearing (see [Bon5], [McM], [EMM]).

The first strand occurs in the proof in [SW] for Theorem 1.2: there standard geometric analytic techniques were applied to the curvature equation to understand how the λ-grafting map changed under small perturbations of the Riemann surface. Such techniques might be applicable to the analogous problem (of the main Theorem 1.1) of understanding how the X-grafting map varies under small perturbations of the measured lamination, but it would necessarily be more involved, due to the local structure of the space $\mathcal{ML}(S)$ being more complicated than that of the space $\mathcal{T}(S)$.

Fortunately, most of the required details for this study of $\mathcal{ML}(S)$ are already in the literature: here we make heavy use of Bonahon's work (following Thurston [Thu3]) on the deformation theory of $\mathcal{ML}(S)$ (see [Bon2] [Bon3] [Bon4] [Bon5]). In particular, the crux of our proof relies on Bonahon's observation that there is a sense in which infinitesimal grafting is complex linear. This complex analyticity, in keeping with the second tradition discussed above, implies that a study of the effect on grafting of infinitesimally changing the measured lamination is, by duality, a study of infinitesimally shearing the hyperbolic surface. Thus we may apply the analysis in the proof of Theorem 1.2 to the problem of the main Theorem 1.1.

Grafting rays. In a final section, we study the coordinate system on Teichmüller space induced by the X-grafting homeomorphism $\mathcal{ML}(S) \to \mathcal{T}(S)$, and analyze the behavior of extremal and hyperbolic length functions on *grafting rays*–paths in $\mathcal{T}(S)$ of the form $t \mapsto \mathrm{gr}_{t\lambda} X$. We show:

Theorem 1.4. *For each $X \in \mathcal{T}(S)$ and $\lambda \in \mathcal{ML}(S)$ the extremal length of λ on $\mathrm{gr}_{t\lambda} X$ is monotone decreasing for all $t \gg 0$ and is asymptotic to $\frac{\ell(\lambda,X)}{t}$, where $\ell(\lambda, X)$ is the hyperbolic length of λ on X.*

Theorem 1.5. *For each $X \in \mathcal{T}(S)$ and any simple closed hyperbolic geodesic $\gamma \in \mathcal{ML}(S)$, the hyperbolic length of γ on $\mathrm{gr}_{t\gamma} X$ is monotone decreasing for all $t \gg 0$ and is asymptotic to $\pi\frac{\ell(\lambda,X)}{t}$.*

The monotonicity and asymptotic behavior described in Theorems 1.4 and 1.5 are combined with explicit estimates on the derivatives of length functions in Theorems 6.2 and 6.6 below.

Organization of the paper.

§2 presents the infinitesimal version of the main theorem (Theorem 2.6), after introducing the necessary background on measured laminations and grafting. The reduction to an infinitesimal statement is modeled on the argument of Scannell-Wolf in [SW], and uses Tanigawa's properness theorem for grafting (Theorem 2.1).

§3 describes shearing deformations of hyperbolic surfaces, closely following the work of Bonahon on shearing coordinates for Teichmüller space. Here we present the crucial complex-linearity result of Bonahon (Theorem 3.4) that is used in the proof of Theorem 2.6.

§4 is devoted to the proofs of Theorems 2.6 and 1.1, which follow using the theory developed in §§2-3.

§5 collects some applications of the main theorem, including the proof of Corollary 1.3 and a rigidity result for quasi-Fuchsian manifolds.

§6 discusses the grafting coordinates for Teichmüller space and the asymptotic behavior of extremal and hyperbolic length functions on grafting rays; Theorems 1.4 and 1.5 and associated derivative estimates are proved here.

Acknowledgements. The authors thank Francis Bonahon, Howard Masur, Yair Minsky, and Robert Penner for stimulating discussions related to this work.

2. Grafting and infinitesimal grafting

We begin with some background on measured laminations, grafting, and tangentiability, which are needed to formulate the main technical result (Theorem 2.6).

Laminations. As in the introduction, S denotes a compact smooth surface of genus $g > 1$ and $\mathcal{T}(S)$ is the Teichmüller space of marked hyperbolic (or conformal) structures on S. We often use $X \in \mathcal{T}(S)$ to represent a particular hyperbolic surface in a given marked equivalence class.

Let $\mathcal{S}$ denote the set of free homotopy classes of simple closed curves on S; we implicitly identify $\gamma \in \mathcal{S}$ with its geodesic representative on a hyperbolic surface $X \in \mathcal{T}(S)$.

A *geodesic lamination* Λ on a hyperbolic surface X is a foliation of a closed subset of X by complete, simple geodesics. Examples of geodesic laminations include simple closed hyperbolic geodesics $\gamma \in \mathcal{S}$ and disjoint unions thereof.

The notion of a geodesic lamination is actually independent of the particular choice of X, in that a geodesic lamination on X determines a geodesic lamination for any other hyperbolic structure $Y \in \mathcal{T}(S)$ in a canonical way (see for example [Bon3, §1]). Thus we speak of a geodesic lamination on S, suppressing the choice of a particular metric. Let $\mathcal{GL}(S)$ denote the set of all geodesic laminations on S with the topology of Hausdorff convergence of closed sets.

A geodesic lamination $\Lambda \in \mathcal{GL}(S)$ is *maximal* if it is not properly contained in another geodesic lamination, in which case the complement of Λ in S is a union of ideal triangles. Every geodesic lamination is contained in a maximal one, though not necessarily uniquely.

Measured laminations. A *transverse measure* μ on a geodesic lamination Λ is an assignment of a positive Borel measure to each compact transversal to Λ in a manner compatible with splitting and isotopy of transversals. Such a measure μ has *full support* if there is no proper sublamination $\Lambda' \subset \Lambda$ such that μ assigns the zero measure to transversals disjoint from Λ'.

Let $\mathcal{ML}(S)$ denote the space of *measured geodesic laminations* on S, i.e. pairs $\lambda = (\Lambda, \mu)$ where $\Lambda \in \mathcal{GL}(S)$ and μ is a transverse measure on Λ of full support. We denote by $\lambda(\tau)$ the total measure assigned to a transversal τ by $\lambda \in \mathcal{ML}(S)$.

The topology on $\mathcal{ML}(S)$ is that of weak-$*$ convergence of measures on compact transversals. The underlying geodesic lamination of $\lambda \in \mathcal{ML}(S)$ is the *support* of λ, written $\operatorname{supp}(\lambda) \in \mathcal{GL}(S)$. The space $\mathcal{ML}(S)$ has an action of $\mathbb{R}^+$ by multiplication of transverse measures; the empty lamination $0 \in \mathcal{ML}(S)$ is the unique fixed point of this action.

For any simple closed geodesic $\gamma \in \mathcal{S}$, there is a measured geodesic lamination (also γ) that assigns to a transversal τ the counting measure on $\tau \cap \gamma$. The rays $\{t\gamma \mid t \in \mathbb{R}^+, \ \gamma \in \mathcal{S}\}$ determined by simple closed curves are dense in $\mathcal{ML}(S)$.

The space $\mathcal{ML}(S)$ is a contractible topological manifold homeomorphic to $\mathbb{R}^{6g-6}$, but it does not have a natural smooth structure. Its natural structure is that of a piecewise linear (PL) manifold, with charts corresponding to train tracks.

Detailed discussion of the space $\mathcal{ML}(S)$ can be found in [Thu1] [EM] [PH] [Ota].

Grafting. As mentioned in the introduction, Thurston showed that grafting along simple closed curves has a natural extension to measured laminations, giving a projective grafting homeomorphism $\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$ and a conformal grafting map $\mathrm{gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$. Tanigawa showed that the latter is a proper map when either one of the two parameters is fixed:

Theorem 2.1 (Tanigawa [Tan])**.** *For any $\lambda \in \mathcal{ML}(S)$, the λ-grafting map $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ is proper. For any $X \in \mathcal{T}(S)$, the X-grafting $\mathrm{gr}_{\bullet} X : \mathcal{ML}(S) \to \mathcal{T}(S)$ is proper.*

The properness of the restricted grafting maps is used in the proofs of Theorems 1.1 (in §4) and 1.2 (in [SW]) to reduce a global statement to a local one, which is then attacked using infinitesimal methods. In the case of λ-grafting, the infinitesimal analysis is possible because $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ is differentiable, and even real-analytic [McM, Cor. 2.11]. (A related real-analyticity property along rays in $\mathcal{ML}(S)$ is discussed in §6 below.)

The main step [SW] was to show that the differential map $d\mathrm{gr}_\lambda : T_X\mathcal{T}(S) \to T_{\mathrm{gr}_\lambda X}\mathcal{T}(S)$ is an isomorphism. Once that is established, Theorem 1.2 follows easily, since gr_λ is then a proper local diffeomorphism of $\mathcal{T}(S)$, hence a covering map of the simply connected space $\mathcal{T}(S)$.

We will follow an analogous outline in the proof of Theorem 1.1, but the infinitesimal analysis is complicated by lack of smooth structure on $\mathcal{ML}(S)$, so the derivative of X-grafting does not exist in the classical sense. Instead we must use a weak notion of differentiability based on one-sided derivatives, which we now discuss.

Tangentiability. A *tangentiable* map $f : U \to V$ between open sets in $\mathbb{R}^n$ is a map with "one-sided" directional derivatives everywhere; in other words, for each $x \in U$ and $v \in \mathbb{R}^n$, the limit

$$\left.\frac{d}{dt}\right|_{t=0^+} f(x+tv) = \lim_{t\to 0^+} \frac{f(x+tv)-f(x)}{t} \tag{2.1}$$

exists, and the convergence is locally uniform in v (for equivalent conditions, see [Bon5, §2]. This convergence allows us to define $T_x f : \mathbb{R}^n \to \mathbb{R}^n$, the *tangent map of f at x*, by

$$T_x f(v) = \left.\frac{d}{dt}\right|_{t=0^+} f(x+tv)$$

Of course if f is a differentiable, then $T_x f$ is just the derivative of f at x, a linear map. When f is only tangentiable, the map $T_x f$ is continuous and homogeneous of degree 1 [Bon5, §1].

A *tangentiable manifold* is one whose transition functions are tangentiable maps; examples include smooth manifolds and PL manifolds. Thus $\mathcal{ML}(S)$ has a natural tangentiable structure. The tangent space $T_x M$ at a point x of a tangentiable manifold is not naturally a vector space, but has the structure of a cone. The notion of tangentiable map extends naturally to tangentiable manifolds.

We will say that a homeomorphism between two tangentiable manifolds is a *tangentiable diffeomorphism* if it and its inverse are tangentiable, and if the tangent maps are everywhere homeomorphisms. A convenient criterion for this is provided by the

Lemma 2.2 (Bonahon [Bon5, Lem. 4])**.** *Let $f : M \to N$ be a homeomorphism between tangentiable manifolds. If f is tangentiable, and all of its tangent maps are injective, then f is a tangentiable diffeomorphism.*

Bonahon showed that grafting is compatible with that tangentiable structure of $\mathcal{ML}(S)$ in the following sense:

Theorem 2.3 (Bonahon [Bon5, Thm. 3])**.** *The grafting map* $\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$ *is a tangentiable diffeomorphism. In particular, the conformal grafting map* $\mathrm{gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$ *is tangentiable, and for each $X \in \mathcal{T}(S)$, the X-grafting map* $\mathrm{gr}_{\bullet} X : \mathcal{ML}(S) \to \mathcal{T}(S)$ *is tangentiable.*

In [Bon5], Bonahon actually computes the tangent map of grafting to show that grafting is tangentiable. After developing the shearing coordinates in §3, Bonahon's description of the tangent map (from which Theorem 2.3 is derived) is given in Theorem 3.4.

Differentiability. A curious feature of the tangentiability of grafting with respect to $\mathcal{ML}(S)$ is that some fragments of classical differentiability remain. For example, the inverse of the projective grafting map $\mathrm{Gr}^{-1} : \mathcal{P}(S) \to \mathcal{ML}(S) \times \mathcal{T}(S)$ factors into the two projections $p_{\mathcal{ML}}$ and $p_{\mathcal{T}}$ (as described in the introduction). By Theorem 2.3, these too are tangentiable maps, but since both the domain and range of $p_{\mathcal{T}}$ are smooth manifolds, it makes sense to ask if this map is differentiable in the usual sense. Extending Theorem 2.3, Bonahon shows

Theorem 2.4 (Bonahon [Bon5, Thm. 3])**.** *The map $p_{\mathcal{T}} : \mathcal{P}(S) \to \mathcal{T}(S)$ is C^1 but not C^2.*

Finally we observe that $p_{\mathcal{T}}(\mathrm{Gr}_\lambda X) = X$, so for each $\lambda \in \mathcal{ML}(S)$ we have $p_{\mathcal{T}} \circ \mathrm{Gr}_\lambda = \mathrm{Id}$, that is, the map $\mathrm{Gr}_\lambda : \mathcal{T}(S) \to \mathcal{P}(S)$ is a smooth section of $p_{\mathcal{T}}$. Since $\mathrm{Gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$ is a homeomorphism, these sections fill up $\mathcal{P}(S)$, and we conclude

Corollary 2.5. *The map $p_{\mathcal{T}}$ is a C^1 submersion.*

Infinitesimal X-grafting. Using the tangentiability of grafting, we can formulate the infinitesimal statement that will be our main tool in the proof of Theorem 1.1:

Theorem 2.6. *The tangent map $T_\lambda \, \mathrm{gr}_{\bullet} X$ of the X-grafting map has no kernel. That is, if λ_t is a tangentiable family of measured laminations and $\frac{d}{dt}\big|_{t=0^+} \mathrm{gr}_{\lambda_t} X = 0$, then $\frac{d}{dt}\big|_{t=0^+} \lambda_t = 0$.*

In the next two sections, we develop machinery to prove this result about the derivative of grafting, then strengthen it to a local injectivity result in order to prove Theorem 1.1. Complications arise in both steps because the maps under consideration are tangentiable rather than smooth.

3. Shearing

In this section we describe the machinery of shearing cocycles for geodesic laminations on a hyperbolic surface, borrowing heavily from the papers of Bonahon [Bon4], [Bon2].

Cocycles. Let G be an abelian group and $\Lambda \in \mathcal{GL}(S)$. A *$G$-valued cocycle on* Λ is a map α that assigns to each transversal τ to Λ an element $\alpha(\tau) \in G$ in a manner compatible with splitting and transversality-preserving isotopy. The G-module of all G-valued cocycles on Λ is denoted $\mathcal{H}(\Lambda, G)$.

Of particular interest for our purposes is the vector space of cocycles for *maximal* laminations with values in $\mathbb{R}$. While it is perhaps not clear from the definition, this vector space is finite dimensional, and the dimension is the same for all maximal laminations:

Theorem 3.1 (Bonahon [Bon2, Prop. 1])**.** *Let $\Lambda \in \mathcal{GL}(S)$ be a maximal lamination. Then $\mathcal{H}(\Lambda, \mathbb{R}) \simeq \mathbb{R}^{6g-6}$.*

The vector space $\mathcal{H}(\Lambda, \mathbb{R})$ carries a natural alternating bilinear form $\omega : \mathcal{H}(\Lambda, \mathbb{R}) \times \mathcal{H}(\Lambda, \mathbb{R}) \to \mathbb{R}$, the *Thurston symplectic form*, which comes from the cup product on $H^1(S, \mathbb{R})$ (see [PH] [Bon2, §3] [SB]). When Λ is maximal, the form ω is nondegenerate, making $\mathcal{H}(\Lambda, \mathbb{R})$ a symplectic vector space. While we do not use this symplectic structure directly in the proof of the main theorem, it is relevant to some of the constructions and examples in the sequel.

If $\lambda \in \mathcal{ML}(S)$ is a measured lamination with $\mathrm{supp}(\lambda) \subset \Lambda \in \mathcal{GL}(S)$, then the total measure $\tau \mapsto \lambda(\tau)$ defines a real-valued cocycle on Λ, which we also denote by $\lambda \in \mathcal{H}(\Lambda, \mathbb{R})$. Cocycles arising from measures in this way take only nonnegative values on transversals; Bonahon showed that the converse is also true.

Theorem 3.2 (Bonahon [Bon4, Prop. 18])**.** *A transverse cocycle $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$ arises from a transverse measure for Λ if and only if $\alpha(\tau) \geq 0$ for every transversal τ.*

We therefore define $\mathcal{M}(\Lambda) \subset \mathcal{H}(\Lambda, \mathbb{R})$, the *cone of transverse measures* for Λ, to be the set of cocycles α satisfying $\alpha(\tau) \geq 0$ for all transversals τ. The set $\mathcal{M}(\Lambda)$ is a convex cone in the vector space $\mathcal{H}(\Lambda, \mathbb{R})$.

While positive real-valued cocycles on Λ correspond to transverse measures, there is an essential difference between a real-valued cocycle $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$ (whose value on a transversal is a real number) and a *signed transverse measure* on Λ which assigns a countably additive signed measure to each transversal τ (see Examples 1-2 below).

An analogous situation is the set function $[a, b] \mapsto (f(b) - f(a))$, where f is a real-valued function; this is a finitely additive function on intervals, but it only arises from a signed Borel measure if f has bounded variation (which is automatic if f is monotone). The connection between this example and real-valued cocycles for a lamination can be seen through the "distribution function" $f(x) = \alpha(\tau_x)$ where $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$, $\tau : [0, 1] \to S$ is a transversal, and $\tau_x = \tau|_{[0,x]}$. This function is defined for a.e. $x \in [0, 1]$ and monotonicity (for every τ) is equivalent to α being nonnegative.

The difference between measures and cocycles is also apparent from the dimension of the span of $\mathcal{M}(\Lambda)$, which is often 1 (by the solution to the Keane conjecture, see [Mas2] [Vee] [Ree] [Ker1]) and is never more than $(3g - 3)$ (because for maximal Λ, the space $\mathcal{H}(\Lambda, \mathbb{R}) \simeq \mathbb{R}^{6g-6}$ is a symplectic vector space in which $\mathrm{span}(\mathcal{M}(\Lambda))$ is isotropic, see [Pap] [Lev] [Kat]). It follows that $\mathcal{M}(\Lambda)$ has positive codimension when, for example, Λ is maximal.

Shearing coordinates. Let $\Lambda \in \mathcal{GL}(S)$ be a maximal lamination, realized as a partial foliation of $X \in \mathcal{T}(S)$ by hyperbolic geodesics. Its lift $\widetilde{\Lambda}$ to the universal cover $\widetilde{X} \simeq \mathbb{H}^2$ determines a (not necessarily locally finite) tiling of $\mathbb{H}^2$ by ideal triangles.

A transversal $\tau : [a, b] \to \mathbb{H}^2$ to $\widetilde{\Lambda}$ determines a pair of ideal triangles T_a and T_b which are the complementary regions of $\widetilde{\Lambda}$ containing $\tau(a)$ and $\tau(b)$.

In [Bon2], Bonahon constructs a *shearing cocycle* $\sigma(X) = \sigma^{\Lambda}(X) \in \mathcal{H}(\Lambda, \mathbb{R})$ from these data with the property that $\sigma(X)(\tau)$ measures the "relative shear" of the triangles T_a and T_b in $\mathbb{H}^2$. For example, the relative shear of two ideal triangles that share an edge is the signed distance between the feet of the altitudes based on the common side.

Remarkably, $\sigma(X)$ determines the metric X, and the set of such cocycles admits an explicit description. Let $\mathcal{C}(\Lambda) \subset \mathcal{H}(\Lambda, \mathbb{R})$ denote the set of $\mathbb{R}$-valued cocycles that arise as shearing cocycles of hyperbolic metrics, and recall that $\omega : \mathcal{H}(\Lambda, \mathbb{R}) \times \mathcal{H}(\Lambda, \mathbb{R}) \to \mathbb{R}$ is the Thurston symplectic form.

Theorem 3.3 (Bonahon [Bon2, Thm. A,B])**.** *A cocycle $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$ is the shearing cocycle of a hyperbolic metric if and only if only if $\omega(\alpha, \mu) > 0$ for all $\mu \in \mathcal{M}(\Lambda)$, and thus $\mathcal{C}(\Lambda)$ is an open convex cone with finitely many faces. Furthermore $\sigma : \mathcal{T}(S) \to \mathcal{C}(\Lambda)$ is a real-analytic diffeomorphism.*

The condition $\omega(\alpha, \mu) > 0$ in Theorem 3.3 is necessary because for each $\mu \in \mathcal{M}(\Lambda)$, the Thurston pairing $\omega(\mu, \sigma(X))$ is the hyperbolic length of μ on X [Bon2, Thm. 9].

While the convex cone $\mathcal{C}(\Lambda)$ has finitely many faces and is a union of open rays in $\mathcal{H}(\Lambda, \mathbb{R})$, the zero cocycle $0 \in \mathcal{H}(\Lambda, \mathbb{R})$ is *not* an extreme point of $\mathcal{C}(\Lambda)$. In fact, the vector space of signed transverse measures (i.e. $\operatorname{span} \mathcal{M}(\Lambda)$) is ω-isotropic (see [Pap]), and therefore

$$\operatorname{span} \mathcal{M}(\Lambda) \subset \partial \mathcal{C}(\Lambda). \tag{3.1}$$

A schematic representation $\mathcal{C}(\Lambda)$ appears in Figure 2, where $\partial\mathcal{C}(\Lambda)$ contains a one-dimensional subspace of $\mathcal{H}(\Lambda, \mathbb{R})$.

Shearing maps. We now use the shearing embedding $\sigma : \mathcal{T}(S) \to \mathcal{H}(\Lambda, \mathbb{R})$ to turn translation in the vector space $\mathcal{H}(\Lambda, \mathbb{R})$ into a (locally-defined) map of Teichmüller space.

Let $X \in \mathcal{T}(S)$ and $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$. If the sum $\sigma(X) + \alpha$ is the shearing cocycle of a hyperbolic surface, we call this hyperbolic surface $\operatorname{sh}_\alpha X$, the *shearing of X by α*. Thus $\operatorname{sh}_\alpha X$ is defined by the condition

$$\sigma(\operatorname{sh}_\alpha X) = \sigma(X) + \alpha. \tag{3.2}$$

Since $\sigma(\mathcal{T}(S)) = \mathcal{C}(\Lambda)$ is open, there is a neighborhood $U \subset \mathcal{H}(\Lambda, \mathbb{R}) \times \mathcal{T}(S)$ of $\{0\} \times \mathcal{T}(S)$ in which the shearing map $\operatorname{sh} : U \to \mathcal{T}(S)$ is well-defined.

In particular, for any fixed $X \in \mathcal{T}(S)$ and $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$ there is some $\epsilon > 0$ such that $\operatorname{sh}_{t\alpha} X$ is defined for all $|t| < \epsilon$ (see Figure 2), and

$$\frac{d}{dt}\sigma\left(\operatorname{sh}_{t\alpha} X\right) = \alpha.$$

We consider a few examples of shearing maps to highlight the role of $\mathcal{C}(\Lambda)$ and the fact that $\operatorname{sh}_\alpha X$ is not defined for all pairs (α, X). First, if Λ contains a simple closed geodesic γ, then for each $t \in \mathbb{R}$ the cocycle $t\gamma$ is a signed transverse measure. Furthermore, for all $X \in \mathcal{T}(S)$, we have $(\sigma(X) + t\gamma) \in \mathcal{C}(\Lambda)$, since $t\gamma \subset \mathcal{M}(\Lambda)$ and the span of $\mathcal{M}(\Lambda)$ is ω-isotropic;

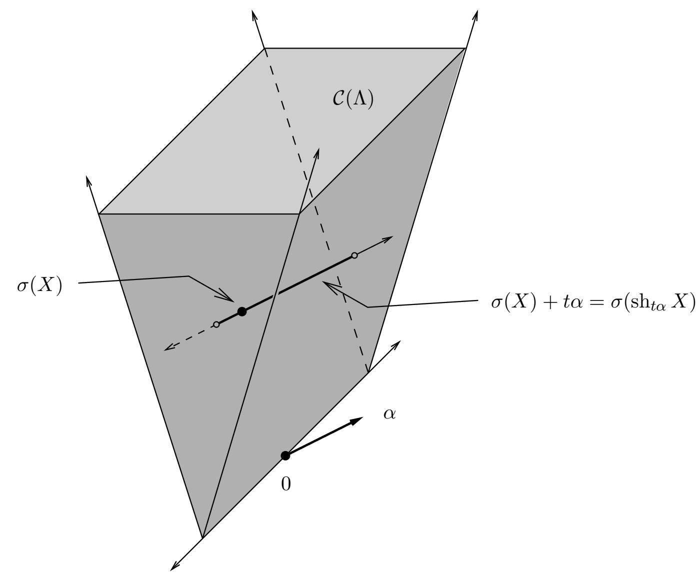

FIGURE 2. The shearing map is a translation in the shearing embedding of $\mathcal{T}(S)$ in $\mathcal{H}(\Lambda, \mathbb{R})$.

thus $\mathrm{sh}_{t\gamma} X$ is defined for all $t \in \mathbb{R}$. Concretely, the hyperbolic surface $\mathrm{sh}_{t\gamma} X$ is obtained from X by cutting along the geodesic γ and then gluing the two boundary components with a twist (by signed distance t).

This twisting example has a natural generalization: given a cocycle $\lambda \in \mathcal{H}(\Lambda, \mathbb{R})$ representing a transverse measure for Λ, the shearing $\mathrm{sh}_{t\lambda}(X)$ is again defined for all $t \in \mathbb{R}$ and the resulting map $\mathrm{sh}_{t\lambda} : \mathcal{T}(S) \to \mathcal{T}(S)$ is called an *earthquake*. For further discussion of earthquakes, see [Thu2] [Ker3] [EM] [Bon1] [McM].

As a final example, consider shearing a surface X using a cocycle $\alpha \in \mathcal{H}(\Lambda, \mathbb{R})$ that is itself the shearing cocycle of a hyperbolic surface, i.e. $\alpha \in \mathcal{C}(\Lambda)$. For any transverse measure $\lambda \in \mathcal{M}(\Lambda)$, the hyperbolic length of λ on $\mathrm{sh}_{t\alpha} X$ can be computed using the Thurston intersection form ω (cf. [Bon2, Thm. 9]), and we have

$$\ell(\lambda, \mathrm{sh}_{t\alpha} X) = \omega(\lambda, \sigma(X) + t\alpha) = \ell(\lambda, X) + t\,\omega(\lambda, \alpha) = A + Bt \tag{3.3}$$

where $A, B > 0$. Since the length of a measured lamination is positive, it follows that the set of t for which $\sigma(X) + t\alpha$ is the shearing cocycle of a hyperbolic metric (and hence those for which $\mathrm{sh}_{t\alpha} X$ exists) is a subset of $\{t > -(A/B)\}$.

Tangent cocycles. Let $\lambda_t \in \mathcal{ML}(S)$, $t \in [0, \epsilon)$ be a tangentiable ray of measured laminations. We will represent the tangent vector $\frac{d}{dt}\big|_{t=0^+} \lambda_t$ by a transverse cocycle to a certain geodesic lamination (as in [Bon3]). We first describe the underlying geodesic lamination.

The *essential support of* λ_t *at* $t = 0^+$ is a geodesic lamination $\Lambda \in \mathcal{GL}(S)$ that reflects how the support of λ_t is changing for small positive values of t. For a PL family of measured laminations λ_t, the essential support is the Hausdorff limit $\lim_{t\to 0^+} \operatorname{supp}(\lambda_t)$ of the supporting geodesic laminations [Bon3, Prop. 4]. For the general case, we only sketch the construction, and refer the reader to [Bon3, §2] for details.

First lift λ_t to a family $\widetilde{\lambda}_t$ of measured geodesic laminations in $\mathbb{H}^2$. Define a set $\widetilde{\Lambda}$ of geodesics in $\mathbb{H}^2$ as follows: a geodesic γ belongs to $\widetilde{\Lambda}$ if and only if for every smooth transversal $\tau : [-\epsilon, \epsilon] \to \mathbb{H}^2$ with $\tau(0) \in \gamma$, the total transverse measure of τ with respect to λ_t is at least Ct for some $C > 0$ (depending on τ and γ) and all t sufficiently small. Then $\widetilde{\Lambda}$ is the lift of a geodesic lamination $\Lambda \in \mathcal{GL}(S)$, the essential support of λ_t.

The tangent vector $\frac{d}{dt}\big|_{t=0^+} \lambda_t$ defines a real-valued transverse cocycle $\dot{\lambda} \in \mathcal{H}(\Lambda, \mathbb{R})$ on the essential support as follows:

$$\dot{\lambda}(\tau) = \lim_{t\to 0^+} \frac{1}{t} \left(\lambda_t(\tau) - \lambda_0(\tau)\right).$$

This is the *tangent cocycle* of λ_t at $t = 0^+$. Clearly the same formula defines a cocycle for a lamination containing the essential support of λ_t, so as a convenience we may assume Λ is maximal. We illustrate this construction with two examples:

Example 1. *The cocycle determined by a measured lamination* $\lambda \in \mathcal{M}(\Lambda)$ *is the tangent cocycle of the "ray"* $\lambda_t = (1 + t)\lambda \in \mathcal{ML}(S)$ *at* $t = 0^+$.

Example 2. *Let* S *be a punctured torus with meridian* α *and longitude* β *(putting aside our assumption that* S *is compact for a moment), and consider the family* $\lambda_t \in \mathcal{ML}(S)$ *defined by the conditions* $\lambda_t(\alpha) = 1$ *and* $\lambda_t(\beta) = t$. *Thus for each* $n \in \mathbb{N}$, *the measured lamination* $\lambda_{1/n}$ *is a simple closed curve with homology class* $(n[\alpha] + [\beta])$ *and weight* $\frac{1}{n}$.

The essential support of λ_t *at* $t = 0^+$ *is the geodesic lamination* $\Lambda = \alpha \cup \eta$, *where* η *is an infinite simple geodesic that spirals toward* α *in each direction. The derivative* $\dot{\lambda} \in \mathcal{H}(\Lambda, \mathbb{R})$ *is a cocycle with full support and indefinite sign (compare Bonahon's example* [Bon3, p. 104]*).*

This second example shows that the tangent cocycle may not be a transverse measure, and the essential support may not admit a measure of full support. This illustrates some of the difficulties of using differential methods on the space of measured laminations and of adapting the methods in [SW] to prove the main theorem.

Following Thurston, Bonahon showed that the association of a cocycle $\dot{\lambda}$ to a tangentiable family λ_t provides a linear model for each PL face of the tangent space $T_{\lambda_0}\mathcal{ML}(S)$ (see [Bon3], [Thu3, §6]). When $\operatorname{supp}(\lambda_0)$ is not maximal, however, no single maximal lamination Λ can be chosen to contain the essential support of *every* family λ_t, and so there is no embedding $T_{\lambda_0}\mathcal{ML}(S) \to \mathcal{H}(\Lambda, \mathbb{R})$.

In contrast, when $\mathrm{supp}(\lambda_0) = \Lambda$ is maximal (a generic situation that excludes, for example, closed leaves), the tangent cocycle construction defines a homeomorphism $T_{\lambda_0}\mathcal{ML}(S) \simeq \mathcal{H}(\Lambda, \mathbb{R})$, giving the tangent space a canonical linear structure. This was observed in [Thu3].

Complex linearity. So far we have seen real-valued cocycles on geodesic laminations arise in two different contexts: first as shearing cocycles providing coordinates for $\mathcal{T}(S)$, and then as tangent vectors to families of measured laminations. Building on these two constructions, the following result of Bonahon will allow us to connect the derivative of grafting with respect to $\mathcal{T}(S)$ and $\mathcal{ML}(S)$:

Theorem 3.4 (Bonahon [Bon5, Prop. 5] [Bon2, §10])**.** *Let $X \in \mathcal{T}(S)$ and $\lambda \in \mathcal{ML}(S)$. For each maximal geodesic lamination $\Lambda \in \mathcal{GL}(S)$ containing the support of λ, there is a complex-linear map $L = L(\Lambda, \lambda, X) : \mathcal{H}(\Lambda, \mathbb{C}) \to T_{\mathrm{Gr}_\lambda X}\mathcal{P}(S)$ which determines the tangent map of* Gr *in tangent directions carried by Λ, in the following sense:*

Let $\lambda_t \in \mathcal{ML}(S)$ be a tangentiable family of measured laminations with $\lambda_0 = \lambda$ and with essential support contained in Λ, and let $\dot\lambda = \frac{d}{dt}\big|_{t=0^+} \lambda_t \in \mathcal{H}(\Lambda, \mathbb{R})$. Let $X_t \in \mathcal{T}(S)$ be a smooth family of hyperbolic structures with $X_0 = X$ whose derivative in the Λ shearing embedding is $\dot\sigma = \frac{d}{dt}\big|_{t=0^+} \sigma(X_t)$.

Then $t \mapsto \mathrm{Gr}_{\lambda_t} X_t$ is a tangentiable curve in $\mathcal{P}(S)$ and

$$\frac{d}{dt}\bigg|_{t=0^+} \mathrm{Gr}_{\lambda_t} = L(\dot\sigma + i\dot\lambda)$$

Similarly $\mathrm{gr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$ *is tangentiable and its derivative has the same complex linearity property.*

In terms of the piecewise linear structure of $\mathcal{ML}(S)$, Theorem 3.4 says that on each linear face of the tangent space $T_{(\lambda,X)}(\mathcal{ML}(S) \times \mathcal{T}(S))$, the tangent map $T_{(\lambda,X)}\,\mathrm{gr}$ is the restriction of a complex-linear map $\mathcal{H}(\Lambda, \mathbb{C}) \to T_{\mathrm{gr}_\lambda X}\mathcal{T}(S)$ (see [Bon5, §2] [Bon2, §10]).

4. Proof of the main theorem

With the necessary background in place, we can now show that the tangent map of $\mathrm{gr}_{\bullet} X$ has no kernel.

Proof of Theorem 2.6. Fix X and suppose that $\frac{d}{dt}\big|_{t=0^+} \mathrm{gr}_{\lambda_t} X = 0$. Let $\dot\lambda \in \mathcal{H}(\Lambda, \mathbb{R})$ denote the derivative of λ_t at $t = 0^+$, with $\Lambda \in \mathcal{GL}(S)$ maximal. We must show that $\dot\lambda = 0$.

For all t sufficiently small, the shearing $X_t = \mathrm{sh}_{t\dot\lambda} X \in \mathcal{T}(S)$ is defined and satisfies $X_0 = X$ and $\dot\lambda = \frac{d}{dt}\sigma(X_t)$. Therefore by Theorem 3.4,

$$i\left[\frac{d}{dt}\bigg|_{t=0^+} \mathrm{gr}_{\lambda_0} X_t\right] = iL(\dot\lambda) = L(i\dot\lambda) = \frac{d}{dt}\bigg|_{t=0^+} \mathrm{gr}_{\lambda_t} X_0 = 0,$$

where L is the complex-linear map representing the tangent map of gr in tangent directions carried by Λ.

By Theorem 1.2, gr_{λ_0} is an immersion, and so $\frac{d}{dt}\big|_{t=0^+} X_t = 0$. Thus

$$\dot{\lambda} = \frac{d}{dt}\sigma(X_t) = 0.$$

□

If the grafting map were continuously differentiable in the usual sense, the proof of the main theorem would now be straightforward, using linearity of the derivative and the inverse function theorem to conclude that $\mathrm{gr}_{\bullet} X$ is a local diffeomorphism. We will follow this general outline, but we will need to use additional properties of the grafting maps to strengthen the infinitesimal result to a local one.

In fact, some argument specific to grafting is necessary at this point. In general, a tangent map that has no kernel need not be injective (consider $\mathbb{C} \to \mathbb{R}^+$ by $z \mapsto |z|$ at $z = 0$), and even if the tangent map is injective, tangentiability does not imply the continuous variation of derivatives needed for the inverse function theorem.

With these potential problems in mind, we analyze the X-grafting map $\mathrm{gr}_{\bullet} X$ as the composition of the projective X-grafting map $\mathrm{Gr}_{\bullet} X$ and the smooth projection $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$. Let $N = \dim \mathcal{T}(S) = \dim \mathcal{ML}(S) = \frac{1}{2} \dim \mathcal{P}(S) = 3g - 3$.

Let $G_X = \{\mathrm{Gr}_\lambda X | \lambda \in \mathcal{ML}(S)\} \subset \mathcal{P}(S)$ denote the image of $\mathrm{Gr}_{\bullet} X$. Recall the map $p_{\mathcal{T}} : \mathcal{P}(S) \to \mathcal{T}(S)$ is defined by $p_{\mathcal{T}}(\mathrm{Gr}_\lambda X) = X$, and so $G_X = p_{\mathcal{T}}^{-1}(X)$ is a fiber of this map. Since $p_{\mathcal{T}}$ is a C^1 submersion (by Corollary 2.5), the set G_X is actually a C^1 submanifold of $\mathcal{P}(S)$ of dimension $N = \dim \mathcal{P}(S) - \dim \mathcal{T}(S)$. In particular, G_X is smoother than its tangentiable parameterization by $\mathcal{ML}(S)$ would suggest.

We can recast Theorem 2.6 as a result about the tangent space to G_X as follows:

Theorem 4.1. *For any $X \in \mathcal{T}(S)$, the C^1 submanifold $G_X \subset \mathcal{P}(S)$ is transverse to the map $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$; that is, for any $Z \in G_X$, we have $T_Z G_X \cap (\ker d\pi) = \{0\}$. Equivalently, the distributions $\ker d\pi$ and $\ker dp_{\mathcal{T}}$ in $T\mathcal{P}(S)$ are transverse.*

Proof. Suppose not, i.e. that there exists $v \in (T_Z G_X \cap \ker d\pi)$ with $v \neq 0$. Since $Z \in G_X$, we have $Z = \mathrm{Gr}_\lambda X$ for some $\lambda \in \mathcal{ML}(S)$. Choose a C^1 path Z_t in G_X with $Z_0 = Z$ and $\frac{d}{dt}\big|_{t=0^+} Z_t = v$, and let $\mathrm{Gr}^{-1}(Z_t) = (\lambda_t, X)$. Since Gr^{-1} is a tangentiable diffeomorphism, the family λ_t is tangentiable and satisfies $\frac{d}{dt}\big|_{t=0^+} \lambda_t \neq 0$. On the other hand, $\frac{d}{dt}\big|_{t=0^+} \mathrm{gr}_{\lambda_t} X = \frac{d}{dt}\big|_{t=0^+} \pi(Z_t) = d\pi(\frac{d}{dt}\big|_{t=0^+} Z_t) = 0$, contradicting Theorem 2.6. □

While a general tangent map can have no kernel and yet fail to be injective, the fact that G_X is C^1 (and thus has linear tangent spaces) rules out this behavior for $\mathrm{Gr}_{\bullet} X$:

Theorem 4.2. *The tangent map $T_\lambda \operatorname{gr}_{\bullet} X$ of the X-grafting map is a homeomorphism.*

Proof. Fix $\lambda \in \mathcal{ML}(S)$ and for brevity let $f = T_\lambda \operatorname{gr}_{\bullet} X$ be the tangent map of the conformal X-grafting map, and let $F = T_\lambda \operatorname{Gr}_{\bullet} X$ be the tangent map of the projective X-grafting map. Then $f = d\pi \circ F$, where $d\pi$ is the differential of $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ at $Z = \operatorname{Gr}_\lambda X$. Note that F and f are homogeneous maps of degree 1, while $d\pi$ is linear.

Since Gr is a tangentiable diffeomorphism, its restriction $\operatorname{Gr}_{\bullet} X : \mathcal{ML}(S) \to \mathcal{P}(S)$ is a tangentiable immersion, i.e. its tangent map F is injective, and is a homogeneous homeomorphism onto its image. Since G_X is a C^1 submanifold of $\mathcal{P}(S)$, the image of F is the linear subspace $T_Z G_X \subset T_Z \mathcal{P}(S)$.

First we show that f injective. Suppose on the contrary there exist $\dot\lambda_1, \dot\lambda_2 \in T_\lambda \mathcal{ML}(S)$, distinct and nonzero, and $f(\dot\lambda_1) = f(\dot\lambda_2)$. Then $v = F(\dot\lambda_1) - F(\dot\lambda_2) \in T_Z G_X$ is nonzero since F is injective, and so $v \in \ker d\pi$, which contradicts Theorem 4.1.

Thus $f : T_\lambda \mathcal{ML}(S) \to T_{\operatorname{gr}_\lambda X} \mathcal{T}(S)$ is injective. Since f is also a homogeneous map of degree 1 between cones of the same dimension, it is a homeomorphism. □

Remark. The fact that $T_\lambda \operatorname{Gr}_{\bullet} X$ is a homeomorphism onto a linear subspace of $T_{\operatorname{Gr}_\lambda X} \mathcal{P}(S)$ is also observed by Bonahon in the proof of Prop. 12 in [Bon5].

Finally we complete our study of $\operatorname{Gr}_{\bullet} X$ and $\operatorname{gr}_{\bullet} X$ by proving the main theorem:

Proof of Theorem 1.1. Let us consider the restriction of the forgetful map $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$ to the C^1 submanifold $G_X \subset \mathcal{P}(S)$. Since G_X is N-dimensional, this projection is a local diffeomorphism at $Z \in G_X$ if the subspaces $T_Z G_X$ and $\ker d\pi$ of $\mathcal{T}_Z \mathcal{P}(S)$ are transverse (by the inverse function theorem). By Theorem 4.1, this is true for every $Z \in G_X$, so $\pi|_{G_X}$ is a local C^1 diffeomorphism.

Thus the conformal X-grafting map is the composition of the homeomorphism $\operatorname{Gr}_{\bullet} X : \mathcal{ML}(S) \to G_X$ and the local homeomorphism $\pi|_{G_X} : G_X \to \mathcal{T}(S)$, so $\operatorname{gr}_{\bullet} X$ is a local homeomorphism. As we noted in Section 2, by the properness of $\operatorname{gr}_{\bullet} X$ (Theorem 2.1), it follows that this map is a homeomorphism. Thus X-grafting is a tangentiable homeomorphism with injective tangent maps (Theorem 4.2), which by Lemma 2.2 is a tangentiable diffeomorphism. □

5. Applications

In this section, we collect some applications of the main theorem itself and of the techniques used in its proof, and we discuss some related questions about grafting coordinates and $\mathbb{CP}^1$ structures.

Projections. We begin by proving the main corollary of Theorems 1.1 and 1.2 about the grafting coordinates for a fiber $P(X) = \pi^{-1}(X) \subset \mathcal{P}(S)$:

Corollary 1.3. *For each $X \in \mathcal{T}(S)$, the space $P(X)$ of $\mathbb{CP}^1$ structures on X is a graph over each factor in the grafting coordinate system. In fact, we have:*

1. *The projection $p_{\mathcal{T}}|_{P(X)} : P(X) \to \mathcal{T}(S)$ is a C^1 diffeomorphism.*
2. *The projection $p_{\mathcal{ML}}|_{P(X)} : P(X) \to \mathcal{ML}(S)$ is a tangentiable diffeomorphism.*

Proof. First we consider the regularity of the maps. Bonahon showed that $p_{\mathcal{T}}$ is C^1 (Theorem 2.4), and $p_{\mathcal{ML}}$ is tangentiable because it is the composition of Gr^{-1}, a tangentiable diffeomorphism (Theorem 2.3), and the projection to one factor of a product of tangentiable manifolds.

(1) *Proof that $p_{\mathcal{T}}|_{P(X)}$ is a diffeomorphism.* First of all, the map $p_{\mathcal{T}}|_{P(X)}$ is a homeomorphism because it has inverse map
$$Y \mapsto \mathrm{Gr}_{(\mathrm{gr}_{\bullet} X)^{-1}(Y)} Y$$
where the map $(\mathrm{gr}_{\bullet} X)^{-1} : \mathcal{T}(S) \to \mathcal{ML}(S)$ exists by Theorem 1.1.

Thus it suffices to show that $p_{\mathcal{T}}|_{P(X)}$ is a local diffeomorphism. But the kernel of $dp_{\mathcal{T}}|_{P(X)}$ is the intersection of $\ker d\pi$ and $\ker dp_{\mathcal{T}}$ in $T\mathcal{P}(S)$, which is zero by Theorem 4.1. So the derivative of $p_{\mathcal{T}}|_{P(X)}$ is an isomorphism, and by the inverse function theorem this map is a local diffeomorphism.

(2) *Proof that $p_{\mathcal{ML}}|_{P(X)}$ is a tangentiable diffeomorphism.* As in (1), we first show that $p_{\mathcal{ML}}|_{P(X)}$ is a homeomorphism by exhibiting an inverse map,
$$\lambda \mapsto \mathrm{Gr}_\lambda(\mathrm{gr}_\lambda^{-1}(Y))$$
where gr_λ^{-1} exists by Theorem 1.2.

By Lemma 2.2, we need only show that the tangent map of $p_{\mathcal{ML}}|_{P(X)}$ is everywhere injective. That this tangent map has no kernel also follows easily from Theorem 1.2, but to show injectivity we will use an argument modeled on the proofs of Theorems 2.6 and 4.2.

Suppose on the contrary that two distinct, nonzero tangent vectors $v_1, v_2 \in T_Z P(X)$ have the same image in $T_\lambda \mathcal{ML}(S)$ under the tangent map of $p_{\mathcal{ML}}$, where $Z = \mathrm{Gr}_\lambda Y$. Then differentiable paths in $P(X)$ with tangent vectors v_1 and v_2 are mapped by Gr^{-1} to tangentiable paths $\mathrm{Gr}_{\lambda_t^{(1)}} Y_t^{(1)}$ and $\mathrm{Gr}_{\lambda_t^{(2)}} Y_t^{(2)}$, respectively, where $Y_0^{(k)} = Y$ and $\lambda_0^{(k)} = \lambda$ for $k = 1, 2$.

In this notation, the image of v_k by the tangent map $T_Z p_{\mathcal{ML}}$ is $\frac{d}{dt}\big|_{t=0^+} \lambda_t^{(k)}$, so we have $\frac{d}{dt}\big|_{t=0^+} \lambda_t^{(1)} = \frac{d}{dt}\big|_{t=0^+} \lambda_t^{(2)} = \dot{\lambda}$. In particular there is a single geodesic lamination Λ containing the essential support of both families $\lambda_t^{(k)}$, and their common tangent vector defines a cocycle $\dot{\lambda} \in \mathcal{H}(\Lambda, \mathbb{R})$. Using the shearing embedding of Teichmüller space in $\mathcal{H}(\Lambda, \mathbb{R})$ gives cocycles $\dot{\sigma}_k = \frac{d}{dt}\big|_{t=0^+} (\sigma^\Lambda(Y_t^{(k)}))$, and since $v_1 \neq v_2$ we have $\dot{\sigma}_1 \neq \dot{\sigma}_2$.

By Theorem 3.4, there is a complex-linear map $L : \mathcal{H}(\Lambda, \mathbb{C}) \to T_X\mathcal{T}(S)$ that gives the tangent map of Gr for tangent vectors to $\mathcal{ML}(S) \times \mathcal{T}(S)$ at (λ, X) representable by complex-valued cocycles on Λ, so $v_k = L(\dot{\sigma}_k + i\dot{\lambda})$. Now consider $\dot{\sigma} = \dot{\sigma}_1 - \dot{\sigma}_2 \neq 0$, which is the tangent vector to the shearing family $Y_t = \mathrm{sh}_{t\dot{\sigma}} Y$; we have

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0^+} \mathrm{Gr}_\lambda Y_t &= L(\dot{\sigma}) \\ &= L\left((\dot{\sigma}_1 + i\dot{\lambda}) - (\dot{\sigma}_2 + i\dot{\lambda})\right) \\ &= v_1 - v_2 \in T_Z P(X), \end{aligned}$$

and so $\frac{d}{dt}\big|_{t=0^+} \mathrm{gr}_\lambda Y_t = 0$, which by Theorem 1.2 implies that $\frac{d}{dt}\big|_{t=0^+} Y_t = 0$, and so $\dot{\sigma} = \frac{d}{dt}\big|_{t=0^+} \sigma^\Lambda(Y_t) = 0$, a contradiction. Thus the tangent map of $p_{\mathcal{ML}}$ is injective, as required.

□

Pruning. Since the λ-grafting map $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ is a homeomorphism (Theorem 1.2), there is an inverse map $\mathrm{pr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ which we call *pruning by* λ. Roughly speaking, grafting by λ inserts a Euclidean subsurface along the leaves of λ, and pruning by λ removes it. Allowing λ to vary, we obtain the pruning map $\mathrm{pr} : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$, and fixing X we have the X-pruning map $\mathrm{pr}_{\bullet} X : \mathcal{ML}(S) \times \{X\} \to \mathcal{T}(S)$.

We can reformulate Corollary 1.3 in terms of pruning as follows:

Corollary 5.1. *For each $X \in \mathcal{T}(S)$, the X-pruning map $\mathrm{pr}_{\bullet} X : \mathcal{ML}(S) \to \mathcal{T}(S)$ is a tangentiable diffeomorphism.*

Proof. The graph of the X-pruning map consists of the pairs (λ, Y) such that $\mathrm{gr}_\lambda Y = X$, which is simply the fiber $P(X) \subset \mathcal{P}(S) \simeq \mathcal{ML}(S) \times \mathcal{T}(S)$. In terms of the projections projections $p_{\mathcal{ML}} : P(X) \to \mathcal{ML}(S)$ and $p_{\mathcal{T}} : P(X) \to \mathcal{T}(S)$, we have $\mathrm{pr}_{\bullet} X = p_{\mathcal{T}} \circ p_{\mathcal{ML}}^{-1}$, which is a tangentiable diffeomorphism by Corollary 1.3. □

Previously it was known that $\mathrm{pr}_{\bullet} X$ is a "rough homeomorphism", i.e. a proper map of degree 1, and that is has a natural extension to the Thurston compactification of $\mathcal{T}(S)$ and the projective compactification of $\mathcal{ML}(S)$ by $\mathbb{P}\mathcal{ML}(S) = (\mathcal{ML}(S) - \{0\})/\mathbb{R}^+$. Furthermore, the resulting boundary map $\mathbb{P}\mathcal{ML}(S) \to \mathbb{P}\mathcal{ML}(S)$ is the *antipodal map* relative to X. For details, see [D1].

Internal Coordinates for the Bers slice. Let $\mathcal{QF} = \mathcal{QF}(S)$ denote the space of marked quasi-Fuchsian hyperbolic 3-manifolds homeomorphic to $S \times \mathbb{R}$ (see [Ber1] [Nag]). Each such manifold $M \in \mathcal{QF}$ has ideal boundary

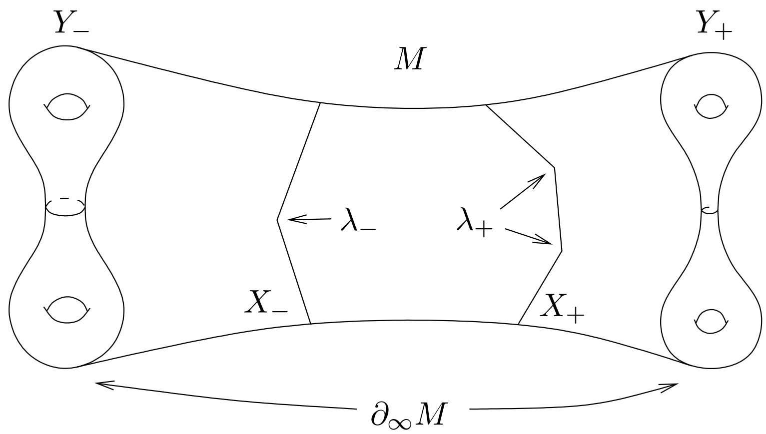

FIGURE 3. Geometric data associated to a quasi-Fuchsian manifold $M \in \mathcal{QF}$. In a Bers slice, Y_- is fixed.

$\partial_\infty M = Y_+ \sqcup Y_-$ with conformal structures $Y_\pm \in \mathcal{T}(S)$ and convex core boundary surfaces $X_\pm \in \mathcal{T}(S)$ with bending laminations $\lambda_\pm \in \mathcal{ML}(S)$ (represented schematically in Figure 3). Furthermore, the convex core and ideal boundary surfaces satisfy $\operatorname{gr}_{\lambda_\pm} Y_\pm = X_\pm$ ([McM, Thm. 2.8], see also [KT]).

In his celebrated holomorphic embedding [Ber2] of the Teichmüller space $\mathcal{T}(S)$ into $\mathbb{C}^{3g-3}$, Bers focused on the slice $\mathcal{B}_Y \subset \mathcal{QF}$ of those quasi-Fuchsian manifolds with a fixed ideal boundary surface $Y_- = Y$. Bers showed that a quasi-Fuchsian manifold $M \in \mathcal{B}_Y$ is uniquely determined by its other conformal boundary surface $Y_+ \in \mathcal{T}(S)$, which can be chosen arbitrarily (the *simultaneous uniformization theorem* [Ber1]).

A corollary of our main theorem is that $\mathcal{B}_Y$ may also be parameterized by the hyperbolic structure X_-:

Corollary 5.2. *Let $M, M' \in \mathcal{QF}$ be marked quasi-Fuchsian manifolds, and suppose an end of M (respectively M') has ideal boundary Y (resp. Y') and the associated convex core boundary surface has hyperbolic metric X (resp. X'). If Y and Y' are conformally equivalent and X and X' are isometric, then M is isometric to M'.*

Proof. By hypothesis $Y' = Y$ and $X' = X$ as points in Teichmüller space, so the bending measures λ and λ' of the convex core boundaries satisfy $\operatorname{gr}_\lambda X = \operatorname{gr}_{\lambda'} X = Y$. By Theorem 1.1, we have $\lambda = \lambda'$.

A quasi-Fuchsian manifold M is uniquely determined up to isometry by the hyperbolic metric X and bending lamination λ of one of its convex core boundary surfaces, since one can use X and λ to construct the associated equivariant pleated plane in $\mathbb{H}^3$ and its holonomy group $\pi_1 M \subset \mathrm{PSL}_2(\mathbb{C})$ (see [EM]). As M and M' share these data, they are isometric. $\square$

Remarks.

1. In [SW], it was observed that a manifold $M \in \mathcal{B}_Y$ is also determined by the bending lamination $\lambda_- \in \mathcal{ML}(S)$ on the same side as the fixed conformal structure Y. It is not known whether either the bending lamination λ_+ or the hyperbolic structure X_+ determine elements of $\mathcal{B}_Y$.
2. More generally, one can ask what geometric data determine $M \in \mathcal{QF}$ up to isometry. Bonahon and Otal [BO] showed that if λ_+ and λ_- bind the surface and are supported on simple closed curves, then the pair (λ_+, λ_-) determines $M \in \mathcal{QF}$ uniquely. Recently, Bonahon [Bon6] showed that this restriction to simple closed curve may be lifted for elements of $\mathcal{QF}$ which are sufficiently close to the Fuchsian subspace $\mathcal{F} \subset \mathcal{QF}$, and Series [Ser1] proved that λ_+ and λ_- determine M when S is a once-punctured torus.

6. Grafting coordinates and rays

In this final section we discuss how the ray structure of $\mathcal{ML}(S)$ is transported to $\mathcal{T}(S)$ by the X-grafting map.

Recall (from Section 2) that the action of $\mathbb{R}^+$ on $\mathcal{ML}(S)$ by scaling transverse measures gives this space the structure of a cone, with the empty lamination 0 as its base point. By Theorem 1.1, for each $X \in \mathcal{T}(S)$ we can use the X-grafting map to parameterize the Teichmüller space by $\mathcal{ML}(S)$, providing a global system of "polar coordinates" centered at X. In this coordinate system, the ray $\mathbb{R}^+\lambda \in \mathcal{ML}(S)$ corresponds to the *grafting ray* $\{\mathrm{gr}_{t\lambda} X \,|\, t \in \mathbb{R}^+\}$, a properly embedded path starting at X, and Teichmüller space is the union of these rays.

It would be interesting to understand the geometry of this coordinate system, and especially the grafting rays. Thus we ask:

Question. What is the behavior of the grafting ray $t \mapsto \mathrm{gr}_{t\gamma} X$ and how does it depend on λ and X?

Naturally, one first wonders about the regularity of grafting rays, since the grafting map itself exhibits a combination of tangentiable and differentiable behavior. However, along rays the grafting map is as smooth as possible:

Theorem 6.1 (McMullen [McM])**.** *The grafting ray $t \mapsto \mathrm{gr}_{t\lambda}$ is a real-analytic map from $\mathbb{R}^+$ to $\mathcal{T}(S)$; in fact, it is the restriction of a real-analytic map $\{t \geq -\epsilon\} \to \mathcal{T}(S)$ for some $\epsilon > 0$ (depending on λ). Furthermore, if $\lambda_n \to \lambda$ in $\mathcal{ML}(S)$, then the λ_n grafting rays converge in C^ω to the λ grafting ray.*

Remarks.

1. In [McM] it is shown that for any $\lambda \in \mathcal{ML}(S)$, the *complex earthquake map* $\mathrm{eq}_\lambda : \mathbb{H} \to \mathcal{T}(S)$ is holomorphic and extends to an open neighborhood of $\mathbb{H} \cup \mathbb{R}$ (Proposition 2.6 and Theorem 2.10). Furthermore these maps vary continuously with $\lambda \in \mathcal{ML}(S)$ (Theorem 2.5),

as do their derivatives, since they are holomorphic. Since the grafting ray is the restriction of the complex earthquake to $i\mathbb{R}$, Theorem 6.1 follows immediately.

2. The regularity of grafting rays and complex earthquakes is closely related to (and in part, an application of) the analyticity of quakebend deformations of surface group representations in $\mathrm{PSL}_2(\mathbb{C})$ established by Epstein and Marden [EM].

Length estimates. We now study the variation of the hyperbolic length and extremal length of the grafting lamination along a grafting ray. When λ is supported on a simple closed geodesic γ, it is clear that a large grafting will result in a surface in which γ is short in terms of either hyperbolic or extremal length, because $\mathrm{gr}_{t\gamma} X$ contains an annulus of large modulus homotopic to γ. In the remainder of this section we refine this intuition, starting with extremal length:

Theorem 6.2. *For each $X \in \mathcal{T}(S)$, the extremal length of λ is of order $1/t$ on the λ-grafting ray $t \mapsto \mathrm{gr}_{t\lambda} X$ and is monotone decreasing for all $t \gg 0$. Specifically, we have*

(1) $E(\lambda, \mathrm{gr}_{t\lambda} X) = \frac{\ell(\lambda,X)}{t} + O(t^{-2})$, *and*

(2) $\frac{d}{dt} E(\lambda, \mathrm{gr}_{t\lambda} X) = \frac{-\ell(\lambda,X)}{t^2} + O(t^{-3})$

where $E(\lambda, Y)$ denotes the extremal length of λ on the Riemann surface Y. The implicit constants depend only on $\chi(S)$.

Note that Theorem 6.2 includes the results stated in the introduction as Theorem 1.4. Before giving the proof of Theorem 6.2, we fix notation and recall some concepts from Teichmüller theory used therein.

Annuli. Let A be an annular Riemann surface of modulus M and let $E(A) = 1/M = E(\gamma, A)$ be the extremal length of γ, the nontrivial isotopy class of simple closed curves on A. Then A is isomorphic to a rectangle $R_A = [0, E(A)] \times (0, 1) \subset \mathbb{C}$ with its vertical sides identified. We call the complex local coordinate z on A coming from this realization the *natural coordinate* for A. Similarly, the induced flat metric $|dz|$ on A is the *natural metric*, with respect to which A is a Euclidean cylinder of height 1 and circumference $E(A)$.

Jenkins-Strebel differentials. For any isotopy class γ of simple closed curves on a compact Riemann surface X, there is a unique embedded annulus $A_\gamma \subset X$ homotopic to γ of maximum modulus $\mathrm{Mod}(A_\gamma) = 1/E(\gamma, X)$. The annulus A_γ is dense in Y, and if z is the natural coordinate for A_γ, the quadratic differential dz^2 on A extends holomorphically to a quadratic differential on X, the *Jenkins-Strebel* differential for γ.

Foliations. A holomorphic quadratic differential ϕ on a Riemann surface X has an associated singular *horizontal foliation* $\mathcal{F}(\phi)$ whose leaves integrate the distribution of tangent vectors v satisfying $\phi(v) \geq 0$. Integration of

$|\operatorname{Im}\sqrt{\phi}|$ gives a transverse measure on $\mathcal{F}(\phi)$. Similarly $\mathcal{F}(-\phi)$ is the *vertical foliation*, whose transverse measure comes from $|\operatorname{Re}\sqrt{\phi}|$.

When ϕ is a Jenkins-Strebel differential on a compact surface, the nonsingular leaves of $\mathcal{F}(\phi)$ are closed and homotopic to γ; in the realization of the Jenkins-Strebel annulus A_γ as a rectangle with identifications, these are the horizontal lines, while leaves of $\mathcal{F}(-\phi)$ are vertical lines. The transverse measures for $\mathcal{F}(\phi)$ and $\mathcal{F}(-\phi)$ are given by $|dy|$ and $|dx|$ in the rectangle, respectively. Thus the closed leaves of $\mathcal{F}(\phi)$ have total measure $E(A_\gamma) = E(\gamma, X)$ with respect to the transverse measure of the vertical foliation $\mathcal{F}(-\phi)$. Furthermore these closed leaves realize the minimum transverse measure among all curves homotopic to γ (see [Gar, Lem. 11.5.3]).

Pairing and extremal length. The natural pairing between Beltrami differentials $\mu = \mu(z)\frac{d\bar{z}}{dz}$ (with $\mu(z) \in L^\infty$) and integrable holomorphic quadratic differentials $\phi = \phi(z)dz^2$ on a Riemann surface X is given by

$$\langle \mu, \phi \rangle = \operatorname{Re} \int_X \mu\phi = \int_X \mu(z)\phi(z)\, |dz|^2.$$

When $X = A$ is an annulus with natural coordinate z_A, pairing a Beltrami differential with $\phi = dz_A^2$ gives the infinitesimal change in extremal length $E(A)$ (see [Gar, §1.9]); that is, if A_t is a family of annuli identified by a family of quasiconformal maps with derivative μ_t, then

$$\frac{d}{dt} E(A_t) = 2\langle \mu(t), dz_{A_t}^2 \rangle. \tag{6.1}$$

Similarly, when ϕ is a Jenkins-Strebel differential on a compact surface X, pairing with ϕ gives the differential of the extremal length function on Teichmüller space:

Theorem 6.3 (Gardiner [Gar, Thm. 11.8.5])**.** *Let $t \mapsto X_t \in \mathcal{T}(S)$ be a differentiable path whose tangent vector is represented by the Beltrami differential $\mu(t)$ on X_t. Let γ be an isotopy class of simple closed curves and $E(t) = E(\gamma, X_t)$ its extremal length on X_t. Then*

$$E'(t) = 2\langle \mu(t), \phi(t) \rangle$$

where $\phi(t)$ is the Jenkins-Strebel differential for γ on X_t.

Stretching annuli. Let X be a compact Riemann surface and $A \subset X$ an annulus in the homotopy class of γ, a simple closed curve. The natural coordinate z on A gives a Beltrami differential $\frac{d\bar{z}}{dz}$ on A, which extends to a Beltrami differential on X by setting it to zero on $(X - A)$. This differential represents an infinitesimal affine stretch of A. We will be interested in the extent to which $\frac{d\bar{z}}{dz}$ affects the extremal length of γ, as estimated in

Lemma 6.4. *Let $A \subset X$, γ, and z be as above, and let ϕ be the Jenkins-Strebel differential on X for γ. Then*

$$E(X) \geq \langle \frac{d\bar{z}}{dz}, \phi \rangle \geq \frac{2E(X)^2}{E(A)} - E(X)$$

where $E(A) = E(\gamma, A)$ and $E(X) = E(\gamma, X)$ are the extremal lengths of γ on these two surfaces.

Remarks.

1. When $A = A_\gamma$ is the Jenkins-Strebel annulus (of maximum modulus), we have $E(A) = E(X)$ and both inequalities in Lemma 6.4 become equalities. However this is clear since $\frac{d\overline{z}}{dz} = \overline{\phi}/|\phi|$ if z is the natural coordinate of the Jenkins-Strebel annulus. The point of the Lemma is that we also have tight bounds for the pairing when A has *nearly* maximum modulus.
2. A related estimate for nearly maximal annuli is used in Kerckhoff's proof of that that foliation map $\mathcal{F} : Q(X) \to \mathcal{ML}(S)$ is a homeomorphism, see [Ker2, Lem. 3.2].

Proof. Throughout the proof we use the natural coordinate z to identify A with a rectangle $R_A \subset \mathbb{C}$ whose vertical sides are identified.

Writing the restriction of ϕ to A in terms of the natural coordinate, we have

$$\phi = \phi(z)dz^2 = (\alpha + i\beta)^2 dz^2,$$

where α and β are real-valued functions defined locally up to a common sign away from the zeros of ϕ; in particular the functions α^2, β^2 and $|\alpha|$ are well-defined almost everywhere. We want to estimate

$$\begin{aligned}\langle \frac{d\overline{z}}{dz}, \phi \rangle &= \operatorname{Re} \int \frac{d\overline{z}}{dz}\, \phi(z) dz^2 \\ &= \int_A \operatorname{Re}\left((\alpha + i\beta)^2\right)\ |dz|^2 \\ &= \int_0^1 \int_0^{E(A)} (\alpha^2 - \beta^2) dx dy.\end{aligned}$$

Since A is a subset of X, we have

$$\int_0^1 \int_0^{E(A)} (\alpha^2 + \beta^2) dx dy = \int_A |\phi| \leq \int_X |\phi| = E(X). \tag{6.2}$$

Since $\beta^2 \geq 0$, this gives the desired upper bound on the pairing of $\frac{d\overline{z}}{dz}$ with ϕ:

$$\langle \frac{d\overline{z}}{dz}, \phi \rangle = \int_0^1 \int_0^{E(A)} (\alpha^2 - \beta^2) dx dy \leq \int_0^1 \int_0^{E(A)} (\alpha^2 + \beta^2) dx \leq E(X)$$

The horizontal lines in R_A represent closed curves in X homotopic to γ, so the total transverse measure of any one of these with respect to $\mathcal{F}(-\phi)$ is at least $E(X)$. Integrating over the unit interval of such horizontal curves in R_A we have

$$\int_0^1 \int_0^{E(A)} |\operatorname{Re} \sqrt{\phi}| dx dy = \int_0^1 \int_0^{E(A)} |\alpha| dx dy \geq E(X),$$

and applying the Cauchy-Schwarz inequality we obtain

$$\int_0^1 \int_0^{E(A)} \alpha^2 dx dy \geq \frac{E(X)^2}{E(A)}. \tag{6.3}$$

Multiplying (6.3) by 2 and subtracting (6.2) we have

$$\langle \frac{d\overline{z}}{dz}, \phi \rangle = \int_0^{E(A)} (\alpha^2 - \beta^2) dx dy \geq \frac{2E(X)^2}{E(A)} - E(X).$$

□

Note that the proof of Lemma 6.4 is essentially a calculation on the annulus A and uses little about the enclosing surface X except that it is foliated by closed trajectories of ϕ. Indeed, the same argument can be applied with the compact surface X replaced by an annulus B and with $\phi = dz_B^2$ the natural quadratic differential on B, and it is this version we will need in the proof of Theorem 6.6:

Lemma 6.5. *Let B be an annular Riemann surface of finite modulus and $A \subset B$ a homotopically essential subannulus. Then*

$$E(B) \geq \langle \frac{d\overline{z_A}}{dz_A}, dz_B^2 \rangle \geq \frac{2E(B)^2}{E(A)} - E(B)$$

Extremal length and grafting rays. Using Lemma 6.4 as the main technical tool, we are now ready to give the

Proof of Theorem 6.2. By Theorem 6.1, the grafting rays and their derivatives vary continuously over $\mathcal{ML}(S)$. Since multiples of simple closed geodesics are dense in $\mathcal{ML}(S)$, and estimates (1) and (2) of Theorem 6.2 are homogeneous, it suffices to consider the case when $\lambda = \gamma$ is a simple closed geodesic with unit weight. For brevity let $Y_t = \mathrm{gr}_{t\gamma} X$; we abbreviate $E(t) = E(\gamma, Y_t)$ and $\ell = \ell(\gamma, X)$.

The proof of (1) follows the usual pattern for an extremal length estimate (see [Ker2, §3]): a particular annulus homotopic to γ bounds $E(t)$ from above, while a particular conformal metric on the surface bounds $E(t)$ from below. In this case the annulus is the grafting cylinder $A_t \subset Y_t$ of modulus t/ℓ, and the conformal metric on Y_t is the *Thurston metric*–the union of the product metric on $A_t = [0, t] \times \gamma$ and the hyperbolic metric of X (see [Tan, §2.1]). Applying the geometric and analytic definitions of extremal length gives

$$\frac{\ell}{t} > E(\gamma, Y_t) > \frac{\ell^2}{t\ell + A} > \frac{\ell}{t} - \frac{A}{t^2}$$

where $A = 4\pi(g-1)$ and $(t\ell + A)$ is the area of the Thurston metric; thus (1) follows.

Now we estimate the derivative of extremal length. For all $s, t > 0$, there is a natural quasiconformal map from Y_t to Y_s that is affine on A_t, stretching it vertically in the natural coordinate, and conformal on $(Y_t - A_t)$. The derivative of this family of maps at $s = t$ is the Beltrami differential

$$\mu(t) = -(2t)^{-1}\frac{d\overline{z_t}}{dz_t}$$

where z_t is the natural coordinate on the grafting annulus A_t and the Beltrami differential $\frac{d\overline{z_t}}{dz_t}$ is understood to be identically zero outside A_t.

By Theorem 6.3, the derivative of extremal length along the grafting ray is

$$\frac{d}{dt}E(t) = 2\langle \mu(t), \phi(t)\rangle = -t^{-1}\langle \frac{d\overline{z_t}}{dz_t}, \phi(t)\rangle. \tag{6.4}$$

We estimate the pairing $\langle \frac{d\overline{z_t}}{dz_t}, \phi(t)\rangle$ using Lemma 6.4; starting with the upper bound, we have

$$\langle \frac{d\overline{z_t}}{dz_t}, \phi(t)\rangle \leq E(t) \leq \frac{\ell}{t} \tag{6.5}$$

while the lower bound from the lemma gives

$$\langle \frac{d\overline{z_t}}{dz_t}, \phi(t)\rangle \geq \frac{2E(t)^2}{E(A_t)} - E(t) = \frac{2t}{\ell}E(t)^2 - E(t).$$

Using $E(t) \geq \frac{\ell}{t} - \frac{A}{t^2}$ on the first term and $E(t) \leq \frac{\ell}{t}$ on the second, we obtain

$$\langle \frac{d\overline{z_t}}{dz_t}, \phi(t)\rangle \;\geq\; \frac{2t}{\ell}\left(\frac{\ell}{t} - \frac{A}{t^2}\right)^2 - \frac{\ell}{t} \;\geq\; \frac{\ell}{t} - \frac{4A}{t^2}. \tag{6.6}$$

Multiplying (6.5) and (6.6) by $-1/t$ and using the formula (6.4) for the derivative of extremal length, we have

$$-\frac{\ell(\gamma, X)}{t^2} \leq \frac{d}{dt}E(\gamma, \mathrm{gr}_{t\gamma} X) \leq -\frac{\ell(\gamma, X)}{t^2} + \frac{4A}{t^3},$$

which gives part (2) of Theorem 6.2. □

As a consequence of Theorem 6.2, extremal length decreases along grafting rays outside of a compact set in $\mathcal{T}(S)$ of the form $\{\mathrm{gr}_\lambda X \mid \ell(\lambda, X) \leq C\}$.

Hyperbolic length. The hyperbolic length of λ along a grafting ray is more difficult to control than the extremal length, but for the case of a single curve with large weight, the same techniques used in the proof of Theorem 6.2 give:

Theorem 6.6. *Let $X \in \mathcal{T}(S)$ and let γ be a simple closed hyperbolic geodesic on X. Then the hyperbolic length of γ is of order $1/t$ on the γ-grafting ray and is monotone decreasing for all $t \gg 0$. Specifically, we have*

(1) $\ell(\gamma, \mathrm{gr}_{t\gamma} X) = \frac{\pi\ell(\gamma,X)}{t} + O(t^{-2})$
(2) $\frac{d}{dt}\ell(\gamma, \mathrm{gr}_{t\gamma} X) = -\frac{\pi\ell(\gamma,X)}{t^2} + O(t^{-3})$

as $t \to \infty$, where the implicit constants depend on X and γ.

Note that Theorem 6.6 includes the results stated in the introduction as Theorem 1.5.

Proof. As before let $Y_t = \mathrm{gr}_{t\gamma} X$, and abbreviate $\ell(t) = \ell(\gamma, Y_t)$ and $E(t) = \ell(\gamma, Y_t)$.

A standard argument using the collar lemma shows that the extremal and hyperbolic length of a curve are asymptotically proportional when the length is small; specifically, we have

$$\pi E(t) < \ell(t) < \pi E(t) + CE(t)^2 \tag{6.7}$$

for all t such that $E(t) < 1$, where C is a universal constant. Since $E(t) < 1$ for all $t > \ell(\gamma, X)$, substituting the estimate for $E(t)$ from Theorem 6.2 gives (1).

Now we establish the derivative estimate (2). Let $\hat{Y}_t$ denote the cover of Y_t corresponding to the subgroup $\langle\gamma\rangle \subset \pi_1(Y_t)$. Since $\hat{Y}_t$ is conformally equivalent to an annulus of modulus $\pi/\ell(t)$, we have $\ell'(t) = \pi \frac{d}{dt} E(\hat{Y}_t)$.

Recall that $\mu(t) = (-2t)^{-1} \frac{d\overline{z_t}}{dz_t}$ represents the derivative of Y_t, where z_t is the natural coordinate on A_t. Thus the derivative of the annular covers $\hat{Y}_t$ is represented by the pullback Beltrami differential $p^*(\mu(t))$ where $p : \hat{Y}_t \to Y_t$ is the covering projection. Let w_t denote the natural coordinate on $\hat{Y}_t$. By (6.1) we have

$$\ell'(t) = \pi \frac{d}{dt} E(\hat{Y}_t) = 2\pi \left\langle p^*(\mu(t)), dw_t^2 \right\rangle. \tag{6.8}$$

To estimate this pairing, we analyze the differential $p^*(\mu(t))$; its support is the preimage of the grafting cylinder $p^{-1}(A_t) \subset \hat{Y}_t$, which consists of

(i) A homotopically essentially annulus $\hat{A}_t$ such that $p|_{\hat{A}_t} : \hat{A}_t \to A_t$ is a conformal isomorphism, and

(ii) A complementary set $\Omega = p^{-1}(A_t) - \hat{A}_t$ that is a disjoint union of countably many simply connected regions $\Sigma_i \subset \hat{Y}_t$ such that the restriction of p to any one of them gives a universal covering $p|_{\Sigma_i} : \Sigma_i \to A_t$,

as depicted in Figure 4. Therefore we have

$$\langle p^*(\mu(t)), dw_t^2 \rangle = \int_{\hat{A}_t} p^*(\mu(t)) dw_t^2 + \int_{\Omega} p^*(\mu(t)) dw_t^2, \tag{6.9}$$

and we can analyze these two terms individually.

By the length estimates of part (1), the reciprocal moduli $E(\hat{A}_t) = E(A_t) = \ell(0)/t$ and $E(\hat{Y}_t) = \ell(t)/\pi$ differ by $O(t^{-2})$. It follows that the subannulus $\hat{A}_t$ accounts for nearly all of the area of $\hat{Y}_t$. In fact, restricting the metric $|dw_t|$ to $\hat{A}_t$ and using the analytic definition of extremal length gives

$$\mathrm{Area}(\hat{A}_t, |dw_t|) \geq \frac{\ell(\gamma, |dw_t|)^2}{E(\hat{A}_t)} = \frac{E(\hat{Y}_t)^2}{E(\hat{A}_t)} \geq \frac{\ell(0)}{t} - \frac{C}{t^2},$$

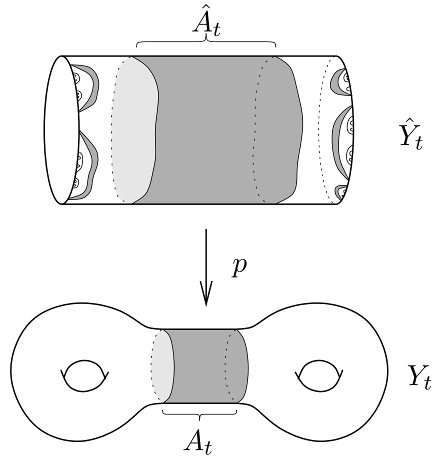

FIGURE 4. The grafting annulus and its lifts to the annular cover.

where C depends on γ and X. Since $\text{Area}(\hat{Y}_t, |dw_t|) = \ell(0)/t$ and Ω is disjoint from $\hat{A}_t$, we have $\text{Area}(\Omega, |dw_t|) \leq C/t^2$.

This area estimate implies that the second term in (6.9) is negligible, i.e.

$$\begin{aligned} \left| \int_\Omega p^*(\mu(t)) dw_t^2 \right| &\leq \|p^*(\mu(t))\|_\infty \text{Area}(\Omega, |dw_t|) \\ &\leq \left(\frac{1}{2t}\right)\left(\frac{C_2}{t^2}\right) = O(t^{-3}). \end{aligned} \tag{6.10}$$

Now we consider the first term in (6.9). Since $p|_{\hat{A}_t}$ is a conformal isomorphism, we have $p^*(\mu(t))|_{\hat{A}_t} = (-2t)^{-1}\frac{d\overline{\hat{z}_t}}{d\hat{z}_t}$ where $\hat{z}_t$ is the natural coordinate of $\hat{A}_t$. Applying Lemma 6.5 we have

$$E(\hat{Y}_t) \geq \langle \frac{d\overline{\hat{z}_t}}{d\hat{z}_t}, dw_t^2 \rangle \geq \frac{2E(\hat{Y}_t)^2}{E(\hat{A}_t)} - E(\hat{Y}_t).$$

As before we substitute $E(\hat{Y}_t) = \ell(t)/\pi$, $E(\hat{A}_t) = \ell(0)/t$, and apply the estimates for $\ell(t)$ to obtain

$$\int_{\hat{A}_t} p^*(\mu(t)) dw_t^2 = (-2t)^{-1} \langle \frac{d\overline{\hat{z}_t}}{d\hat{z}_t}, dw_t^2 \rangle = -\frac{\ell(0)}{2t^2} + O(t^{-3}).$$

Thus we have estimates for both terms in (6.9), and applying the formula (6.8) for $\ell'(t)$ gives the desired result:

$$\ell'(t) = -\frac{\pi \ell(0)}{t^2} + O(t^{-3}).$$

□

Remark. In [McM, Cor. 3.2], McMullen shows that $\ell(\lambda, \operatorname{gr}_{t\lambda} X) < \ell(\lambda, X)$ for all $X \in \mathcal{T}(S)$, $\lambda \in \mathcal{ML}(S)$, and $t > 0$; it is also mentioned that this upper bound can be strengthened to

$$\ell(\gamma, \operatorname{gr}_{t\gamma} X) \leq \frac{\pi}{\pi + t} \ell(\gamma, X).$$

for any simple closed curve γ. Theorem 6.6 shows that this upper bound is asymptotically sharp.

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