MINIMAL SURFACES OF LEAST TOTAL CURVATURE AND MODULI SPACES OF PLANE POLYGONAL ARCS

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Introduction. The first major goal of this paper is to prove the existence of complete minimal surfaces of each genus $p > 1$ which minimize the total curvature (equivalently, the degree of the Gauß map) for their genus. The genus zero version of these surfaces is known as Enneper's surface (see [Oss2]) and the genus one version is due to Chen-Gackstatter ([CG]). Recently, experimental evidence for the existence of these surfaces for genus $p \leq 35$ was found by Thayer ([Tha]); his surfaces, like those in this paper, are hyperelliptic surfaces with a single end, which is asymptotic to the end of Enneper's surface.

Our methods for constructing these surfaces are somewhat novel, and as their development is the second major goal of this paper, we sketch them quickly here. As in the construction of other recent examples of complete immersed (or even embedded) minimal surfaces in $\mathbf{E}^3$, our strategy centers around the Weierstraß representation for minimal surfaces in space, which gives a parametrization of the minimal surface in terms of meromorphic data on the Riemann surface which determine three meromorphic one-forms on the underlying Riemann surface.

The art in finding a minimal surface via this representation lies in finding a Riemann surface and meromorphic data on that surface so that the representation is well-defined, i.e., the local Weierstraß representation can be continued around closed curves without changing its definition. This latter condition amounts to a condition on the imaginary parts of some periods of forms associated to the original Weierstraß data.

In many of the recent constructions of complete minimal surfaces, the geometry of the desired surface is used to set up a space of possible Weierstraß data and Riemann surfaces,

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and then to consider the period problem as a purely analytical one. This approach is very effective as long as the dimension of the space of candidates remains small. This happens for instance if enough symmetry of the resulting surface is assumed so that the moduli space of candidate (possibly singular) surfaces has very small dimension – in fact, there is sometimes only a single surface to consider. Moreover, the candidate quotient surfaces (for instance, a thrice punctured sphere) often have a relatively well-understood function theory which serves to simplify the space of possibilities, even if the (quite difficult) period problem for the Weierstraß data still remains.

In our situation however, the dimension of the space of candidates grows with the genus. Our approach is to first view the periods and the conditions on them as defining a geometric object (and inducing a construction of a pair of Riemann surfaces), and to then prove analytically that the Riemann surfaces are identical, employing methods from Teichmüller theory.

Generally speaking, our approach is to construct two different Riemann surfaces, each with a meromorphic one-form, so that the period problem would be solved if only the surfaces would coincide. To arrange for a situation where we can simultaneously define a Riemann surface, and a meromorphic one-form on that surface with prescribed periods, we exploit the perspective of a meromorphic one-form as defining a singular flat structure on the Riemann surface, which we can develop onto $\mathbf{E}^2$.

In particular, we first assume sufficient symmetry of the Riemann surface so that the quotient orbifold flat structure has a fundamental domain in $\mathbf{E}^2$ which is bounded by a properly embedded arc composed of $2p+2$ horizontal and vertical line segments with the additional properties that the segments alternate from horizontal segments to vertical segments, with the direction of travel also alternating between left and right turns. We call such an arc a 'zigzag'; further we restrict our attention to 'symmetric zigzags', those zigzags which are symmetric about the line $\{y = x\}$ (see Figure 1).

A crucial observation is that we can turn this construction around. Observe that a zigzag Z bounds two domains, one, $\Omega_{\mathrm{NE}}(Z)$, on the northeast side, and one, $\Omega_{\mathrm{SW}}(Z)$, on the southwest side. When we double each of these domains and then take a double cover of the resulting surface, branched over each of the images of the vertices of Z, we have two hyperelliptic Riemann surfaces, $\mathcal{R}_{\mathrm{NE}}(Z)$ and $\mathcal{R}_{\mathrm{SW}}(Z)$, respectively. Moreover, the form dz when restricted to $\Omega_{\mathrm{NE}}(Z)$ and $\Omega_{\mathrm{SW}}(Z)$, lifts to meromorphic one-forms $\omega_{\mathrm{NE}}(Z)$ on $\Omega_{\mathrm{NE}}(Z)$ and $\omega_{\mathrm{SW}}(Z)$ on $\Omega_{\mathrm{SW}}(Z)$ both of whose sets of periods are integral linear combinations of the periods of dz along the horizontal and vertical arcs of Z.

Then, suppose for a moment that we can find a zigzag Z so that $\Omega_{\mathrm{NE}}(Z)$ is conformally equivalent to $\Omega_{\mathrm{SW}}(Z)$ with the conformal equivalence taking vertices to vertices (where ∞ is considered a vertex). (We call such a zigzag *reflexive*.) Then $\mathcal{R}_{\mathrm{NE}}(Z)$ would be conformally equivalent to $\mathcal{R}_{\mathrm{SW}}(Z)$ in a way that $e^{i\pi/4}\omega_{\mathrm{NE}}(Z)$ and $e^{-i\pi/4}\omega_{\mathrm{SW}}(Z)$ have conjugate periods. As these forms will represent gdh and $g^{-1}dh$ in the classical Weierstraß representation $X = \mathrm{Re}\int(\frac{1}{2}\left(g - \frac{1}{g}\right), \frac{i}{2}\left(g + \frac{1}{g}\right), 1)\ dh$, it will turn out that this conformal equivalence is just what we need for the Weierstraß representation based on $\omega_{\mathrm{NE}}(Z)$ and $\omega_{\mathrm{SW}}(Z)$ to be well-defined.

This construction is described precisely in §3.

We are left to find such a zigzag. Our approach is non-constructive in that we consider the space $\mathcal{Z}_p$ of all possible symmetric zigzags with $2p+2$ vertices and then seek, within that space $\mathcal{Z}$, a symmetric zigzag Z_0 for which there is a conformal equivalence between $\Omega_{\mathrm{NE}}(Z_0)$ and $\Omega_{\mathrm{SW}}(Z_0)$ which preserves vertices. The bulk of the paper, then, is an analysis of this moduli space $\mathcal{Z}_p$ and some functions on it, with the goal of finding a certain fixed point within it.

Our methods, at least in outline, for finding such a symmetric zigzag are quite standard in contemporary Teichmüller theory. We first find that the space $\mathcal{Z}_p$ is topologically a cell, and then we seek an appropriate height function on it. This appropriate height function should be proper, so that it has an interior critical point, and it should have the feature that at its critical point $Z_0 \in \mathcal{Z}_p$, we have the desired vertex-preserving conformal equivalence between $\Omega_{\mathrm{NE}}(Z_0)$ and $\Omega_{\mathrm{SW}}(Z_0)$.

One could imagine that a natural height function might be the Teichmüller distance between $\mathcal{R}_{\mathrm{NE}}(Z)$ and $\mathcal{R}_{\mathrm{SW}}(Z)$, but it is easy to see that there is a family Z_t of zigzags, some of whose vertices are coalescing, so that the Teichmüller distance between $\mathcal{R}_{\mathrm{NE}}(Z_t)$ and $\mathcal{R}_{\mathrm{SW}}(Z_t)$ tends to a finite number. We thus employ a different height function $D(\cdot)$ that, in effect, blows up small scale differences between $\mathcal{R}_{\mathrm{NE}}(Z_t)$ and $\mathcal{R}_{\mathrm{SW}}(Z_t)$ for a family of zigzags $\{Z_t\}$ that leave all compacta of $\mathcal{Z}_p$.

We discuss the space $\mathcal{Z}_p$ of symmetric zigzags, and study degeneration in that space in §4. We show that the map between the marked extremal length spectra for $\mathcal{R}_{\mathrm{NE}}(Z)$ and $\mathcal{R}_{\mathrm{SW}}(Z)$ is not real analytic at infinity in $\mathcal{Z}_p$, and thus there must be small scale differences between those extremal length spectra. We do this by first observing that both extremal lengths and Schwarz-Christoffel integrals can be computed using generalized hypergeometric functions; we then show that the well-known monodromy properties of these functions lead to a crucial sign difference in the asymptotic expansions of the Schwarz-Christoffel maps at regular singular points. Finally, these sign differences are exploited to yield the desired non-analyticity.

Our height function $D(Z)$ while not the Teichmüller distance between $\mathcal{R}_{\mathrm{NE}}(Z)$ and $\mathcal{R}_{\mathrm{SW}}(Z)$, is still based on differences between extremal lengths on those surfaces, and in effect, we follow the gradient flow dD on $\mathcal{Z}_p$ from a convenient initial point in $\mathcal{Z}_p$ to a solution of our problem. There are two aspects to this approach. First, it is especially convenient that we know a formula ([Gar]) for $d[\mathrm{Ext}_{[\gamma]}(\mathcal{R})]$ where $\mathrm{Ext}_{[\gamma]}(\mathcal{R})$ denotes the extremal length of the curve family $[\gamma]$ on a given Riemann surface $\mathcal{R}$. This gradient of extremal length is given in terms of a holomorphic quadratic differential $2\Phi_{[\gamma]}(\mathcal{R}) = d\,\mathrm{Ext}_{[\gamma]}(\mathcal{R})$ and can be understood in terms of the horizontal measured foliation of that differential providing a 'direction field' on $\mathcal{R}$ along which to infinitesimally deform $\mathcal{R}$ as to infinitesimally increase $\mathrm{Ext}_{[\gamma]}(\mathcal{R})$. We then show that grad $D(\cdot)\,\big|_Z$ can be understood in terms of a pair of holomorphic quadratic differentials on $\mathcal{R}_{\mathrm{NE}}(Z)$ and $\mathcal{R}_{\mathrm{SW}}(Z)$, respectively, whose (projective) measured foliations descend to a well-defined projective class of measured foliations on $\mathbf{C} = \Omega_{\mathrm{NE}}(Z) \cup Z \cup \Omega_{\mathrm{SW}}(Z)$. This foliation class then indicates a direction in which to infinitesimally deform Z so as to infinitesimally decrease $D(\cdot)$, as long as Z is not critical for $D(\cdot)$. Thus, a minimum for $D(\cdot)$ is a symmetric zigzag Z_0 for which $\Omega_{\mathrm{NE}}(Z_0)$ is conformally equivalent to $\Omega_{\mathrm{SW}}(Z)$ in a vertex preserving way. Second, it is technically convenient to flow along a path in $\mathcal{Z}_p$ in which the form of the height function simplifies. In fact, we will flow from a genus $p-1$ solution in $\mathcal{Z}_{p-1} \subset \partial\overline{\mathcal{Z}}$ along a path $\mathcal{Y}$ in $\mathcal{Z}_p$ to a genus p solution in $\mathcal{Z}_p$. Here the technicalities are that some Teichmüller theory and the symmetry we have imposed on the zigzags allow us to invoke the implicit function theorem at a genus $p-1$ solution in $\partial\overline{\mathcal{Z}}$ to find such a good path $\mathcal{Y} \subset \mathcal{Z}_p$. Thus, formally, we find a solution in each genus inductively, by showing that given a reflexive zigzag of genus $p-1$, we can 'add a handle' to obtain a solution of genus p.

We study the gradient flow of the height function $D(\cdot)$ in §5.

Combining the results in sections 4 and 5, we conclude

Main Theorem B. *There exists a reflexive symmetric zigzag of genus p for $p \geq 0$ which is isolated in $\mathcal{Z}_p$.*

When we interpret this result about zigzags as a result on Weierstraß data for minimally immersed Riemann surfaces in $\mathbf{E}^3$, we derive as a corollary

Main Theorem A. *For each $p \geq 0$, there exists a minimally immersed Riemann surfaces in $\mathbf{E}^3$ with one Enneper-type end and total curvature $-4\pi(p+1)$. This surface has at most eight self-isometries.*

In §6, we adapt our methods slightly to prove the existence of minimally immersed surfaces of genus $p(k-1)$ with one Enneper-type end of winding order $2k-1$: these surfaces extend and generalize examples of Karcher ([Kar]) and Thayer ([Tha]), as well as those constructed in Theorem A.

While we were preparing this manuscript several years ago we received a copy of a preprint by K. Sato [S] which also asserts Theorem A. Our approach is different than that of Sato, and possibly more general, as it is possible to assign zigzag configurations to a number of families of putative minimal surfaces. We discuss further applications of this technique in a forthcoming paper [WW].

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§2. Background and Notation.

2.1 Minimal Surfaces and the Weierstraß Representation. Here we recall some well-known facts from the theory of minimal surfaces and put our result into context.

Locally, a minimal surface can always be described by *Weierstraß data*, i.e. there are always a simply connected domain U, a holomorphic function g and a holomorphic 1-form dh in U such that the minimal surface is locally given by

$$z \mapsto \operatorname{Re} \int_{\cdot}^{z} \begin{matrix} \frac{1}{2}(g - \frac{1}{g})dh \\ \frac{i}{2}(g + \frac{1}{g})dh \\ dh \end{matrix}$$

For instance, $g(z) = z$ and $dh = \frac{dz}{z}$ will lead to the catenoid, while $g(z) = z$ and $dh = zdz$ yields the Enneper surface.

It is by no means clear how global properties of a minimal surface are related to this local representation.

However, two global properties together have very strong consequences on the Weierstraß data. One is the metrical completeness, and the other the total (absolute) Gaußian curvature of the surface R, defined by

$$\mathcal{K} := \int_R |K| dA = 4\pi \cdot \text{degree of the Gauß map}$$

We will call a complete minimal surface of finite absolute Gaußian curvature a *finite* minimal surface.

Then by a famous theorem of R. Osserman, every finite minimal surface (see [Oss1, Oss2, Laws]) can be represented by Weierstraß data which are defined on a compact Riemann surface R, punctured at a finite number of points. Furthermore, the Weierstraß data extend to meromorphic data on the compact surface. Thus, the construction of such surfaces is reduced to finding meromorphic Weierstraß data on a compact Riemann surface such that the above representation is well defined, i.e. such that all three 1-forms showing up there have purely imaginary periods. This is still not a simple problem.

From now on, we will restrict our attention to finite minimal surfaces.

Looked at from far away, the most visible parts of a finite minimal surface will be the ends. These can be seen from the Weierstraß data by looking at the singularities P_j of the Riemannian metric which is given by the formula

$$ds = \left(|g| + \frac{1}{|g|} \right) |dh| \tag{2.0}$$

An end occurs at a puncture P_j if and only if ds becomes infinite in the compactified surface at P_j.

To each end is associated its winding or spinning number d_j, which can be defined geometrically by looking at the intersection curve of the end with a very large sphere which will be close to a great circle and by taking its winding number, see [Gack, J-M].

This winding number is always odd: it is 1 for the catenoid end 3 for the Enneper end. There is one other end of winding number 1, namely the planar end which of course occurs as the end of the plane, but also as one end of the Costa surface, see [Cos1]. For finite minimal surfaces, there is a Gauß-Bonnet formula relating the total curvature to genus and winding numbers:

$$\int_R K dA = 2\pi \left(2(1-p) - r - \sum_{j=1}^{r} d_j \right)$$

where p is the genus of the surface, r the number of ends and d_j the winding number of an end. For a proof, see again [Gack, J-M].

From this formula one can conclude that for a non planar surface $\int_R |K| dA \leq 4\pi(p+1)$ which raises the question of finding for each genus a (non-planar) minimal surface for which equality holds. This is the main goal of the paper.

The following is known:

- $p = 0$: The Enneper surface and the catenoid are the only non-planar minimal surfaces with $\mathcal{K} = 4\pi$, see e.g. [Oss2].
- $p = 1$: The only surface with $\mathcal{K} = 8\pi$ is the Chen-Gackstatter surface (which is defined on the square torus), see [CG, Lop, Blo].
- $p = 2$: An example with $\mathcal{K} = 12\pi$ was also constructed in [CG]. Uniqueness is not known here.
- $p = 3$: An example with $\mathcal{K} = 16\pi$ was constructed by do Espírito-Santo ([Esp]).
- $p \leq 35$: E. Thayer has solved the period problem numerically and produced pictures of surfaces with minimal $\mathcal{K}$.

Note that all these surfaces are necessarily not embedded: For given genus, a finite minimal surface of minimal $\mathcal{K}$ could, by the winding number formula, have only either one end of Enneper type (winding number 3) which is not embedded or two ends of winding number 1. But by a theorem of R. Schoen ([Sch]), an embedded finite minimal surface with only two ends has to be the catenoid. Hence if one looks for embedded minimal surfaces, one has to allow more $\mathcal{K}$. For the state of the art here, see [Ho-Ka].

If one allows even more total curvature and permits non-embeddedness, some general methods are available, as explained in [Kar].

2.2. Zigzags. A *zigzag* Z of genus p is an open and properly embedded arc in C composed of alternating horizontal and vertical subarcs with angles of $\pi/2$, $3\pi/2$, $\pi/2$, $3\pi/2, \ldots, \pi/2$ between consecutive sides, and having $2p+1$ vertices ($2p+2$ sides, including an initial infinite vertical side and a terminal infinite horizontal side.) A *symmetric* zigzag of genus p is a zigzag of genus p which is symmetric about the line $\{y = x\}$. The space $\mathcal{Z}_p$ of genus p zigzags consists of all symmetric zigzags of genus p up to similarity; it is equipped with the topology induced by the embedding of $\mathcal{Z}_p \longrightarrow \mathbf{R}^{2p}$ which associates to a zigzag Z the $2p$-tuple of its lengths of sides, in the natural order.

A symmetric zigzag Z divides the plane $\mathbf{C}$ into two regions, one which we will denote by $\Omega_{\mathrm{NE}}(Z)$ which contains large positive values of $\{y = x\}$, and the other which we will denote by $\Omega_{\mathrm{SW}}(Z)$. (See Figure 1.)

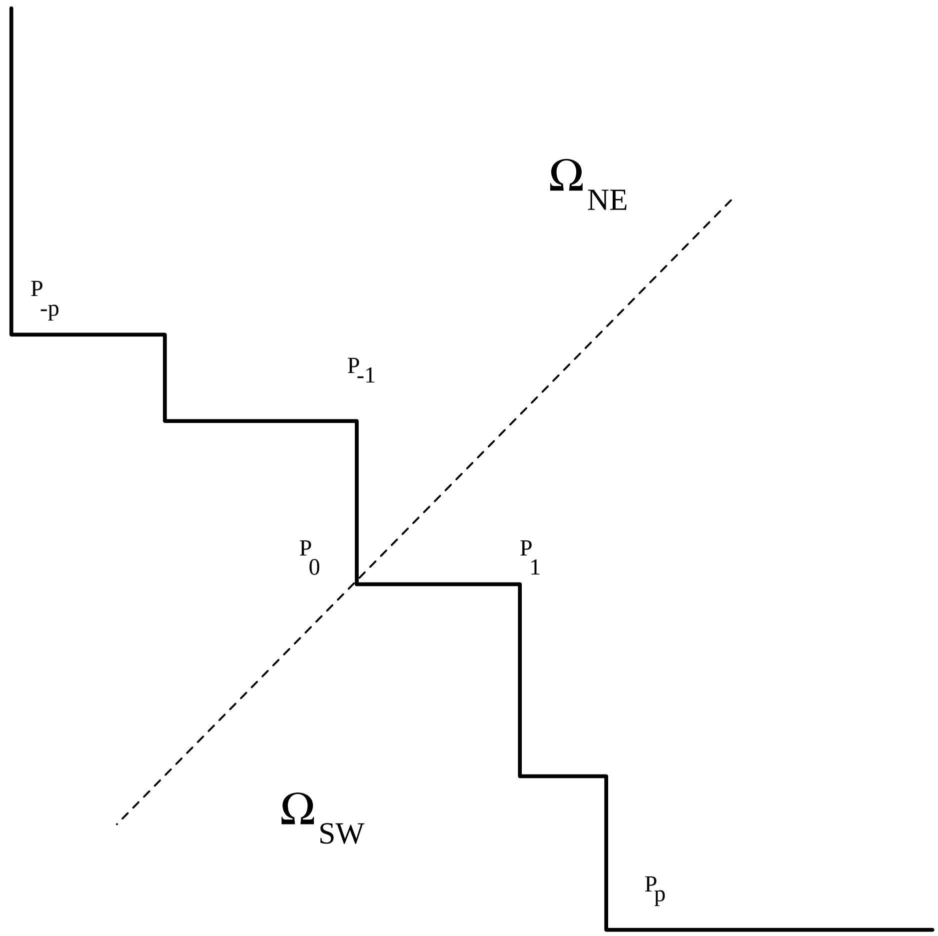

Figure 1

Definition 2.2.1. A symmetric zigzag Z is called *reflexive* if there is a conformal map $\phi : \Omega_{\mathrm{NE}}(Z) \to \Omega_{\mathrm{SW}}(Z)$ which takes vertices to vertices.

Examples 2.2.2. There is only one zigzag of genus 0, consisting of the positive imaginary and positive real half-axes. It is automatically symmetric and reflexive.

Every symmetric zigzag of genus 1 is also automatically reflexive.

2.3. Teichmüller Theory. For M a smooth surface, let Teich (M) denote the Teichmüller space of all conformal structures on M under the equivalence relation given by pullback by diffeomorphisms isotopic to the identity map id: $M \longrightarrow M$. Then it is well-known that Teich (M) is a smooth finite dimensional manifold if M is a closed surface.

There are two spaces of tensors on a Riemann surface $\mathcal{R}$ that are important for the Teichmüller theory. The first is the space $\mathrm{QD}(\mathcal{R})$ of holomorphic quadratic differentials, i.e., tensors which have the local form $\Phi = \varphi(z)dz^2$ where $\varphi(z)$ is holomorphic. The second is the space of Beltrami differentials $\mathrm{Belt}(\mathcal{R})$, i.e., tensors which have the local form $\mu = \mu(z)d\bar{z}/dz$.

The cotangent space $T^*_{[\mathcal{R}]}(\text{Teich }(M))$ is canonically isomorphic to $\mathrm{QD}(\mathcal{R})$, and the tangent space is given by equivalence classes of (infinitesimal) Beltrami differentials, where

μ_1 is equivalent to μ_2 if

$$\int_{\mathcal{R}} \Phi(\mu_1 - \mu_2) = 0 \qquad \text{for every} \quad \Phi \in \mathrm{QD}(\mathcal{R}).$$

If $f : \mathbf{C} \to \mathbf{C}$ is a diffeomorphism, then the Beltrami differential associated to the pullback conformal structure is $\nu = \frac{f_{\bar z}}{f_z}\frac{d\bar z}{dz}$. If f_ϵ is a family of such diffeomorphisms with $f_0 = \mathit{identity}$, then the infinitesimal Beltrami differential is given by $\frac{d}{d\epsilon}\Big|_{\epsilon=0} \nu_{f_\epsilon} = \left(\frac{d}{d\epsilon}\Big|_{\epsilon=0} f_\epsilon\right)_{\bar z}$. We will carry out an example of this computation in §5.2.

A holomorphic quadratic differential comes with a picture that is a useful aid to one's intuition about them. The picture is that of a pair of transverse measured foliations, whose properties we sketch briefly (see [FLP], [Ke], and [Gar] for more details). We next define a measured foliation on a (possibly) punctured Riemann surface; to set notation, in what follows, the Riemann surface $\mathcal{R}$ is possibly punctured, i.e. there is a closed Riemann surface $\overline{\mathcal{R}}$, and a set of points $\{q_1, \dots, q_m\}$, so that $\mathcal{R} = \overline{\mathcal{R}} - \{q_1, \dots, q_m\}$.

A measured foliation $(\mathcal{F}, \mu)$ on a Riemann surface $\mathcal{R}$ with singularities $\{p_1, \dots, p_l\}$ (where some of the singularities might also be elements of the puncture set $\{q_1, \dots, q_m\}$) consists of a foliation $\mathcal{F}$ of $\mathcal{R} - \{p_1, \dots, p_l\}$ and a measure μ as follows. If the foliation $\mathcal{F}$ is defined via local charts $\phi_i : U_i \longrightarrow \mathbb{R}^2$ (where $\{U_i\}$ is a covering of $\mathcal{R} - \{p_1, \dots, p_l\}$) which send the leaves of $\mathcal{F}$ to horizontal arcs in $\mathbb{R}^2$, then the transition functions $\phi_{ij} : \phi_i(U_i) \longrightarrow \phi_j(U_j)$ on $\phi_i(U_i) \subset \mathbb{R}^2$ are of the form $\phi_{ij}(x, y) = (h(x, y), c \pm y)$; here the function h is an arbitrary continuous map, but c is a constant. We require that the foliation in a neighborhood (in $\overline{\mathcal{R}}$) of the singularities be topologically equivalent to those that occur at the origin in $\mathbb{C}$ of the integral curves of the line field $z^k dz^2 > 0$ where $k \geq -1$. (There are easy extensions to arbitrary integral k, but we will not need those here.)

We define the measure μ on arcs $A \subset \mathcal{R}$ as follows: the measure $\mu(A)$ is given by

$$\mu(A) = \int_A |dY|$$

where $|dY|$ is defined, locally, to be the pullback $|dY|_{U_i} = \phi_i^*(|dy|)$ of the horizontal transverse measure $|dy|$ on $\mathbb{R}^2$. Because of the form of the transition functions ϕ_{ij} above, this measure is then well-defined on arcs in $\mathcal{R}$.

An important feature of this measure (that follows from its definition above) is its "translation invariance". That is, suppose $A_0 \subset \mathcal{R}$ is an arc transverse to the foliation $\mathcal{F}$, with ∂A_0 a pair of points, one on the leaf l and one on the leaf l'; then, if we deform A_0 to A_1 via an isotopy through arcs A_t that maintains the transversality of the image of A_0 at every time, and also keeps the endpoints of the arcs A_t fixed on the leaves l and l', respectively, then we observe that $\mu(A_0) = \mu(A_1)$.

Now a holomorphic quadratic differential Φ defines a measured foliation in the following way. The zeros $\Phi^{-1}(0)$ of Φ are well-defined; away from these zeros, we can choose a canonical conformal coordinate $\zeta(z) = \int^z \sqrt{\Phi}$ so that $\Phi = d\zeta^2$. The local measured foliations $(\{\operatorname{Re}\zeta = \text{const}\}, |d\operatorname{Re}\zeta|)$ then piece together to form a measured foliation known as the vertical measured foliation of Φ, with the translation invariance of this measured foliation of Φ following from Cauchy's theorem.

Work of Hubbard and Masur ([HM]) (see also alternate proofs in [Ke], [Gar] and [Wo]), following Jenkins ([J]) and Strebel ([Str]), showed that given a measured foliation $(\mathcal{F}, \mu)$ and a Riemann surface $\mathcal{R}$, there is a unique holomorphic quadratic differential Φ_μ on $\mathcal{R}$ so that the horizontal measured foliation of Φ_μ is equivalent to $(\mathcal{F}, \mu)$.

Extremal length. The extremal length $\mathrm{Ext}_{\mathcal{R}}([\gamma])$ of a class of arcs , on a Riemann surface $\mathcal{R}$ is defined to be the conformal invariant

$$\sup_\rho \frac{\ell_\rho^2(,\ )}{\mathrm{Area}(\rho)}$$

where ρ ranges over all conformal metrics on $\mathcal{R}$ with areas $0 < \mathrm{Area}(\rho) < \infty$ and $\ell_\rho(,\ )$ denotes the infimum of ρ-lengths of curves $\gamma \in$, . Here , may consist of all curves freely homotopic to a given curve, a union of free homotopy classes, a family of arcs with endpoints in a pair of given boundaries, or even a more general class. Kerckhoff ([K]) showed that this definition of extremal lengths of curves extended naturally to a defintion a extremal lengths of measured foliations.

For a class , consisting of all curves freely homotopic to a single curve $\gamma \subset M$, (or more generally, a measured foliation $(\mathcal{F}, \mu)$ we see that $\mathrm{Ext}_{(\cdot)}(,\ )$ (or $\mathrm{Ext}_{(\cdot)}(\mu)$) can be construed as a real-valued function $\mathrm{Ext}_{(\cdot)}(,\ )$: $\mathrm{Teich}(M) \longrightarrow \mathcal{R}$. Gardiner ([Gar]) showed that $\mathrm{Ext}_{(\cdot)}(\mu)$ is differentiable and Gardiner and Masur ([GM]) showed that $\mathrm{Ext}_{(\cdot)}(\mu) \in C^1$ ($\mathrm{Teich}(M)$). [In our particular applications, the extremal length functions on our moduli spaces will be real analytic: this will be explained in §4.5.] Moreover Gardiner computed that

$$d\,\mathrm{Ext}_{(\cdot)}(\mu)\ \Big|_{[\mathcal{R}]} = 2\Phi_\mu$$

so that

$$\left(d\,\mathrm{Ext}_{(\cdot)}(\mu)\ \Big|_{[\mathcal{R}]}\right)[\nu] = 4\,\mathrm{Re}\int_{\mathcal{R}} \Phi_\mu \nu. \tag{2.1}$$

Teichmüller maps, Teichmüller distance. (This material will only be used in an extended digression in §5.5) Recall that points in Teichmüller space can also be defined to be equivalence classes of Riemann surface structures R on M, the structure $\mathcal{R}_1$ being equivalent to the structure $\mathcal{R}_2$ if there is a homeomorphism $h : M \to M$, homotopic to the identity, which is a conformal map of the structures $\mathcal{R}_1$ and $\mathcal{R}_2$.

We define the Teichmüller distance $d(\{\mathcal{R}_1\}, \{\mathcal{R}_2\})$ by

$$d_{Teich}(\{\mathcal{R}_1\}, \{\mathcal{R}_2\}) = \frac{1}{2}\log\inf_h K(h)$$

where $h : \mathcal{R}_1 \to \mathcal{R}_2$ is a quasiconformal homeomorphism homotopic to the identity on M and $K[h]$ is the maximal dilatation of h. This metric is well-defined, so we may unambiguously write $\mathcal{R}_1$ for $\{\mathcal{R}_1\}$.

An extraordinary fact about this metric is that the extremal maps, known as Teichmüller maps, admit an explicit description, as does the family of maps which describe a geodesic.

Teichmüller's theorem asserts that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are distinct points in T_g, then there is a unique quasiconformal $h : \mathcal{R}_1 \to \mathcal{R}_2$ with h homotopic to the identity on M which minimizes the maximal dilatation of all such h. The complex dilatation of h may be written $\mu(h) = k\frac{\bar{q}}{|q|}$ for some non-trivial $q \in \mathrm{QD}(\mathcal{R}_1)$ and some k, $0 < k < 1$, and then

$$d_{Teich}(\mathcal{R}_1, \mathcal{R}_2) = \frac{1}{2}\log(1+k)/(1-k).$$

Conversely, for each $-1 < k < 1$ and non-zero $q \in \mathrm{QD}(S_1)$, the quasiconformal homeomorphism h_k of $\mathcal{R}_1$ onto $h_k(\mathcal{R}_2)$, which has complex dilatation $k\bar{q}/|q|$, is extremal in its homotopy class. Each extremal h_k induces a quadratic differential q_k' on $h_k(\mathcal{R}_1)$, with critical points of q and q_k' corresponding under h_k; furthermore, to the natural parameter w for q near $p \in S_1$ there is a natural parameter w_k' near $h_k(p)$ so that

$$\operatorname{Re} w_k' = K^{1/2}\operatorname{Re} w \quad \text{and} \quad \operatorname{Im} w_k' = K^{-1/2}\operatorname{Im} w,$$

where $K = (1+k)/(1-k)$. In particular, the horizontal (and vertical) foliations for q and q_k' are equivalent.

The map h_k is called the Teichmüller extremal map determined by q and k; the differential q is called the initial differential and the differential q_k is called the terminal differential. We can assume all quadratic differentials are normalized in the sense that

$$||q|| = \int |q| = 1.$$

The Teichmüller geodesic segment between S_1 and S_2 consists of all points $h_s(\mathcal{R}_1)$ where the h_s are Teichmüller maps on $\mathcal{R}_1$ determined by the quadratic differential $q \in \mathrm{QD}(\mathcal{R}_1)$ corresponding to the Teichmüller map $h : \mathcal{R}_1 \to \mathcal{R}_2$ and $s \in [0, \|\mu(h)\|_\infty]$.

Kerckhoff [K] has given a characterization of the Teichmüller metric $d_{Teich}(\mathcal{R}_1, \mathcal{R}_2)$ in terms of the extremal lengths of corresponding curves on the surfaces. He proves

$$d_{Teich}(\mathcal{R}_1, \mathcal{R}_2) = \frac{1}{2}\log\sup_{\gamma} \frac{\mathrm{Ext}_{\mathcal{R}_1}(\gamma)}{\mathrm{Ext}_{\mathcal{R}_2}(\gamma)} \tag{2.2}$$

where the supremum ranges over all simple closed curves on M.

§3. From Zigzags to minimal surfaces.

Let Z be a zigzag of genus p dividing the plane into two regions Ω_{NE} and Ω_{SW}. We denote the vertices of Ω_{NE} consecutively by $P_{-p}, \ldots, P_p$ and set $P_\infty = \infty$. The vertices of Ω_{SW} however are labeled in the opposite order $Q_j := P_{-j}$ and $Q_\infty = \infty$. We double both regions to obtain punctured spheres S_{NE} and S_{SW} whose punctures are also called P_j and Q_j. Finally we take hyperelliptic covers $\mathcal{R}_{\text{NE}}$ over S_{NE}, branched over the P_j, and $\mathcal{R}_{\text{SW}}$ over S_{SW}, branched over the Q_j, to obtain two hyperelliptic Riemann surfaces of genus p, punctured at the Weierstraß points which will still be called P_j and Q_j. The degree 2 maps to the sphere are called $\pi_{\text{NE}} : \mathcal{R}_{\text{NE}} \to S_{\text{NE}}$ and $\pi_{\text{SW}} : \mathcal{R}_{\text{SW}} \to S_{\text{SW}}$.

Example 3.1. For a genus 1 zigzag, the Riemann surfaces $\mathcal{R}_{\text{NE}}$ and $\mathcal{R}_{\text{SW}}$ will be square tori punctured at the three half-period points and the one full-period point.

Now suppose that the zigzag Z is reflexive. Then there is a conformal map $\phi : \Omega_{\text{NE}} \to \Omega_{\text{SW}}$ such that $\phi(P_j) = Q_j$. Clearly ϕ lifts to conformal maps $\phi : S_{\text{NE}} \to S_{\text{SW}}$ and $\phi : \mathcal{R}_{\text{NE}} \to \mathcal{R}_{\text{SW}}$ which again take punctures to punctures.

The surface $\mathcal{R}_{\text{NE}}$ will be the Riemann surface on which we are going to define the Weierstraß data. The idea is roughly as follows: If we look at the Weierstraß data for the Enneper surface, it is evident that the 1-forms $g\,dh$ and $\frac{1}{g}dh$ have simpler divisors than their linear combinations which actually appear in the Weierstraß representation as the first two coordinate differentials. Thus we are hunting for these two 1-forms, and we want to define them by the geometric properties of the (singular) flat metrics on the surface for which they specify the line elements, because this will encode the information we need to solve the period problem.

To do this, we look at the flat metrics on $\mathcal{R}_{\text{NE}}$ and $\mathcal{R}_{\text{SW}}$ which come from the following construction. First, the domains Ω_{NE} and Ω_{SW} obviously carry the flat euclidean metric ($ds = |dz|$) metrics. Doubling these regions defines flat (singular) metrics on the spheres S_{NE} and S_{SW} (with cone points at the lifts of the vertices of the zigzags and cone angles of alternately π and $3\pi/2$). These metrics are then lifted to $\mathcal{R}_{\text{NE}}$ and $\mathcal{R}_{\text{SW}}$ by the respective covering projections.

The exterior derivatives of the multivalued developing maps define single valued holomorphic nonvanishing 1-forms ω_{NE} on $\mathcal{R}_{\text{NE}}$ and ω_{SW} on $\mathcal{R}_{\text{SW}}$, because the flat metrics on the punctured surfaces have trivial linear holonomy. Furthermore, the behavior of these 1-forms at a puncture is completely determined by the cone angle of the flat metric at the puncture. Indeed, in a suitable local coordinate, the developing map of the flat metric near a puncture with cone angle $2\pi k$ is given by z^k. Hence the exterior derivative of the developing map will have a zero (or pole) of order $k-1$ there. Note that these considerations are valid for the point P_∞ as well if we allow negative cone angles. All this is well known in the context of meromorphic quadratic differentials, see [Str].

Examples 3.2. For the genus 0 zigzag, we obtain a 1-form ω_{NE} on the sphere $\mathcal{R}_{\text{NE}}$ with a pole of order 2 at P_∞ which we can call dz and a 1-form ω_{SW} on the sphere $\mathcal{R}_{\text{SW}}$ with a zero of order 2 at P_0 and a pole of order 4 at P_∞ which then is z^2dz.

For a symmetric genus 1 zigzag, ω_{NE} will be closely related to the Weierstraß $\wp$-function on the square torus, as it is a 1-form with double zero in P_0 and double pole at P_∞.

Furthermore, ω_{SW} is a meromorphic 1-form with double order zeroes in $P_{\pm 1}$ and fourth order pole at P_∞ which also can be written down in terms of classical elliptic functions.

In general, we can write down the divisors of our meromorphic 1-forms as:

$$(\omega_{\mathrm{NE}}) = \begin{cases} P_0^2 \cdot P_{\pm 2}^2 \cdot P_{\pm 4}^2 \cdot \ldots \cdot P_{\pm(p-1)}^2 \cdot P_\infty^{-2} & \text{p odd} \\ P_{\pm 1}^2 \cdot P_{\pm 3}^2 \cdot \ldots \cdot P_{\pm(p-1)}^2 \cdot P_\infty^{-2} & \text{p even} \end{cases}$$

$$(\omega_{\mathrm{SW}}) = \begin{cases} Q_{\pm 1}^2 \cdot Q_{\pm 3}^2 \cdot \ldots \cdot Q_{\pm p}^2 \cdot Q_\infty^{-4} & \text{p odd} \\ Q_0^2 \cdot Q_{\pm 2}^2 \cdot Q_{\pm 4}^2 \cdot \ldots \cdot Q_{\pm p}^2 \cdot Q_\infty^{-4} & \text{p even} \end{cases}$$

$$(d\pi_{\mathrm{NE}}) = P_0 \cdot P_{\pm 1} \cdot P_{\pm 2} \cdot \ldots \cdot P_{\pm p} \cdot P_\infty^{-3}$$

Now denote by $\alpha := e^{-\pi i/4} \cdot \omega_{\mathrm{NE}}$, $\beta := e^{-\pi i/4} \cdot \phi^* \omega_{\mathrm{SW}}$ and $dh := const \cdot d\pi_{\mathrm{NE}}$, where we choose the constant such that $\alpha \cdot \beta = dh^2$ which is possible because the divisors coincide. Now we can write down the Gauß map of the Weierstraß data on $\mathcal{R}_{\mathrm{NE}}$ as $g = \frac{\alpha}{dh}$, and we check easily that the line element (2.0) is regular everywhere on $\mathcal{R}_{\mathrm{NE}}$ except at the lift of P_∞.

One can check that the thus defined Weierstraß data coincide for the reflexive genus 0 and genus 1 zigzags with the data for the Enneper surface and the Chen-Gackstatter surface.

We can now claim

Theorem 3.3. If Z is a symmetric reflexive zigzag of genus p, then $(\mathcal{R}_{\mathrm{NE}}, g, dh)$ as above define a Weierstraß representation of a minimal surface of genus p with one Enneper-type end and total curvature $-4\pi(p+1)$.

Proof. The claim now is that the 1-forms in the Weierstraß representation all have purely imaginary periods. For dh this is obvious, because the form dh is even exact and so all periods even vanish. Because of $(g - \frac{1}{g})dh = \alpha - \beta$ and $i(g + \frac{1}{g})dh = i(\alpha + \beta)$ this is equivalent to the claim that α and β have complex conjugate periods. To see this, we first construct a basis for the homology on $\mathcal{R}_{\mathrm{NE}}$ and then compute the periods of α and β using their geometric definitions.

To define $2p$ cycles B_j on $\mathcal{R}_{\mathrm{NE}}$, we take curves b_j in Ω_{NE} connecting a boundary point slightly to the right of P_{j+1} with a boundary point slightly to the left of P_j for $j = -p, \ldots, p-1$. We double this curve to obtain a closed curve B_j on $\mathcal{S}_{\mathrm{NE}}$ which encircles exactly P_j and P_{j+1}. These curves have closed lifts B_j to $\mathcal{R}_{\mathrm{NE}}$ and form a homology basis. Now to compute a period of our 1-forms, observe that a period is nothing other than the image of the closed curve under the developing map of the flat metric which defines the 1-form. But this developing map can be read off from the zigzag — one only has to observe that developing a curve around a vertex (regardless whether the angle there is $\frac{\pi}{2}$ or $3\frac{\pi}{2}$) will change the direction of the curve there by 180°. Doing this yields

$$\begin{aligned}
\int_{B_j} \alpha &= \int_{B_j} e^{-\pi i/4} \cdot \omega_{\mathrm{NE}} = 2e^{-\pi i/4} \cdot (P_j - P_{j+1}) \\
\int_{B_j} \beta &= \int_{B_j} e^{-\pi i/4} \cdot \phi^* \omega_{\mathrm{SW}} = \int_{\phi(B_j)} e^{-\pi i/4} \cdot \omega_{\mathrm{SW}} = 2e^{-\pi i/4} \cdot (Q_j - Q_{j+1}) \\
&= 2e^{-\pi i/4} \cdot (P_{-j} - P_{-j-1})
\end{aligned}$$

which yields the claim by the symmetry of the zigzag.

Finally, we have to compute the total absolute curvature of the minimal surface. By the definition of the Gauß map we have

$$(g) = \begin{cases} P_0^{+1} \cdot P_{\pm 1}^{-1} \cdot P_{\pm 2}^{+1} \cdot \ldots \cdot P_{\pm p}^{-1} \cdot P_\infty & \text{p odd} \\ P_0^{-1} \cdot P_{\pm 1}^{+1} \cdot P_{\pm 2}^{-1} \cdot \ldots \cdot P_{\pm p}^{-1} \cdot P_\infty & \text{p even} \end{cases}$$

and thus $\deg g = p + 1$ which implies $\int_R K dA = -4\pi(p+1)$ as claimed. □

Remark 3.4. We close by making some comments on the amount of symmetry involved in this approach. Usually in the construction of minimal surfaces the underlying Riemann surface is assumed to have so many automorphisms that the moduli space of possible conformal structures is very low dimensional (in fact, it consists only of one point in many examples). This helps solving the period problem (if it is solvable) because this will then be a problem on a low dimensional space. In our approach, the dimension of the moduli space grows with the genus, and the use of symmetries has other purposes: It allows us to construct for given periods a pair of surfaces with one 1-form on each which would solve the period problem if only the surfaces would coincide. Indeed we observe

Lemma 3.5. *The minimal surface of genus p constructed below has only eight isometries, and at most eight conformal or anticonformal automorphisms that fix the end, independently of genus.*

Proof. Observe that as the end is unique, any isometry of the minimal surface fixes the end. As the isometry necessarily induces a conformal or anti-conformal automorphism, it is sufficient to prove only the latter statement of the lemma. Because of the uniqueness of the hyperelliptic involution, this automorphism descends to an automorphism of the punctured sphere which fixes the image of infinity and permutes the punctures (the images of the Weierstraß points). As there are at least three punctures, all lying on the real line, we see that the real line is also fixed (setwise). After taking the reflection (an anti-conformal automorphism) of the sphere across the real line, which fixes all the punctures, we are left to consider the conformal automorphisms of the domain Ω_{NE}. Finally, this domain has only two conformal symmetries, the identity and the reflection about the diagonal. The lemma follows by counting the automorphisms we have identified in the discussion. □

§4. The Height Function on Moduli Space.

4.1 The space of zigzags $\mathcal{Z}_p$ and a natural compactification $\partial\overline{\mathcal{Z}}$. We recall the space $\mathcal{Z}_p$ of equivalence classes of symmetric genus p zigzags constructed in Section 2.2; here the equivalence by similarity was defined so that two zigzags Z and Z' would be equivalent if and only if both of the pairs of complementary domains $(\Omega_{\mathrm{NE}}(Z), \Omega_{\mathrm{NE}}(Z'))$ and $(\Omega_{\mathrm{SW}}(Z), \Omega_{\mathrm{SW}}(Z'))$ were conformally equivalent. Label the finite vertices of the zigzag by $P_{-p}, \dots, P_0, \dots, P_p$. Thus, we may choose a unique representative for each class in $\mathcal{Z}_p$ by setting the vertices $P_0 \in \{y = x\}$, $P_p = 1$, $P_{-p} = i$ and $P_k = i\overline{P_{-k}}$ for $k = 0, \dots, p$; here all the vertices $P_{-p}, \dots, P_p$ are required to be distinct. The topology of $\mathcal{Z}_p$ defined in Section 2.2 then agrees with the topology of the space of canonical representatives induced by the embedding of $\mathcal{Z}_p \to \mathbf{C}^{p-1}$ by $Z \mapsto (P_1, \dots, P_{p-1})$. With these normalizations and this last remark on topology, it is evident that $\mathcal{Z}_p$ is a cell of dimension $p-1$.

We have interest in the natural compactification of this cell, obtained by attaching a boundary $\partial\overline{\mathcal{Z}}_p$ to $\mathcal{Z}$. This boundary will be composed of zigzags where some proper consecutive subsets of $\{P_0, \dots, P_p\}$ (and of course the reflections of these subsets across $\{y = x\}$) are allowed to coincide; the topology on $\overline{\mathcal{Z}}_p = \mathcal{Z}_p \cup \partial\overline{\mathcal{Z}}_p$ is again given by the topology on the map of coordinates of normalized representatives $Z \in \mathcal{Z}_p \mapsto (P_1, \dots, P_{p-1}) \in C^{p-1}$.

Evidently, $\partial\overline{\mathcal{Z}}_p$ is stratified by unions of zigzag spaces $\mathcal{Z}_p^k$ of real dimension k, with each component of $\mathcal{Z}_p^k$ representing the (degenerate) zigzags that result from allowing k distinct vertices to remain in the (degenerate) zigzag after some points $P_0, \dots, P_p$ have coalesced. For instance $\mathcal{Z}_p^0$ consists of the zigzags where all of the points $P_0, \dots, P_p$ have coalesced to either $P_0 \in \{y = x\}$ or $P_1 = 1$, and the faces $\mathcal{Z}_p^{p-2}$ are the loci in $\overline{\mathcal{Z}}_p$ where only two consecutive vertices have coalesced.

Observe that each of these strata is naturally a zigzag space in its own right, and one can look for a reflexive symmetric zigzag of genus $k+1$ within $\mathcal{Z}_p^k$.

4.2 Extremal length functions on $\mathcal{Z}$. Consider the punctured sphere S_{NE} in §3, where we labelled the punctures $P_{-p}, \dots, P_0, \dots, P_p$, and P_∞ and observed that S_{NE} had two reflective symmetries: one about the image of Z and one about the image of the curve $\{y = x\}$ on Ω_{NE}. Let $[B_k]$ denote the homotopy class of simple curves which encloses the punctures P_k and P_{k+1} for $k = 1, \dots, p-1$ and $[B_{-k}]$ the homotopy class of simple curves which encloses the punctures P_{-k} and P_{-k-1} for $k = 1, \dots, p-1$. Let $[\gamma_k]$ denote the pair of classes $[B_k] \cup [B_{-k}]$. Under the homotopy class of maps which connects S_{NE} to S_{SW} (lifted from $\phi : \Omega_{\mathrm{NE}} \to \Omega_{\mathrm{SW}}$, the vertex preserving map), there are corresponding homotopy classes of curves on S_{SW}, which we will also label $[\gamma_k]$.

Set $E_{\mathrm{NE}}(k) = \mathrm{Ext}_{S_{\mathrm{NE}}}([\gamma_k])$ and $E_{\mathrm{SW}}(k) = \mathrm{Ext}_{S_{\mathrm{SW}}}([\gamma_k])$ denote the extremal lengths of $[\gamma_k]$ in S_{NE} and S_{SW}, respectively.

Let $T_{0,2p+2}^{\mathrm{symm}}$ denote a subspace of the Teichmüller space of $2p+2$ punctured spheres whose points are equivalence classes of $2p+2$ punctured spheres (with a pair of involutions) coming from one complementary domain $\Omega_{\mathrm{NE}}(Z)$ of a symmetric zigzag Z. This $T_{0,2p+2}^{\mathrm{symm}}$ is a $p-1$ dimensional subspace of the Teichmüller space $T_{0,2p+2}$ of $2p+2$ punctured spheres.

Consider the map $E_{\mathrm{NE}} : T_{0,2p+2}^{\mathrm{symm}} \to \mathbf{R}_+^{p-1}$ given by $S_{\mathrm{NE}} \mapsto (E_{\mathrm{NE}}(1), \dots, E_{\mathrm{NE}}(p-1))$.

Proposition 4.2.1. *The map* $E_{\mathrm{NE}} : T^{\mathrm{symm}}_{0,2p+2} \to \mathbf{R}^{p-1}_{+}$ *is a homeomorphism onto* $\mathbf{R}^{p-1}_{+}$.

Proof. It is clear that E_{NE} is continuous. To see injectivity and surjectivity, apply a Schwarz-Christoffel map $SC : \Omega_{\mathrm{NE}} \to \{\mathrm{Im}\, z > 0\}$ to Ω_{NE}; this map sends Ω_{NE} to the upper half-plane, taking Z to $\mathbf{R}$ so that $SC(P_\infty) = \infty$, $SC(P_0) = 0$, $SC(P_1) = 1$ and $SC(P_{-k}) = -SC(P_k)$. These conditions uniquely determine SC; moreover $E_{\mathrm{NE}}(k) = 2\,\mathrm{Ext}_{\mathbf{H}^2}(\ ,\ _k)$ where $,\ _k$ is the class of pairs of curves in $\mathbf{H}^2$ that connect the real arc between $SC(P_{-k-2})$ and $SC(P_{-k-1})$ to the real arc between $SC(P_{-k})$ and $SC(P_{-k+1})$, and the arc between $SC(P_{k-1})$ and $SC(P_k)$ to the arc between $SC(P_{k+1})$ and $SC(P_{k+2})$. Now, any choice of $p-1$ numbers $x_i = SC(P_i)$ for $2 \le 1 \le p$ uniquely determines a point in $T^{\mathrm{symm}}_{0,2p+2}$, and these choices are parametrized by the extremal lengths $\mathrm{Ext}_{\mathbf{H}^2}(\ ,\ _k) \in (0, \infty)$. This proves the result. $\square$

Let $\mathrm{QD}^{\mathrm{symm}}(S_{\mathrm{NE}})$ denote the vector space of holomorphic quadratic differentials on S_{NE} which have at worst simply poles at the punctures and are real along the image of Z and the line $\{y = x\}$.

Our principal application of Proposition 4.2.1 is the following

Corollary 4.2.2. *The cotangent vectors* $\{dE_{\mathrm{NE}}(k) \mid k = 1, \dots, p-1\}$ *(and* $\{dE_{\mathrm{SW}}(k) \mid k = 1, \dots, p-1\}$*) are a basis for* $T^*_{S_{\mathrm{NE}}} T^{\mathrm{symm}}_{0,2p+2}$*, hence for* $\mathrm{QD}^{\mathrm{symm}}(S_{\mathrm{NE}})$.

Proof. The cotangent space $T^*_{S_{\mathrm{NE}}} T^{\mathrm{symm}}_{0,2p+2}$ to the Teichmüller space $T^{\mathrm{symm}}_{0,2p+2}$ is the space $\mathrm{QD}(S_{\mathrm{NE}})$ of holomorphic quadratic differentials on S_{NE} with at most simple poles at the punctures. A covector cotangent to $T^{\mathrm{symm}}_{0,2p+2}$ must respect the reflective symmetries of the elements of $T^{\mathrm{symm}}_{0,2p+2}$, hence its horizontal and vertical foliations must be either parallel or perpendicular to the fixed sets of the reflections. Thus such a covector, as a holomorphic quadratic differential, must be real on those fixed sets, and hence must lie in $\mathrm{QD}^{\mathrm{symm}}(S_{\mathrm{NE}})$. The result follows from the functions $\{E_{\mathrm{NE}}(k) \mid k = 1, \dots, p-1\}$ being coordinates for $T^{\mathrm{symm}}_{0,2p+2}$. $\square$

4.3 The height function $D(Z) : \mathcal{Z} \to \mathbf{R}$. Let the height function $D(Z)$ be

$$D(Z) = \sum_{j=1}^{p-1} \left[\exp\left(\frac{1}{E_{\mathrm{NE}}(j)}\right) - \exp\left(\frac{1}{E_{\mathrm{SW}}(j)}\right) \right]^2 + \left[E_{\mathrm{NE}}(j) - E_{\mathrm{SW}}(j)\right]^2 . \tag{4.1}$$

We observe that $D(Z) = 0$ if and only if $E_{\mathrm{NE}}(j) = E_{\mathrm{SW}}(j)$, which holds if and only if S_{NE} is conformally equivalent to S_{SW}. We also observe that, for instance, if $E_{\mathrm{NE}}(j)/E_{\mathrm{SW}}(j) \ge C_0$ but both $E_{\mathrm{NE}}(j)$ and $E_{\mathrm{SW}}(j)$ are quite small, then $D(Z)$ is quite large. It is this latter fact which we will exploit in this section.

4.4 Monodromy Properties of the Schwarz-Christoffel Maps. Here we derive the facts about the Schwarz-Christoffel maps we need to prove properness of the height function.

Let $t = (0 < t_1 < t_2 < \dots < t_p)$ be p points on the real line. We put $t_0 := 0, t_\infty := \infty$ and $t_{-k} = -t_k$. Then the Schwarz-Christoffel formula tells us that we can map the upper half plane conformally to a NE-domain such that the $\{t_i\}$ are mapped to vertices by the function

$$f(z) = \int_0^z (t - t_{-p})^{1/2}(t - t_{-p+1})^{-1/2} \cdots (t - t_p)^{1/2} dt$$

and to a SW-domain by

$$g(z) = \int_0^z (t - t_{-p})^{-1/2}(t - t_{-p+1})^{+1/2} \cdots (t - t_p)^{-1/2} dt$$

Note that the exponents alternate sign. We are not interested in normalizing these maps at the moment by introducing some factor, but we have to be aware of the fact that scaling the t_k will scale f and g.

Now introduce the periods $a_k = f(t_{k+1}) - f(t_k)$ and $b_k = g(t_{k+1}) - g(t_k)$ which are complex numbers, either real or purely imaginary. Denote by

$$T := \{t : t_i \in \mathbb{C}, t_j \neq t_k \quad \forall j, k\},$$

the complex-valued configuration space for the $2p+1$-tuples $\{t\}$. It is clear that we can analytically continue the a_k and b_k along any path in T to obtain holomorphic multi-valued functions.

Lemma 4.4.1. *Continue a_k analytically along a path in T defined by moving t_j anti-clockwise around t_{j+1} and denote the continued function by $\tilde{a}_k$, similarly for b_k. Then we have*

$$\tilde{a}_k = \begin{cases} a_k & \text{if } k \neq j-1, j+1 \\ a_k + 2a_{k+1} & \text{if } j = k+1 \\ a_k - 2a_{k-1} & \text{if } j = k-1 \end{cases}$$

with analogous formulas holding for $\tilde{b}_k$.

Proof. Imagine that the defining paths of integration for a_k was made of flexible rubber band which is tied to t_k t_{k+1}. Now moving t_j will possibly drag the rubber band into some new position. The resulting curves are precisely those paths of integration which need to be used to compute $\tilde{a}_k$. If $j \neq k-1, k+1$, the paths remain the same, hence $\tilde{a}_k = a_k$. If $j = k+1$, the rubberband between t_k and t_{k+1} is pulled around t_{k+2} and back to t_{k+1}. Hence a_k changes by the amount of the integral which goes from t_{k+1} to t_{k+2}, loops around t_{k+2} and then back to t_{k+1}. Hence the first part contributes a_{k+1}. Now, for the second part of the path of integration, by the very definition of the Schwarz-Christoffel maps we know that a small interval through t_{k+2} is mapped to a $90°$ hinge, so that a small infinitesimal loop turning around t_{k+2} will be mapped to an infinitesimal straight line segment. In fact, locally near t_{k+2} the Schwarz-Christoffel map is of the form $z \mapsto z^{1/2}$ or $z \mapsto z^{3/2}$. Therefore we get from the integration back to t_{k+1} another contribution of $+a_{k+1}$. The same argument is valid for $j = k-1$ and for the b_k. $\square$

Now denote by $\delta := t_{k+1} - t_k$ and fix all t_j other than t_{k+1}: we regard t_{k+1} as the independent variable.

Corollary 4.4.2. *The functions* $a_k - \frac{\log \delta}{\pi i} a_{k+1}$ *and* $b_k - \frac{\log \delta}{\pi i} b_{k+1}$ *are holomorphic in* δ *at* $\delta = 0$.

Proof. By Lemma 4.4.1, the above functions are singlevalued and holomorphic in a punctured neighborhood of $\delta = 0$. It is easy to see from the explicit integrals defining a_k, b_k, a_{k+1} and b_{k+1} that the above functions are also bounded, hence they extend holomorphically to $\delta = 0$. $\square$

Now for the properness argument, we are more interested in the absolute values of the periods than in the periods themselves: we translate the above statement about periods into a statement about their respective absolute values. This will lead to a crucial difference in the behavior of the extremal length functions on the NE– and SW regions.

Corollary 4.4.3. *Either* $|a_k| - \frac{\log \delta}{\pi} |a_{k+1}|$ *or* $|a_k| + \frac{\log \delta}{\pi} |a_{k+1}|$ *is real analytic in* δ *for* $\delta = 0$. *In the first case,* $|b_k| + \frac{\log \delta}{\pi} |b_{k+1}|$ *is real analytic in* δ*, in the second* $|b_k| - \frac{\log \delta}{\pi} |b_{k+1}|$.

Remark. Note the different signs here! This reflects that we alternate between left and right turns in the zigzag.

Proof. If we follow the images of the t_k in the NE-domain, we turn alternatingly left and right, that is, the direction of a_{k+1} alternates between i times the direction of a_k and $-i$ times the direction of a_k.

This proves the first statement, using corollary 4.4.2. Now if we turn left at P_k in the NE domain, we turn right in the SW domain, and vice versa, because the zigzag is run through in the opposite orientation. This proves the second statement. $\square$

From this we deduce a certain non-analyticity which is used in the properness proof. Denote by s_k, t_k the preimages of the vertices P_k of a zigzag under the Schwarz-Christoffel maps for the NW- and SW-domain respectively. We normalize these maps such that $s_0 = t_0 = 0$ and $s_n = t_n = 1$. Introduce $\delta_{\mathrm{NE}} = s_{k+1} - s_k$ and $\delta_{\mathrm{SW}} = t_{k+1} - t_k$. We can now consider δ_{NE} as a function of δ_{SW}:

Corollary 4.4.4. *The function* δ_{NE} *does not depend real analytically on* δ_{SW}.

Proof. Suppose the opposite is true.We know that either $|b_k| + \frac{\log \delta_{\mathrm{SW}}}{\pi} |b_{k+1}|$ or $|b_k| - \frac{\log \delta_{\mathrm{SW}}}{\pi} |b_{k+1}|$ depends real analytically on δ_{SW}, hence on δ_{NE}, so we may assume with no loss in generality that $|b_k| + \frac{\log \delta_{\mathrm{SW}}}{\pi} |b_{k+1}|$ depends real analytically on δ_{SW}. Then by Corollary 4.4.3, we see that $|a_k| - \frac{\log \delta_{\mathrm{SW}}}{\pi} |a_{k+1}|$ depends analytically on δ_{NE}, hence by assumption on δ_{SW}. Hence $B(\delta_{\mathrm{SW}}) := \frac{|b_k|}{|b_{k+1}|} + \frac{\log \delta_{\mathrm{SW}}}{\pi}$ and $A(\delta_{\mathrm{NE}}) := \frac{|a_k|}{|a_{k+1}|} - \frac{\log \delta_{\mathrm{NE}}}{\pi}$ depends real analytically on δ_{SW} and δ_{NE}, respectively. But

$$\frac{|b_k|}{|b_{k+1}|} = \frac{|a_k|}{|a_{k+1}|}$$

by the assumption on equality of periods, so

$$B(\delta_{\mathrm{SW}}) - \frac{\log \delta_{\mathrm{SW}}}{\pi} = A(\delta_{\mathrm{NE}}) + \frac{\log \delta_{\mathrm{NE}}}{\pi}.$$

But then $\frac{\log(\delta_{\mathrm{SW}}\delta_{\mathrm{NE}})}{\pi} = B(\delta_{\mathrm{SW}}) - A(\delta_{\mathrm{NE}})$ is analytic in δ_{NE}. Of course, the product $\delta_{\mathrm{SW}}\delta_{\mathrm{NE}}$ is analytic in δ_{NE} and non-constant, as $\delta_{\mathrm{SW}}\delta_{\mathrm{NE}}$ tends to zero by the hypothesis on extremal length. But then $log(\delta_{\mathrm{SW}}\delta_{\mathrm{NE}})$ is analytic in δ_{NE}, near $\delta_{\mathrm{NE}} = 0$, which is absurd. $\square$

Remark. Note that the Corollary remains true if we consider zigzags which turn alternatingly left and right by a (fixed) angle other than $\pi/2$. This will only affect constants in Lemma 4.4.1 and Corollaries 4.4.2, 4.4.3. In Corollary 4.4.4, we need only that the coefficients of $\log \delta$ are distinct, and this is also true for the zigzags with non-orthogonal sides. We will use this generalization in section 6.

4.5 An extremal length computation. Here we compute the extremal length of curves separating two points on the real line. This will be needed in the next section. We do this first in a model situation: Let $\lambda < 0 < 1$ and consider the family of curves , in the upper half plane joining the interval $(-\infty, \lambda)$ with the interval $(0, 1)$. For a detailed account on this, see [Oht], p. 179–214. He gives the result in terms of the Jacobi elliptic functions from which it is straightforward to deduce the asymptotic expansions which we need. Because it fits in the spirit of this paper, we give an informal description of what is involved in terms of elliptic integrals of Weierstraß type.

It turns out that the extremal metric for , is rather explicit and can be seen best by considering a slightly different problem: Consider the family $,'$ of curves in S^2 encircling only λ and 0 thus separating them from 1 and ∞. Then the extremal length of $,'$ is twice the one we want.

Lemma 4.5.1. *The extremal metric in this situation is given by the flat cone metric on $S^2 - \{\lambda, 0, 1, \infty\}$ with cone angles π at each of the four vertices.*

Proof. Directly from the length-area method of Beurling, or see [Oht]. $\square$

This metric can be constructed by taking the double cover over $S^2 - \{\lambda, 0, 1, \infty\}$, branched over $\lambda, 0, 1, \infty$ which is a torus T, which has a unique flat conformal metric (up to scaling). This metric descends as the cone metric we want to $S^2 - \{\lambda, 0, 1, \infty\}$. This allows us to compute the extremal length in terms of certain elliptic period integrals. Because the covering projection $p : T \to S^2$ is given by the equation ${p'}^2 = p(p-1)(p-\lambda)$ we compute the periods of T as

$$\omega_i = \int_{\gamma_i} \frac{du}{\sqrt{u(u-1)(u-\lambda)}} \tag{4.2}$$

where γ_1 denotes a curve in $,'$ and γ_2 a curve circling around λ and 0. We conclude that the extremal length we are looking for is given by

Lemma 4.5.2.

$$\mathrm{Ext}(,\ ) = 2\frac{|\omega_1|^2}{\det(\omega_1, \omega_2)} \tag{4.3}$$

Proof. see [Oht] $\square$

Alternatively,

$$\frac{\omega_2}{\omega_1} = -\frac{\int_0^\lambda \frac{du}{\sqrt{u(u-1)(u-\lambda)}}}{\int_0^1 \frac{du}{\sqrt{u(u-1)(u-\lambda)}}} \tag{4.4}$$

This is as explicit as we can get.

It is evident from formulas (4.2), (4.3), and (4.4) that Ext(,) is real analytic on $T_{0,4}$, hence on $T^{\mathrm{symm}}_{0,2p+2}$.

Now we are interested in the asymptotic behavior of the extremal lengths Ext(,) as $\lambda \to 0$.

Lemma 4.5.3.

$$\mathrm{Ext}(,\ ) = O\left(\frac{1}{\log|\lambda|}\right)$$

Proof. The asymptotic behavior of elliptic integrals is well known, but it seems worth observing that all the information which we need is in fact contained in the geometry. For a more formal treatment, we refer to [Oht, Rain]. The period ω_1 is easily seen to be holomorphic in λ by developing the integrand into a power series and integrating term by term; explicitly we can obtain

$$\omega_1(\lambda) = 2\pi\left(1 + \frac{1}{4}\lambda + \frac{9}{32}\lambda^2 + \frac{25}{128}\lambda^3 + \dots\right)$$

but all we need is the holomorphy and $\omega_1(0) \neq 0$. Concerning ω_2, the general theory of ordinary differential equations with regular singular points predicts that any solution ω of the o.d.e. has the general form

$$\omega = c_1\big(\log(\lambda)\omega_1(\lambda) + \lambda f_1(\lambda)\big) + c_2\omega_1(\lambda)$$

with some explicitly known holomorphic function $f_1(\lambda)$.

From a similar monodromy argument as in the above section 4.4 one can obtain that

$$\omega_2(\lambda) - \frac{i}{\pi}\log(-\lambda)\omega_1(\lambda)$$

is holomorphic at $\lambda = 0$ which is simultaneously a more specific but less general statement. Nevertheless this can already be used to deduce the claim, but for the sake of completeness we cite from [Rain] the full expansion:

$$\omega_2(\lambda) = \frac{i}{\pi}\left(\log(-\lambda)\omega_1(\lambda) + \lambda f_1(\lambda)\right) - \frac{4i\log 2}{\pi}\omega_1(\lambda)$$

with the expansion of $\lambda f_1(\lambda)$ given by

$$\lambda f_1(\lambda) = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} \left(H(a,n) + H(b,n) - 2H(1,n)\right) \lambda^n$$

Here

$$\begin{aligned} (a)_n &:= a(a+1)\cdot \ldots \cdot (a+n-1) \\ H(a,n) &= \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} \\ a &= 1/2 \\ b &= 1/2 \end{aligned}$$

From this, one can deduce the claim in any desired degree of accuracy. $\square$

Now we generalize this to 4 arbitary points $t_1 < t_2 < t_3 < t_4$ on the real line and to the family , of curves connecting the arc $t_1 t_2$ to the arc $t_3 t_4$.

We denote the cross ratio of t_1, t_2, t_3, t_4 by $(t_1 : t_2 : t_3 : t_4)$ which is chosen so that $(\infty : \lambda : 0 : 1) = \lambda)$.

Corollary 4.5.4. *For $t_2 \to t_3$, we have*

$$\mathrm{Ext}\,, = O\left(\frac{1}{-\log|(t_1 : t_2 : t_3 : t_4)|}\right)$$

Proof. This follows by applying the Möbiustransformation to the t_i which maps them to $\infty, \lambda, 0, 1$. $\square$

Remark. Here we can already see that to establish properness we need to consider points t_i such that $t_2 \to t_3$ while t_1 and t_4 stay at finite distance away.

4.6 Properness of the Height Function. In this section we prove

Theorem 4.6.1. *The height function $D(Z)$ is proper on $\mathcal{Z}$, for $p > 2$.*

Proof. Let Z_0 be a zigzag in the boundary of $\overline{\mathcal{Z}}$. We can imagine Z_0 as an ordinary zigzag where some (consecutive) vertices have coalesced. We can assume that we have a cluster of coalesced points $P_k = P_{k+1} = \ldots = P_{k+l}$ but $P_{k-1} \neq P_k$ and $P_{k+l} \neq P_{k+l+1}$ (here $k \geq 0$ and $k + l \leq p$).

We first consider the case where $k \geq 1$, taking up the case $k = 0$ later. Denote the family of curves connecting the segment $P_{k-1}P_k$ with the segment $P_{k+l}P_{k+l+1}$ (and their counterparts symmetric about the central point P_0) in the NE domain by $,_{\mathrm{NE}}$ and in the SW domain by $,_{\mathrm{SW}}$ and their extremal lengths in Ω_{NE} (Ω_{SW}, *resp.*) by $\mathrm{Ext}\,,_{\mathrm{NE}}$ ($\mathrm{Ext}\,,_{\mathrm{SW}}$, *resp.*). Now recall that the height function was defined so that

$$D(Z) \geq \sum_{j=1}^{p-1} \left[\exp\left(\frac{1}{E_{\mathrm{NE}}(j)}\right) - \exp\left(\frac{1}{E_{\mathrm{SW}}(j)}\right)\right]^2.$$

where the extremal length were taken of curves encircling two consecutive points. To prove properness it is hence sufficient to prove that at least one pair $E_{\mathrm{NE}}(j), E_{\mathrm{SW}}(j)$ approaches 0 with different rates to some order for any sequence of zigzags $Z_n \to Z_0$. Suppose this is not the case. Then especially all $E_{\mathrm{NE}}(j), E_{\mathrm{SW}}(j)$ with $j = k, \ldots, k+l$ approach zero at the same rate for some $Z_n \to Z_0$. Now conformally the points $t_k, \ldots, t_{k+l+1}$ are determined by the extremal lengths $E_{\mathrm{NE}}(j)$ and $E_{\mathrm{SW}}(j)$ so that under the assumption, Ext , $_{\mathrm{NE}}$ and Ext , $_{\mathrm{SW}}$ approach zero with the same rate. Thus to obtain a contradiction it is sufficient to prove that

$$e^{1/\operatorname{Ext}\Gamma_{\mathrm{NE}}} - e^{1/\operatorname{Ext}\Gamma_{\mathrm{SW}}}$$

is proper in a neighborhood of Z_0 in $\mathcal{Z}$. Such a neighborhood is given by all zigzags Z where distances between coalescing points are sufficiently small. Especially, the quantity $\epsilon := |P_k - P_{k+j}|$ is small.

To estimate the extremal length, we map the NE– and SW domains of a zigzag Z in a neighborhood of Z_0 by the inverse Schwarz-Christoffel maps to the upper half plane and apply then the asymptotic formula of the last section, using that the asymptotics for a symmetric pair of degenerating curve families agree with the asymptotics of a single degenerating curve family.

Denote by δ_{NE} and δ_{SW} the difference $t_{k+1} - t_k$ of the images of P_{k+1} and P_k for NE and SW respectively. Because the Schwarz-Christoffel map is a homeomorphism on the compactified domains, the quantities δ_{NE} and δ_{SW} will go to zero with ϵ, while the distances $t_{k-1} - t_k$ and $t_{k+1} - t_{k+l+1}$ are uniformly bounded away from zero in any compact coordinate patch. Hence by Corollary 4.5.4

$$\begin{aligned}
&\left| e^{\frac{1}{\operatorname{Ext}\Gamma_{\mathrm{NE}}}} - e^{\frac{1}{\operatorname{Ext}\Gamma_{\mathrm{SW}}}} \right| \\
&\qquad = \left| O\left(\frac{1}{|(t_{k-1} : t_k : t_{k+1} : t_{k+l+1})_{\mathrm{NE}}|}\right) - O\left(\frac{1}{|(t_{k-1} : t_k : t_{k+1} : t_{k+l+1})_{\mathrm{SW}}|}\right) \right| \\
&\qquad = O\left(\frac{1}{\delta_{\mathrm{NE}}}\right) - O\left(\frac{1}{\delta_{\mathrm{SW}}}\right)
\end{aligned} \tag{4.5}$$

On the other hand, by corollary 4.4.4, we know that δ_{NE} cannot depend analytically on δ_{SW} so that one term will dominate the other and no cancellation occurs. Finally all occuring constants are uniform in a coordinate patch and δ_{NE} depends there in a uniform way on ϵ. This proves local properness near Z_0 in this case and gives the desired contradiction.

In the case where $P_k = 0, \ldots, P_{k+l}$ are coalescing (here $k+l < p$), we use the other terms in the height function, i.e. the inequality

$$D(Z) \geq \sum_{j=0}^{p-1} \left[E_{\mathrm{NE}}(j) - E_{\mathrm{SW}}(j)\right]^2 \tag{4.6}$$

It is a straightforward exercise in the definition of extremal length (see lemma 4.5.1) that, since γ_{k+l-1} and γ_{k+l} intersect once (geometrically), and the point P_{k+l+1} converges to a finite point distinct from $P_k, ..., P_{k+l}$, we can conclude that $E_{\mathrm{NE}}(k+l) = O(\frac{1}{E_{\mathrm{NE}}(k+l-1)})$ and $E_{\mathrm{SW}}(k+l) = O(\frac{1}{E_{\mathrm{SW}}(k+l-1)})$. (Here we use the hypothesis that $p > 2$ in the final argument to ensure the existence of a second dual curve.) Yet an examination of the argument above (see also Corollary 4.4.4, especially) shows that $E_{\mathrm{NE}}(k+l-1)$ vanishes at a different rate than $E_{\mathrm{SW}}(k+l-1)$, hence $E_{\mathrm{NE}}(k+l)$ grows at a different rate than $E_{\mathrm{SW}}(k+l)$. This term alone then in inequality (4.6) shows the claim. $\square$

§5. The Gradient Flow for the Height Function.

5.1. To find a zigzag Z for which $D(Z) = 0$, we imagine flowing along the vector field $\operatorname{grad} D$ on $\mathcal{Z}$ to a minimum Z_0. To know that this minimum Z_0 represents a reflexive zigzag (i.e., a solution to our problem), we need to establish that, at such a minimum Z_0, we have $D(Z) = 0$. That result is the goal of this section; in the present subsection, we state the result and begin the proof.

Proposition 5.1. *There exists $Z_0 \in \mathcal{Z}$ with $D(Z_0) = 0$.*

Proof. Our plan is to find a good initial point $Z^* \in \mathcal{Z}$ and then follow the flow of $-\operatorname{grad} D$ from Z^*; our choice of initial point will guarantee that the flow will lie along a curve $\mathcal{Y} \subset \mathcal{Z}$ along which $D(Z)\bigm|_{\mathcal{Y}}$ will have a special form. Both the argument for the existence of a good initial point and the argument that the negative gradient flow on the curve $\mathcal{Y}$ is only critical at a point Z with $D(Z) = 0$ involve understanding how a deformation of a zigzag affects extremal lengths on S_{NE} and S_{SW}, so we begin with that in subsection 5.2. In subsection 5.3 we choose our good initial point Z^*, while in subsection 5.4 we check that the negative gradient flow from Z^* terminates at a reflexive symmetric zigzag. This will conclude the proof of the main theorems.

5.2. The tangent space to $\mathcal{Z}$. In this subsection, we compute a variational theory for zigzags appropriate for our problem. In terms of our search for minimal surfaces, we recall that the zigzags (and the resulting Euclidean geometry on the domains Ω_{NE} and Ω_{SW}) are constructed to solve the period problem for the Weierstraß data: since we are left to show that the domains Ω_{NE} and Ω_{SW} are conformally equivalent, we need a formula for the variation of the extremal length (conformal invariants) in terms of the periods.

More particularly, note again that what we are doing throughout this paper is relating the Euclidean geometry of Ω_{NE} (and Ω_{SW}, respectively) with the conformal geometry of Ω_{NE} (and Ω_{SW}, resp.) The Euclidean geometry is designed to control the periods of the one-forms ω_{NE} (and ω_{SW}) and is restricted by the requirements that the boundaries of Ω_{NE} (Ω_{SW}, resp.) have alternating left and right orthogonal turns, and that Ω_{NE} and Ω_{SW} are complementary domains of a zigzag Z in $\mathbf{C}$. Of course, we are interested in the conformal geometry of these domains as that is the focus of the Main Theorem B.

In terms of a variational theory, we are interested in deformations of a zigzag through zigzags: thus, informally, the basic moves consist of shortening or lengthening individual sides while maintaining the angles at the vertices. These moves, of course, alter the conformal structure of the complementary domains, and we need to calculate the effect on conformal invariants (in particular, extremal length) of these alterations; those calculations involve the Teichmüller theory described in section 2.3, and form the bulk of this subsection. We list the approach below, in steps.

Step 1) We consider a self-diffeomorphism f_ϵ of $\mathbf{C}$ which takes a given zigzag Z_0 to a new zigzag Z_ϵ: this is given explicitly in §5.2.1, formulas 5.1. (There will be two cases of this, which will in fact require two different types of diffeomorphisms, which we label f_ϵ and f_ϵ^*; they are related via a symmetry, which will later benefit us through an important cancellation.) These diffeomorphisms will be supported in a neighborhood of a pair of edges; later in Step 4, we will consider the effect of contracting the support onto increasingly smaller neighborhoods of those pair of edges.

Step 2) Infinitesimally, this deformation of zigzag results in infinitesimal changes in the conformal structures of the complementary domains, and hence tangent vectors to the Teichmüller spaces of these domains. As described in the opening of §2.3, those tangent vectors are given by Beltrami differentials $\dot{\mu}_{\mathrm{NE}}$ (and $\dot{\mu}_{\mathrm{SW}}$) on Ω_{NE} (and Ω_{SW}) and it is easy to compute $\dot{\mu}_{\mathrm{NE}}$ (and $\dot{\mu}_{\mathrm{SW}}$) in terms of $\left(\frac{d}{d\epsilon}f_\epsilon\right)_{\bar{z}}$ and $\left(\frac{d}{d\epsilon}f_\epsilon^*\right)_{\bar{z}}$. This is done explicitly in §5.2.1, immediately after the explicit computations of f_ϵ and f_ϵ^*.

Step 3) We apply those formulas for $\dot{\mu}_{\mathrm{NE}}$ and $\dot{\mu}_{\mathrm{SW}}$ to the computation of the derivatives of extremal lengths (e.g. $\frac{d}{d\epsilon}\Big|_{\epsilon=0} E_{\mathrm{NE}}(k)$, in the notation of §4). Teichmüller theory (§2.3) provides that this can be accomplished through formula (2.1), which exhibits gradient vectors $d\,\mathrm{Ext}_{\mathrm{NE}}(k)$ as meromorphic quadratic differentials Φ_k^{NE} (and Φ_k^{SW}) on the sphere S_{NE} (and S_{SW}, resp.) As we described in §2.3, Gardiner [Gar] gives a recipe for constructing these differentials in terms of the homotopy classes of their leaves. We describe these differentials in §5.2.2.

Step 4) We have excellent control on these quadratic differentials along the (lift of the) zigzag. Yet the formula (2.1) requires us to consider an integral over the support of the Beltrami differentials $\dot{\mu}_{\mathrm{NE}}$ and $\dot{\mu}_{\mathrm{SW}}$. It is convenient to take a limit of

$$\int \phi^{\mathrm{NE}}(k)\dot{\mu}_{\mathrm{NE}},$$

and the corresponding SW integral, as the support of $\dot{\mu}_{\mathrm{NE}}$ is contracted towards a single pair of symmetric segments. We take this limit and prove that it is both finite and nonzero in §5.2.3: the limit then clearly has a sign which is immediately predictable based on which segments of the zigzag are becoming longer or shorter, and how those segments meet the curve whose extremal length we are measuring. The main difficulty in taking the limits of these integrals is in controlling the appearance of some apparent singularities: this difficulty vanishes once one invokes the symmetry condition to observe that the apparent singularities cancel in pairs.

We begin our implementation of this outline with some notation. Choose a zigzag Z; let I_k denote the segment of Z connecting the points P_k and P_{k+1}. Our goal is to consider the effect on the conformal geometries of S_{NE} and S_{SW} of a deformation of $\mathcal{Z}$, where I_k (and I_{-k-1}, resp.) move into Ω_{NE}: one of the adjacent sides I_{k-1} and I_{k+1} (I_{-k-2} and I_{-k}, resp.) is shortened and one is lengthened, and the rest of the zigzag is unchanged. (See Figure 2.)

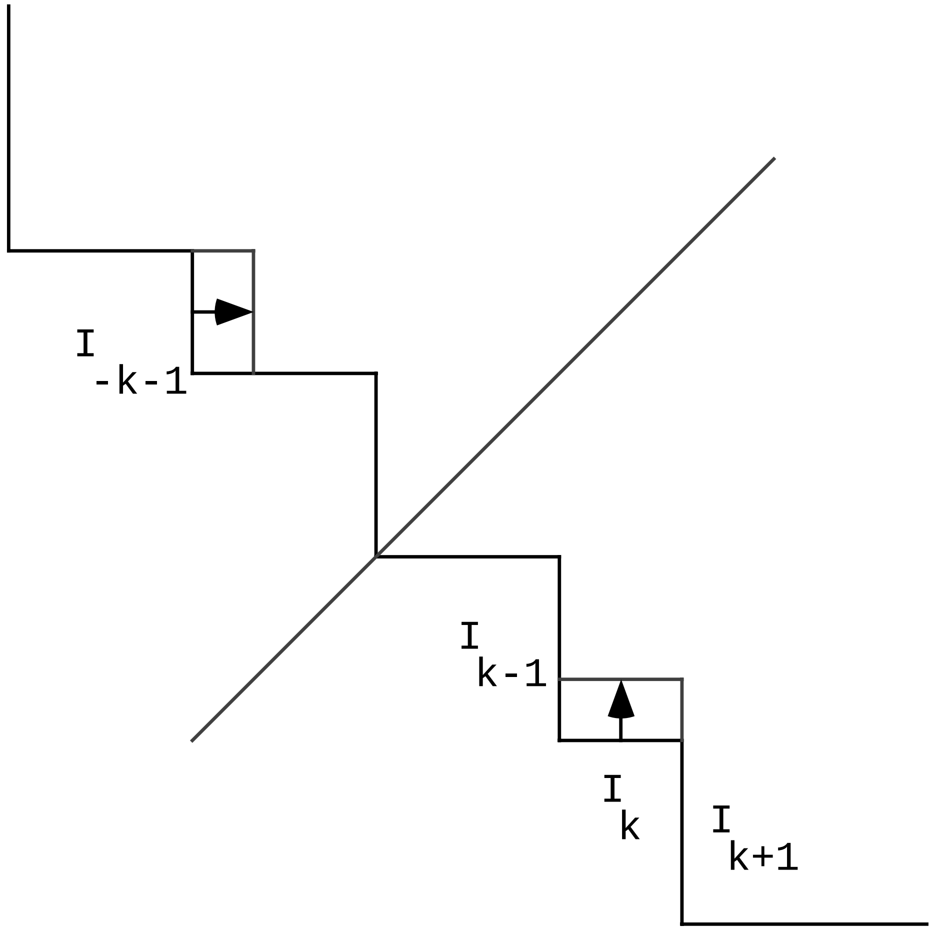

Figure 2

5.2.1. In this subsection, we treat steps 1 and 2 of the above outline. We begin by defining a family of maps f_ϵ which move I_ϵ into Ω_{NE}; we presently treat the case that I_k is horizontal, deferring the vertical case until the next paragraph.

With no loss in generality, we may as well assume that I_{k+1} is vertical; the more general case will just follow from obvious changes in notations and signs. We consider local (conformal) coordinates $z = x+iy$ centered on the midpoint of I_k (i.e., the horizontal segment abutting I_k at the vertex P_k of I_k nearest to the line $\{y = x\}$.) In particular, suppose that I_k is represented by the real interval $[-a, a]$, and define, for $b > 0$ and $\delta > 0$

small, a local Lipschitz deformation $f_\epsilon : \mathbf{C} \to \mathbf{C}$
(5.1a)

$$f_\epsilon(x,y) = \begin{cases} \left(x, \epsilon + \frac{b-\epsilon}{b} y\right), & \{-a \le x \le a, 0 \le y \le b\} = R_1 \\ \left(x, \epsilon + \frac{b+\epsilon}{b} y\right), & \{-a \le x \le a, -b \le y \le 0\} = R_2 \\ \left(x, y + \frac{\epsilon + \frac{b-\epsilon}{b} y - y}{\delta}(x+\delta+a)\right), & \{-a-\delta \le x \le -a, 0 \le y \le b\} = R_3 \\ \left(x, y - \frac{\epsilon + \frac{b-\epsilon}{b} y - y}{\delta}(x-\delta-a)\right), & \{a \le x \le a+\delta, 0 \le y \le b\} = R_4 \\ \left(x, y + \frac{\epsilon + \frac{b+\epsilon}{b} y - y}{\delta}(x+\delta+a)\right), & \{-a-\delta \le x \le -a, -b \le y \le 0\} = R_5 \\ \left(x, y - \frac{\epsilon + \frac{b+\epsilon}{b} y - y}{\delta}(x-\delta-a)\right), & \{a \le x \le a+\delta, -b \le y \le 0\} = R_6 \\ (x,y) & \text{otherwise} \end{cases}$$

where we have defined the regions $R_1, \ldots, R_6$ within the definition of f_ϵ. Also note that here Z_0 contains the arc $\{(-a,y) \mid 0 \le y \le b\} \cup \{(x,0) \mid -a \le x \le a\} \cup \{(a,y) \mid -b \le y \le 0\}$.

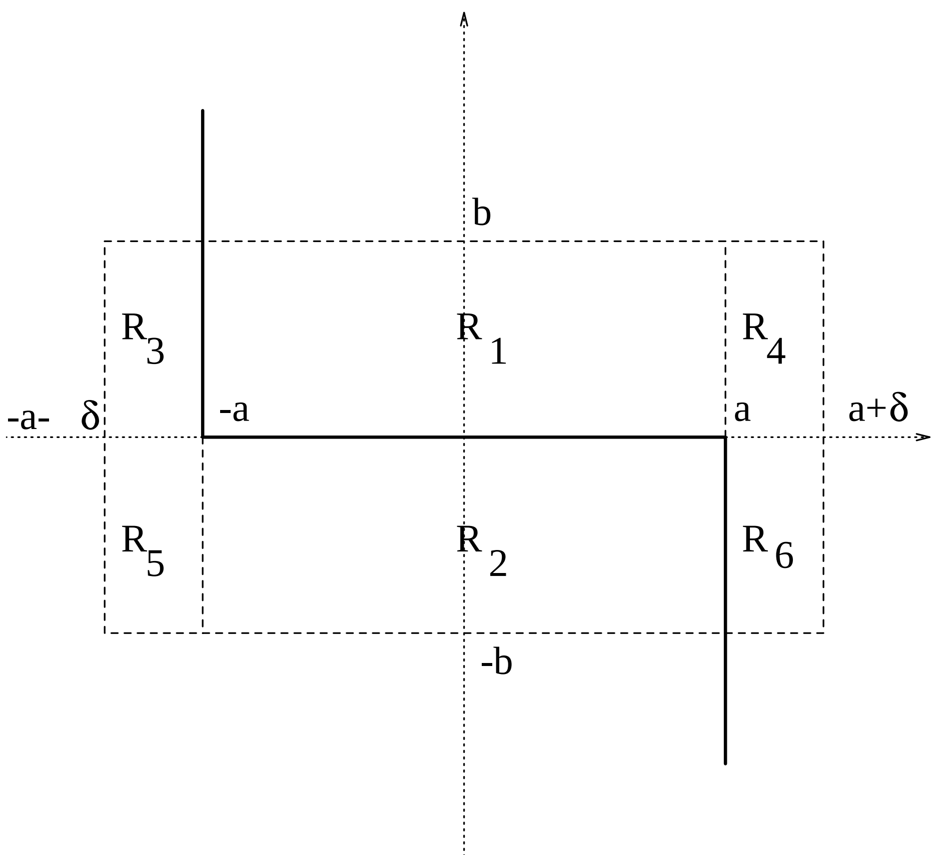

Figure 3

Of course f_ϵ differs from the identity only on a neighborhood of I_k; so that $f_\epsilon(Z_0)$ is a zigzag but no longer a symmetric zigzag. We next modify f_ϵ in a neighborhood of the reflected (across the $y = x$ line) segment I_{-k-1} in an analogous way with a map f_ϵ^* so that $f_\epsilon^* \circ f_\epsilon(Z)$ will be a symmetric zigzag. (Here f_ϵ^* is exactly a reflection of f_ϵ if $k > 1$.

In the case $k = 1$, we require only a small adjustment for the fact that f_ϵ has changed one of the sides adjacent to the segment $\overline{P_{-1}P_0}$, both segments of which lie in $\mathrm{supp}(f^*_\epsilon - \mathrm{id})$.)

Our present conventions are that I_k is horizontal; this forces I_{-k-1} to be vertical and we now write down f^*_ϵ for such a vertical segment; this is a straightforward extension of the description of f_ϵ for a horizontal side, but we present the definition of f^*_ϵ anyway, as we are crucially interested in the signs of the terms. So set

$$
\text{(5.1b)} \quad f^*_\epsilon = \begin{cases}
\left(\epsilon + \frac{b-\epsilon}{b}x, y\right), & \{0 \le x \le b, -a \le y \le a\} = R^*_1 \\
\left(\epsilon + \frac{b+\epsilon}{x}, y\right), & \{-b \le x \le 0, -a \le y \le a\} = R^*_2 \\
\left(x - \frac{\epsilon + \frac{b-\epsilon}{b}x - x}{\delta}(y - \delta - a), y\right), & \{0 \le x \le b, a \le y \le a+\delta\} = R^*_3 \\
\left(x + \frac{\epsilon + \frac{b-\epsilon}{b}x - x}{\delta}(y + \delta + a), y\right), & \{0 \le x \le b, -a-\delta \le y \le -a\} = R^*_4 \\
\left(x - \frac{\epsilon + \frac{b+\epsilon}{b}x - x}{\delta}(y - \delta - a), y\right), & \{-b \le x \le 0, a \le y \le a+\delta\} = R^*_5 \\
\left(x + \frac{\epsilon + \frac{b+\epsilon}{b} - x}{\delta}(y + \delta + a), y\right), & \{-b \le x \le 0, -a-\delta \le y \le -a\} = R^*_6 \\
(x, y) & \text{otherwise}
\end{cases} .
$$

Note that under the reflection across the line $\{y = x\}$, the regions R_1 and R_2 get taken to R^*_1 and R^*_2, but R_4 and R_6 get taken to R^*_3 and R^*_5, while R_3 and R_5 get taken to R^*_4 and R^*_6, respectively.

Let $\nu_\epsilon = \frac{(f_\epsilon)_{\bar z}}{(f_\epsilon)_z}$ denote the Beltrami differential of f_ϵ, and set $\dot\nu = \frac{d}{d\epsilon}\big|_{\epsilon=0} \nu_\epsilon$. Similarly, let ν^*_ϵ denote the Beltrami differential of f^*_ϵ, and set $\dot\nu^* = \frac{d}{d\epsilon}\big|_{\epsilon=0} \nu^*_\epsilon$. Let $\dot\mu = \dot\nu + \dot\nu^*$. Now $\dot\mu$ is a Beltrami differential supported in a bounded domain in $\mathbf{C} = \Omega_{\mathrm{NE}} \cup Z_0 \cup \Omega_{\mathrm{SW}}$ around Z_0, so it restricts to a pair $(\dot\mu_{\mathrm{NE}}, \dot\mu_{\mathrm{SW}})$ of Beltrami differentials on the pair of domains $(\Omega_{\mathrm{NE}}, \Omega_{\mathrm{SW}})$. Thus, this pair of Beltrami differentials lift to a pair $(\dot\mu_{\mathrm{NE}}, \dot\mu_{\mathrm{SW}})$ on the pair $(S_{\mathrm{NE}}, S_{\mathrm{SW}})$ of punctured spheres, where we have maintained the same notation for this lifted pair. But then, as a pair of Beltrami differentials on $(S_{\mathrm{NE}}, S_{\mathrm{SW}}) \subset T^{\mathrm{symm}}_{0,2p+2}$, the pair $(\dot\mu_{\mathrm{NE}}, \dot\mu_{\mathrm{SW}})$ represents a tangent vector to $\mathcal{Z} \subset T^{\mathrm{symm}}_{0,2p+2}$ at Z_0. It is our plan to evaluate $dD(\dot\mu_{\mathrm{NE}}, \dot\mu_{\mathrm{SW}})$ to a precision sufficient to show that $dD(\dot\mu_{\mathrm{NE}}, \dot\mu_{\mathrm{SW}}) < 0$. To do this, we compute $dExt([\gamma])$ for relevant classes of curves $[\gamma]$.

We begin by observing that it is easy to compute that $\dot\nu = \left(\frac{d}{d\epsilon}\big|_{\epsilon=0} f_\epsilon\right)_{\bar z}$ evaluates near I_k to

(5.2a)

$$
\dot\nu = \begin{cases}
\frac{1}{2b}, & z \in R_1 \\
-\frac{1}{2b}, & z \in R_2 \\
\frac{1}{2b}[x + \delta + a]/\delta + i\,(1 - y/b)\,\frac{1}{2d} = \frac{1}{2b\delta}(\bar z + \delta + a + ib), & z \in R_3 \\
-\frac{1}{2b}[x - \delta - a]/\delta - i\,(1 - y/b)\,\frac{1}{2\delta} = \frac{1}{2b\delta}(-\bar z + \delta + a - ib), & z \in R_4 \\
-\frac{1}{2b}[x + \delta + a]/\delta + i\,(1 + y/b)\,\frac{1}{2\delta} = \frac{1}{2b\delta}(-\bar z - \delta - a + ib), & z \in R_5 \\
\frac{1}{2b}[x - \delta - a]/\delta - i\,(1 + y/b)\,\frac{1}{2\delta} = \frac{1}{2b\delta}(\bar z - \delta - a - ib), & z \in R_6 \\
0 & z \notin \mathrm{supp}(f_\epsilon - \mathrm{id})
\end{cases} .
$$

We further compute

$$\dot{\nu}^* = \begin{cases} -\frac{1}{2b}, & R_1^* \\ \frac{1}{2b}, & R_2^* \\ \frac{1}{2b\delta}(i\bar{z} - \delta - a - bi) & R_3^* \\ \frac{1}{2b\delta}(-i\bar{z} - \delta - a + bi) & R_4^* \\ \frac{1}{2b\delta}(-i\bar{z} + \delta + a - bi) & R_5^* \\ \frac{1}{2b\delta}(i\bar{z} + \delta + a + bi) & R_6^* \end{cases} \tag{5.2b}$$

Of course, this then defines the pair $(\dot{\mu}_{\mathrm{NE}}, \dot{\mu}_{\mathrm{SW}})$ by restriction to the appropriate neighborhoods. In particular, $\dot{\mu}_{\mathrm{NE}}$ is supported in the (lifts of) the regions R_1, R_4, R_6, R_1^*, R_3^* and R_5^* while $\dot{\mu}_{\mathrm{SW}}$ is supported in the (lifts of) R_2, R_3, R_5, R_2^*, R_4^* and R_6^*.

5.2.2. We next consider the effect of the variation $\frac{d}{d\epsilon}\big|_{\epsilon=0} f_\epsilon$ upon the conformal geometries of S_{NE} and S_{SW}. We compute the infinitesimal changes of some extremal lengths induced by the variation $\frac{d}{d\epsilon}\big|_{\epsilon=0} f_\epsilon$.

For $[\gamma]$ a homotopy class of (a family of) simple closed curves, the form $d\,\mathrm{Ext}_{(\cdot)}([\gamma]) \in T^*_{(\cdot)}T^{\mathrm{symm}}_{0,2p+2}$ is given by an element of $\mathrm{QD}^{\mathrm{symm}}(\cdot)$. We describe some of these quadratic differentials now; this is step 3 of the outline.

To begin, since the holomorphic quadratic differential $\phi_k^{NE} = \phi_{\gamma_k}^{NE} = d\,\mathrm{Ext}_{S_{\mathrm{NE}}}([\gamma_k])$ is an element of $\mathrm{QD}^{\mathrm{symm}}_{0,2p+2}(S_{\mathrm{NE}})$, it is lifted from a holomorphic quadratic differential ψ_k^{NE} on Ω_{NE} whose horizontal foliation has nonsingular leaves either orthogonal to and connecting the segments I_{-k-2} and I_{-k} or orthogonal to and connecting the segments I_k and I_{k+2}. (The foliation is parallel to the other segments of Z, and the vertices where the foliation changes from orthogonality to parallelism lift to points where the differential ϕ_k^{NE} has a simple pole.)

Now the segments $I_k \in \overline{\Omega_{\mathrm{NE}}}$ corresponds under the map $\phi : \Omega_{\mathrm{NE}} \to \Omega_{\mathrm{SW}}$ to the segment $I_{-k-2} \in \overline{\Omega_{\mathrm{SW}}}$; similarly, $I_{-k-2} \in \overline{\Omega_{\mathrm{NE}}}$ corresponds to $I_k \in \overline{\Omega_{\mathrm{SW}}}$. Thus $d\,\mathrm{Ext}_{S_{\mathrm{SW}}} \in \mathrm{QD}^{\mathrm{symm}}_{0,2p+2}(S_{\mathrm{SW}})$ is lifted from a holomorphic quadratic differential ψ_k^{SW} whose horizontal foliation has nonsingular leaves orthogonal to and connecting the segments I_{-k-2} and I_{-k} and orthogonal to and connecting I_k and I_{k+2}, in an analogous way to ψ_k^{NE}. Now the foliations have characteristic local forms near the support of the divisors of the differentials, and so the foliations of ϕ_k^{NE} (and ϕ_k^{SW}) determine the divisors of these differentials. We collect this discussion, and its implications for the divisors, as

Lemma 5.2. *The horizontal foliations for ψ_k^{NE} and ψ_k^{SW} extend to a foliation of $\mathbf{C} = \Omega_{\mathrm{NE}} \cup Z_0 \cup \Omega_{\mathrm{SW}}$, which is singular only at the vertices of Z. This foliation is parallel to Z except at I_{-k-2}, I_{-k}, I_k and I_{k+2}, where it is orthogonal. The differential ϕ_k^{NE} (and ϕ_k^{SW}) have divisors*

$$(\phi_k^{\mathrm{NE}}) = P_0^2 P_\infty^2 (P_{-k+1}P_{-k}P_{-k-1}P_{-k-2})^{-1}(P_{k-1}P_kP_{k+1}P_{k+2})^{-1} = (\phi_k^{\mathrm{SW}}) \quad \text{if } k > 1$$

$$(\phi_1^{\mathrm{NE}}) = P_\infty^2 (P_{-3}P_{-2}P_{-1}P_1P_2P_3)^{-1} = (\phi_1^{\mathrm{SW}})$$

where P_j refers to the lift of $P_j \in Z$ to S_{NE} and S_{SW}, respectively.

5.2.3. Let ϕ^{NE} denote a meromorphic quadratic differential on S_{NE} (symmetric about the lift of $\{y=x\}$) lifted from a (holomorphic) quadratic differential ψ^{NE} on (the open domain) Ω_{NE}; suppose that ϕ^{NE} represents the covector $d\,\mathrm{Ext.}([\gamma])$ in $T^*_{S_{\mathrm{NE}}}T^{\mathrm{symm}}_{0,2p+2}$ for some class of curves $[\gamma]$. Formula (2.1) says that

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathrm{Ext}_{S^{\epsilon}_{\mathrm{NE}}}([\gamma]) = 2\,\mathrm{Re}\int_{S_{\mathrm{NE}}} \phi^{\mathrm{NE}}\dot{\mu}_{\mathrm{NE}} = 4\,\mathrm{Re}\int_{\Omega_{\mathrm{NE}}} \psi^{\mathrm{NE}}\dot{\nu}_{\mathrm{NE}} \tag{5.3}$$

where $S^{\epsilon}_{\mathrm{NE}}$ is the punctured sphere obtained by appropriately doubling $f_\epsilon(\Omega_{\mathrm{NE}})$.

The formula (5.3) is the basic variational formula that we will use to estimate the changes in the conformal geometries of Ω_{NE} and Ω_{SW} as we vary in $\mathcal{Z}$. However, in order to evaluate these integrals to a precision sufficient to prove Proposition 5.1, we require a lemma. As background to the lemma, note that $\dot{\nu}_{\mathrm{NE}}$ and $\dot{\nu}_{\mathrm{SW}}$ depend upon a choice of small constants $b>0$ and $\delta>0$ describing the size of the neighborhood of I_k and I_{-k-1} supporting $\dot{\nu}_{\mathrm{NE}}$ and $\dot{\nu}_{\mathrm{SW}}$; on the other hand, a hypothesis like the foliation of ψ^{NE} is orthogonal to or parallel to I_k and I_{-k-1} concerns the behavior of ψ^{NE} only at I_k and I_{-k-1} (i.e., when $b=\delta=0$). Thus, to use this information about the foliations in evaluating the right hand sides of formula (5.3), we need to have control on $\mathrm{Re}\int_{\Omega_{\mathrm{NE}}}\psi^{\mathrm{NE}}\dot{\nu}_{\mathrm{NE}}$ as b and δ tend to zero. This is step 4 of the outline we gave at the outset of section 5.2.

Lemma 5.3. $\lim_{b\to0,\delta\to0}\mathrm{Re}\int_{\Omega_{\mathrm{NE}}}\psi^{\mathrm{NE}}\dot{\nu}_{\mathrm{NE}}$ *exists and is finite and non-vanishing. Moreover, if* ψ^{NE} *has foliation either orthogonal to or parallel to* $I_k\cup I_{-k-1}$*, then the sign of the limit equals sgn* $(\psi^{\mathrm{NE}}\dot{\nu}_{\mathrm{NE}}(q))$ *where* q *is a point on the interior of* $I_k\cup I_{-k-1}$*.*

Proof. On the interior of $I_k\cup I_{-k-1}$, the coefficients of both ψ^{NE} and $\dot{\nu}_{\mathrm{NE}}$ have locally constant sign; as we see from ψ^{NE} being either orthogonal or parallel to Z and symmetric, and from the form of $\dot{\nu}_{\mathrm{NE}}$ in (5.2a) and (5.2b). We then easily check that the sign of the product $\psi^{\mathrm{NE}}\dot{\nu}_{\mathrm{NE}}$ is constant on the interior of $I_k\cup I_{-k-1}$, proving the final statement of the lemma.

The only difficulty in seeing the existence of a finite limit as $b+\delta\to0$ is the possible presence of simple poles of ϕ^{NE} at the lifts of endpoints of $I_k\cup I_{-k-1}$.

To understand the singular behavior of ψ^{NE} near a vertex of the zigzag, we begin by observing that on a preimage on S_{NE} of such a vertex, the quadratic differential has a simple pole. Now let ω be a local uniformizing parameter near the preimage of the vertex on S_{NE} and ζ a local uniformizing parameter near the vertex of Z on $\mathbf{C}$. There are two cases to consider, depending on whether the angle in Ω_{NE} at the vertex is (i) $3\pi/2$ or (ii) $\pi/2$. In the first case, the map from Ω_{NE} to a lift of Ω_{NE} in S_{NE} is given in coordinates by $\omega=(i\zeta)^{2/3}$, and in the second case by $\omega=\zeta^2$. Thus, in the first case we write $\phi^{\mathrm{NE}}_k=c\frac{d\omega^2}{\omega}$ so that $\psi^{\mathrm{NE}}_k=-4/9c(i\zeta)^{-4/3}d\zeta^2$, and in the second case we write $\psi^{\mathrm{NE}}_k=4cd\zeta^2$; in both cases, the constant c is real with sign determined by the direction of the foliation.

With these expansions for ψ^{NE}_k and ψ^{SW}_k, we can compute $d\,\mathrm{Ext}([\gamma_k])[\dot{\mu}]$; of course, this quantity is given by formula (2.1) as

$$
\begin{aligned}
d\,\mathrm{Ext}_{S_{\mathrm{NE}}}[(\gamma_k)][\dot\mu] &= 2\,\mathrm{Re}\int_{S_{\mathrm{NE}}}\phi_k\dot\mu \\
&= 4\,\mathrm{Re}\int_{\Omega_{\mathrm{NE}}}\psi_k\dot\nu \\
&= 4\,\mathrm{Re}\left(\int_{R_1\cup R_1'}+\int_{R_4\cup R_3^*}+\int_{R_6\cup R_5^*}\right)\psi_k\dot\nu. \qquad (5.4)
\end{aligned}
$$

Clearly, as $b+\delta\to 0$, as $|\dot\nu| = O\left(\max\left(\frac{1}{b},\frac{1}{\delta}\right)\right)$, we need only concern ourselves with the contribution to the integrals of the singularity at the vertices of Z with angle $3\pi/2$.

To begin this analysis, recall that we have assumed that I_k is horizontal so that Z has a vertex angle of $3\pi/2$ at P_{k+1} and P_{-k-1}. It is convenient to rotate a neighborhood of I_{-k-1} through an angle of $-\pi/2$ so that the support of $\dot\nu$ is a reflection of the support of $\dot\nu^*$ (see equation (5.1)) through a vertical line. If the coordinates of $\operatorname{supp}\dot\nu$ and $\operatorname{supp}\dot\nu^*$ are z and z^*, respectively (with $z(P_{k+1}) = z^*(P_{-k-1}) = 0$), then the maps which lift neighborhoods of P_{k+1} and P_{-k-1}, respectively, to the sphere S_{NE} are given by

$$z\mapsto (iz)^{2/3}=\omega \quad\text{ and }\quad z^*\mapsto (z^*)^{2/3}=\omega^*.$$

Now the poles on S_{NE} have coefficients $c\frac{d\omega^2}{\omega}$ and $-c\frac{d\omega^{*2}}{\omega^*}$, respectively, so we find that when we pull back these poles from S_{NE} to Ω_{NE}, we have $\psi^{\mathrm{NE}}(z) = -\frac{4}{9}c\;dz^2/\omega^2$ while $\psi^{\mathrm{NE}}(z^*) = -\frac{4}{9}c\,dz^2/(\omega^*)^2$ in the coordinates z and z^* for $\operatorname{supp}\dot\nu$ and $\operatorname{supp}\dot\nu^*$, respectively. But by tracing through the conformal maps $z\mapsto\omega\mapsto\omega^2$ on $\operatorname{supp}\dot\nu$ and $z^*\mapsto\omega^*\mapsto(\omega^*)^2$, we see that if z^* is the reflection of z through a line, then

$$\frac{1}{(\omega(z))^2} = 1/\overline{\omega^*(z^*)^2}$$

so that the coefficients $\psi^{\mathrm{NE}}(z)$ and $\psi^{\mathrm{NE}}(z^*)$ of $\psi^{\mathrm{NE}} = \psi^{\mathrm{NE}}(z)dz^2$ near P_{k+1} and of $\psi^{\mathrm{NE}}(z^*)dz^{*2}$ near P_{-k-1} satisfy $\psi^{\mathrm{NE}}(z) = \overline{\psi^{\mathrm{NE}}(z^*)}$, at least for the singular part of the coefficient.

On the other hand, we can also compute a relationship between the Beltrami coefficients $\dot\nu(z)$ and $\dot\nu^*(z^*)$, in the obvious notation, after we observe that $f_\epsilon^*(z^*) = -\overline{f_\epsilon(z)}$. Differentiating, we find that

$$
\begin{aligned}
\dot\nu^*(z^*) &= \dot f^*(z^*)_{\overline{z^*}} \\
&= -\overline{\dot f(z)}_{\overline{z^*}} \\
&= (\overline{\dot f(z)})_z \\
&= \overline{\dot f(z)_{\overline{z}}} \\
&= \overline{\dot\nu(z)}.
\end{aligned}
$$

Combining our computations of $\psi^{\text{NE}}(z^*)$ and $\dot{\nu}(z^*)$ and using that the reflection $z \mapsto z^*$ reverses orientation, we find that (in the coordinates $z^* = x^* + iy^*$ and $z = x + iy$) for small neighborhoods $N_{kappa}(P_{k+1})$ and $N_{kappa}(P_{-k-1})$ of P_{k+1} and P_{-k-1} respectively,

$$\begin{aligned}
&\mathrm{Re} \int\limits_{\mathrm{supp}\, \dot{\nu} \cap N_k appa(P_{k+1})} \psi^{\text{NE}}(z)\dot{\nu}(z)dxdy + \mathrm{Re} \int\limits_{\mathrm{supp}\, \dot{\nu}^* \cap N_k appa(P_{-k-1})} \psi^{\text{NE}}(z^*)\dot{\nu}(z^*)dx^*dy^* \\
&= \mathrm{Re} \int\limits_{\mathrm{supp}\, \dot{\nu} \cap N_k appa(P)} \psi^{\text{NE}}(z)\dot{\nu}(z) - \psi^{\text{NE}}(z^*)\dot{\nu}(z^*)dxdy \\
&= \mathrm{Re} \int\limits_{\mathrm{supp}\, \dot{\nu} \cap N_k appa} \psi^{\text{NE}}(z)\dot{\nu}(z) - \overline{\left[\psi^{\text{NE}}(z) + O(1)\right]}\ \overline{\dot{\nu}(z)}dxdy \\
&= O(b + \delta)
\end{aligned}$$

the last part following from the singular coefficients summing to a purely imaginary term while $\dot{\nu} = O\left(\frac{1}{b} + \frac{1}{\delta}\right)$, and the neighborhood has area $b\delta$. This concludes the proof of the lemma. $\square$

5.3 A good initial point for the flow. In this subsection, we seek a symmetric zigzag Z^* of genus p with the property that $E_{\text{NE}}(k) = E_{\text{SW}}(k)$ for $k = 2, \ldots, p-1$. This will greatly simplify the height function $D(Z)$ at Z^*. Our argument for the existence of Z^* involves the

Assumption 5.4. There is a reflexive symmetric zigzag of genus $p - 1$.

Since Enneper's surface can be represented by the zigzag of just the positive x- and y- axes, and we already have represented the Chen-Gackstatter surface of genus one by a zigzag in §3, the initial step of the inductive proof of this assumption is established.

The effect of the assumption is that on the codimension 1 face of $\partial\overline{\mathcal{Z}}$ consisting of zigzags with $P_{-1} = P_0 = P_1$, there is a degenerate zigzag Z_0^* with $E_{\text{NE}}(k) = E_{\text{SW}}(k)$ for $k = 2, \ldots, p-1$. Our goal in this subsection is the proof of

Lemma 5.5. *There is a family $Z_t^* \subset \mathcal{Z}$ of non-degenerate symmetric zigzags with limit point Z_0^* where each zigzag Z_t^* satisfies $E_{\text{NE}}(k) = E_{\text{SW}}(k)$ for $k = 2, \ldots, p-1$.*

Proof. We apply the implicit function theorem to a neighborhood V of $Z_0^* = (Z_0^*, 0)$ in $\partial\overline{\mathcal{Z}} \times (-\epsilon, \epsilon)$, where we will identify a neighborhood of Z_0^* in $\overline{\mathcal{Z}}$ with a neighborhood U of Z_0^* in $\partial\overline{\mathcal{Z}} \times [0, \epsilon)$. So our argument will proceed in three steps: (i) we first define our embedding of U into V, (ii) we then show that the mapping $\Phi : (Z, t) \mapsto (E_{\text{NE}}(2) - E_{\text{SW}}(2), \ldots, E_{\text{NE}}(p-1) - E_{\text{SW}}(p-1))$ is differentiable and then (iii) finally we show that $d\Phi \big|_{\partial\mathcal{Z} \times \{0\}}$ is an isomorphism onto $\mathbf{R}^{p-2}$. The first two steps are essentially formal, while the last step involves most of the geometric background we have developed so far, and is the key step in our approach to the gradient flow.

For our first step, normalize the zigzags in U (as in section 4.1) so that $P_{-p} = i$ and $P_p = 1$ and for Z in U near Z_0^*, and let $(t(Z), a_2(Z), \ldots, a_{p-1}(Z))$ denote the Euclidean lengths of the segments $\langle I_0, I_1, \ldots, I_{p-2}\rangle$. Then for $Z \in U$, let Z have coordinates $(\psi(Z), t)$ where $\psi(Z) \in \partial\mathcal{Z}$ has normalized Euclidean lengths $a_2(Z), \ldots, a_{p-1}(Z)$. It is easy to see that $\psi : U \to \partial\overline{\mathcal{Z}}$ is a continuous and well-defined map.

Next we verify that the map Φ is differentiable. We can calculate the differential $D\Phi \big|_U$ by applying some of the discussion of the previous subsection 5.2. For instance, the matrix $D\Phi \big|_U (Z)$ can be calculated in terms of $dE_{\mathrm{NE}}(k) \big|_Z [\dot\mu]$ where $\dot\mu$ corresponds to an infinitesimal motion of an edge of Z, as in formula (5.3). Indeed we see that as $\epsilon \to 0$, all of the derivatives $d(E_{\mathrm{NE}}(k) - E_{\mathrm{SW}}(k))[\dot\mu]$ are bounded and converge: this follows easily from observing that the quadratic differentials ϕ_k are bounded and converge as $\epsilon \to 0$ and then applying formulas (5.3) and (5.2). In fact, when $\operatorname{supp}\dot\mu$ meets the lift of $I_0 \cup I_{-1}$, the same argument continues to hold, after we make one observation. We observe that we can restrict our attention to where we are sliding only the segments $I_1, \ldots, I_{p-1}$ (and their reflections) and not I_0 (and I_{-1}), as the tangent space is $p-1$ dimensional; thus these derivatives are bounded as well.

This is all the differentiability we need for the relevant version of the implicit function theorem.

Finally, we show that $d\Phi \big|_{Z_0^*}: T_{Z_0^*}\mathcal{Z}_{p-1} \to \mathbf{R}^{p-2}$ is an isomorphism. To see this it is sufficient to verify that this linear map $d\Phi \big|_{Z_0^*}$ has no kernel. So let $v \in T_{Z_0^*}\mathcal{Z}_{p-1}$, so that

$$v = \sum_{i=1}^{p-1} c_i \dot\nu_i$$

where $c_i \in \mathbf{R}$ and $\dot\nu_i$ refers to an infinitesimal perturbation of I_i and I_{-i-1} into Ω_{NE} (in the notation for zigzags in $\mathcal{Z}_p$: for Z_0^*, we have that I_0 and I_{-1} have collapsed onto P_0.

Suppose, up to looking at $-v$ instead of v, that some $c_i > 0$, and let $\{i_j\}$ be the subset of the index set $\{1, \ldots, p-1\}$ for which $c_{i_j} > 0$. We consider the (non-empty) curve system , of arcs connecting I_{i_j} to the interval $\overline{P_p P_\infty}$ and I_{-i_j-1} to $\overline{P_\infty P_{-p}}$, let φ_v denote a Jenkins-Strebel differential associated to this curve system. By construction, $\operatorname{sgn}\varphi_v$ is constant on the interior of every interval, and $\varphi_v > 0$ on the interior of I_i if and only if the index $i \in \{i_j\}$.

Thus, by Lemma 5.3 and formula (5.2), we see that both

$$d\,\mathrm{Ext}_\Gamma(\Omega_{\mathrm{NE}})[v] = \sum_{i=1}^{p-1} c_i \int_{I_i} \varphi_v \dot\nu_i > 0 \tag{5.5a}$$

and from formulas (5.2) and (5.3), using that the horizontal (and vertical) foliation(s) of φ_v extend to Ω_{SW}, that

$$d\,\mathrm{Ext}_\Gamma(\Omega_{\mathrm{SW}})[v] = \sum_{i=1}^{p-1} c_i \int_{I_i} \varphi_v \dot\nu_i < 0. \tag{5.5b}$$

Thus

$$d\left(\mathrm{Ext}_\Gamma(\Omega_{\mathrm{NE}}) - \mathrm{Ext}_\Gamma(\Omega_{\mathrm{SW}})\right)[v] > 0. \tag{5.6}$$

Now suppose that $d\Phi \Big|_{Z_0^*} [v] = 0$. Then by the definition of Φ, we would have that $d\,\mathrm{Ext}_\Gamma(\Omega_{\mathrm{NE}})[v] = d\,\mathrm{Ext}(\Omega_{\mathrm{SW}})[v]$, and since $\langle \mathrm{Ext}_{\Gamma_i}(\Omega_{\mathrm{NE}})\rangle$ provides local coordinates in Teichmüller space, we would see that the Teichmüller distance between Ω_{NE} and Ω_{SW} would infinitesimally vanish. But that would force $\mathrm{Ext}_{\Omega_{\mathrm{NE}}}(,\ ) - \mathrm{Ext}_{\Omega_{\mathrm{SW}}}(,\ )$ to vanish to first order, which contradicts our computation (5.5).

We conclude that $d\Phi \Big|_{Z_0^*}$ is an isomorphism, so that the implicit function theorem yields the statement of the lemma. $\square$

5.4 The flow from the good initial point and the proofs of the main results. We now consider one of our "good" zigzags $Z_t^* \in \mathcal{Z}_p$ and use it as an initial point from which to flow along $-\operatorname{grad} D(Z)$ to a reflexive symmetric zigzag.

Let $\mathcal{Y} \subset \mathcal{Z}_p$ denote the set of genus p zigzags for which $\mathrm{Ext}_{\Omega_{\mathrm{NE}}}(,\ _i) = \mathrm{Ext}_{\Omega_{\mathrm{SW}}}(,\ _i)$ for $i = 2, \dots, p-1$. As extremal length functions are in $C^1(T_{0,2p+2}^{\mathrm{symm}})$ by Gardiner-Masur [GM], we see that $\mathcal{Y}$ is a piecewise C^1 submanifold of $\mathcal{Z}_p$. (We shall note momentarily that in our case, these extremal length functions are real analytic.) We consider the height function D restricted to the set $\mathcal{Y}$.

Lemma 5.6. *$D \Big|_{\mathcal{Y}}$ is proper and is critical only at points $Z \in \mathcal{Y} \subset \mathcal{Z}$ for which $D(Z) = 0$.*

Proof. The properness of $D \Big|_{\mathcal{Y}}$ follows from the properness of D, as shown in §4. We next show that if $D(Z) \neq 0$ for $Z \in \mathcal{Y}$, then there exists a tangent vector $v \in T_Z\mathcal{Z}$ for which $dD[v] < 0$, and for which $d\left(\mathrm{Ext}_{\Omega_{\mathrm{NE}}}(,\ _i) - \mathrm{Ext}_{\Omega_{\mathrm{SW}}}(,\ _i)\right)[v] = 0$ so that v lies tangent to a fragment of $\mathcal{Y}$ and infinitesimally reduces the height D.

Indeed, we observe that

$$D \Big|_{\mathcal{Y}} = \left[\left(\exp\frac{1}{E_{\mathrm{NE}(1)}}\right) - \left(\exp\frac{1}{E_{\mathrm{SW}(1)}}\right)\right]^2 + \left[E_{\mathrm{NE}(1)} - E_{\mathrm{SW}(1)}\right]^2$$

as the other terms vanish.

Now, observe that $\mathcal{Z}$ is a real analytic submanifold of the real analytic product manifold $T_{0,2p+2}^{\mathrm{symm}} \times T_{0,2p+2}^{\mathrm{symm}}$, being defined in terms of periods of a pair of holomorphic forms on the underlying punctured spheres. Next we observe that $E_{\mathrm{NE}(j)}$ and $E_{\mathrm{SW}(j)}$ are, for each j, real analytic functions on $T_{0,2p+2}^{\mathrm{symm}}$ with non-degenerate level sets. To see this note that the extremal length functions correspond to just the energy of harmonic maps from the punctured spheres to an interval, with the required analyticity coming from Eells-Lemaire [EL], or directly from 4.5; the non-degeneracy follows from Lemma 5.3, if we apply any $\dot{\nu}$ of the form (5.2) to the zigzag (this will be developed in more detail in the following paragraph). Thus, the set $\mathcal{Y}$ acquires the structure of a real analytic submanifold properly embedded in $\mathcal{Z}$. As $\mathcal{Y}$ is one-dimensional near Z_0, it is one-dimensional (with no boundary points) everywhere.

Now, for $Z \in \mathcal{Y} \subset \mathcal{Z}$ which is *not* a zero of D, we have for any tangent vector $\dot{\mu}$ the formula

$$
\begin{aligned}
dD[\dot\mu] = 2\left(\exp\left(\frac{1}{\mathrm{Ext}_{S_{\mathrm{NE}}}}([\gamma_1])\right) - \exp\left(\frac{1}{\mathrm{Ext}_{S_{\mathrm{SW}}}}([\gamma_1])\right)\right)&\\
\Bigg\{ - \exp\left(\frac{1}{\mathrm{Ext}_{S_{\mathrm{NE}}}}([\gamma_1])\right)(\mathrm{Ext}_{S_{\mathrm{NE}}}([\gamma_1]))^{-2}\, d\,\mathrm{Ext}_{S_{\mathrm{NE}}}([\gamma_1])[\dot\mu]&\\
+ \exp\left(\frac{1}{\mathrm{Ext}_{S_{\mathrm{SW}}}}([\gamma_1])\right)(\mathrm{Ext}_{S_{\mathrm{SW}}}([\gamma_1]))^{-2}\, d\,\mathrm{Ext}_{S_{\mathrm{SW}}}([\gamma_1])[\dot\mu]\Bigg\}&\\
+ [\mathrm{Ext}_{S_{\mathrm{NE}}}([\gamma_1]) - \mathrm{Ext}_{S_{\mathrm{SW}}}([\gamma_1])]\,(d\,\mathrm{Ext}_{S_{\mathrm{NE}}}([\gamma_1])[\dot\mu] - d\,\mathrm{Ext}_{S_{\mathrm{SW}}}([\gamma_1])[\dot\mu])&
\end{aligned}
$$

Then if we evaluate this expression for, say, $\dot\mu$ being given by lifting an infinitesimal move of just one side as in formulas (5.1) and (5.2), we find by an argument similar to that for inequalities (5.5) and (5.6) that $dD[\dot\mu] \neq 0$. This concludes the proof of the lemma. $\square$

Conclusion of the proof of Proposition 5.1. We argue by induction. The union of the positive x- and y-axes, is reflexive via the explicit map $z \mapsto iz^3$; this verifies the statement of the proposition for genus $p = 0$. There is also a unique reflexive zigzag for genus $p = 1$, after we make use of the permissible normalization $P_1 = 1$; we can verify that both S_{NE} and S_{SW} are square tori, as in the first paragraph of §3. In general, once we are given a reflexive symmetric zigzag of genus $p-1$, we are able to satisfy Assumption 5.4, so that Lemmas 5.5 and 5.6 guarantee the existence of a one-dimensional submanifold $\mathcal{Y} \subset \mathcal{Z}$ along which the height function is proper. A minimum Z_0 for this height function is critical for D along $\mathcal{Y}$, and hence satisfies $D(Z_0) = 0$ by Lemma 5.6. $\square$

As the discussion of subsection 4.3 shows that if $D(Z_0) = 0$, then Z_0 is reflexive, we conclude from Proposition 5.1 the

Main Theorem B. *There exists a reflexive symmetric zigzag of genus p for $p \geq 0$ which is isolated in $\mathcal{Z}_p$.*

Proof of Main Theorem B. The local uniqueness follows from inequality (5.6) and the argument following it. $\square$

Our main goal then follows.

Proof of Main Theorem A. By Main Theorem B, there exists a symmetric reflexive zigzag of genus p. By Theorem 3.3, and Lemma 3.4 from this zigzag we can find Weierstraß data for the required minimal surface.

5.5 A remark on a different height function. In this subsection, we try to give some context to the methods we adopted in Sections 4 and 5 by considering an alternate and perhaps more natural height function.

Define a new height function

$$H(Z) = \{d_{\mathrm{Teich}}(\mathcal{R}_{\mathrm{NE}}, \mathcal{R}_{\mathrm{SW}}\}^2.$$

Certainly $H(Z) = 0$ if and only if Z is reflexive. Moreover, here the gradient flow is much easier to work with, at least locally in $\mathcal{Z}_p$. Indeed, we observe

Lemma 5.7. *$H(Z) = 0$ if and only if Z is critical for $H(\cdot)$ on $\mathcal{Z}_p$.*

Sketch of Proof. Clearly $H(Z) \geq 0$ and H is C^1 (even real analytic by the proof of Lemma 5.5) on $\mathcal{Z}_p \subset T^{\mathrm{symm}}_{0,2p+2} \times T^{\mathrm{symm}}_{0,2p+2}$, so if $H(Z) = 0$, then Z is critical for H on $\mathcal{Z}_p$.

So suppose that $H(Z) \neq 0$. Then $\mathcal{R}_{\mathrm{NE}}$ is not conformally equivalent to $\mathcal{R}_{\mathrm{SW}}$, so we can look at the unique Teichmüller map in the homotopy class $[\phi : S_{\mathrm{NE}} \to S_{\mathrm{SW}}]$ (see the opening of §3).

From the construction of S_{NE}, S_{SW} and ϕ from Ω_{NE}, Ω_{SW} and ϕ, we can draw many conclusions about the Teichmüller differentials $q \in \mathrm{QD}^{\mathrm{symm}}(S_{\mathrm{NE}})$ and $q' \in \mathrm{QD}^{\mathrm{symm}}(S_{\mathrm{SW}})$. For instance, the foliations are either perpendicular or parallel to the image of Z on S_{NE} and S_{SW}, and by the construction of the Teichmüller maps, there are zeros in the lift of an interval I_k on S_{NE} if and only if there is a corresponding zero in the lift of $\phi(I_k)$ on S_{SW}. Moreover, there is a simple pole at a puncture on S_{NE} if and only if there is a simple pole at a corresponding puncture on S_{SW}.

Finally, we observe that q has only simple poles and as the foliation of q is symmetric about both the images of Z and the line $\{y = x\}$, we see that there cannot be a simple pole at either the lift of P_0 or ∞. Yet by Riemann-Roch, as there are 4 more poles than zeros, counting multiplicity, there must be a pair of intervals $I_k \cup I_{-k-1}$ whose lift has no zeros of q (and whose image under ϕ of lift has no zeros of $q' \in \mathrm{QD}^{\mathrm{symm}}(S_{\mathrm{SW}})$).

Observe next that Kerckhoff's formula (2.2) says that the horizontal measured foliation $(\mathcal{F}_q, \mu_q)$ of q extremizes the quotient on the right hand side of 2.2. However, consider a deformation (5.1) on our zero-free intervals $I_k \cup I_{-k-1}$. By Lemma 5.3 and formula (5.3) we see that either

$$d\,\mathrm{Ext}_{S_{\mathrm{NE}}}(\mu_q)[\dot{\nu}] > 0 \quad \text{and} \quad d\,\mathrm{Ext}_{S_{\mathrm{SW}}}(\mu_q)[\dot{\nu}] < 0$$

or

$$d\,\mathrm{Ext}_{S_{\mathrm{NE}}}(\mu_q)[\dot{\nu}] < 0 \quad \text{and} \quad d\,\mathrm{Ext}_{S_{\mathrm{SW}}}(\mu_q)[\dot{\nu}] > 0.$$

In either case, we see from Kerckhoff's formula that $dH \Big|_Z [\dot{\mu}] \neq 0$, and so Z is not critical. $\square$

So why use our more complicated height function? The answer lies in formula (4.2), which combined with Kerckhoff's formula (2.1) shows that $H(\cdot) : \mathcal{Z}_p \to \mathbf{R}$ is not a proper function on $\mathcal{Z}_p$. Thus, the backwards gradient flow might flow to a reflexive symmetric zigzag, or it may flow to $\partial\overline{\mathcal{Z}}_p$. A better understanding of this height function H on $\mathcal{Z}$ would be interesting both in its own right and if it would lead to a new numerical algorithm for finding minimal surfaces experimentally.

§6. Extensions of the Method: The Karcher-Thayer Surfaces.

Thayer [T], following work of Karcher [K], defined Weierstraß data (depending upon unknown constants) for a family of surfaces $M_{p,k}$ of genus $p(k-1)$ with one Enneper-type end of winding order $2k-1$. In this notation, the surfaces of genus p described in Theorem A are written $M_{p,2}$. Karcher [K] has solved the period problem for the surfaces $M_{0,k}$ and $M_{1,k}$ for $k>2$ and Thayer has solved the period problem for $M_{2,k}$ for $k>2$. He has also found numerical evidence supporting the solvability of the period problem for $p\le 34$, $k\le 9$. Here we prove

Theorem C. *There exists a minimally immersed surface $M_{p,k}$ of genus $p(k-1)$ with one Enneper-type end with winding order $2k-1$.*

Proof. We argue in close analogy with our proof of Theorem A ($k=2$). Let $\mathcal{Z}_{p,k}$ denote the space of equivalence classes of zigzags with $2p+1$ finite vertices and angles at the vertices alternating between π/k and $\frac{2k-1}{k}\pi$. We double the complementary domains Ω^k_{NE} and Ω^k_{SW} of Z to obtain cone-metric spheres with cone angles of alternating $2\pi/k$ and $2\pi/k(2k-1)$. We then take a k-fold cover of those spheres, branched at the images of the vertices of the zigzag on the cone-metric spheres, to obtain Riemann surfaces $\mathcal{R}^k_{\mathrm{NE}}$ and $\mathcal{R}^k_{\mathrm{SW}}$. By pulling back the form dz from Ω^k_{NE} and Ω^k_{SW}, we obtain, as in Section 3, a pair of forms $\alpha=gdh$ and $\beta=g^{-1}dh$ on which we can base our Weierstraß representation. As before, we set $dh=d\pi$, where π is the branched covering map $\pi:\mathcal{R}^k_{\mathrm{NE}}\to\widehat{\mathbf{C}}$, so that dh is exact; as before, the periods of α and β are constructed to be conjugate, as soon as the zigzag is reflexive, and the induced metric (2.0) is regular at the lifts of the finite vertices of the zigzag.

To see that we can find a reflexive zigzag within $\mathcal{Z}_{p,k}$, we observe that by the remark at the end of section 4.4, the same real non-analyticity arguments of Section 4.4 carry over to the present case, once we replace the $\pi/2$ and $3\pi/2$ angles with π/k and $\pi/k(2k-1)$ angles. All of the rest of the arguments of Section 4 carry over without change and we conclude that the height function $D(Z)$ is proper on $\mathcal{Z}_{p,k}$.

For the gradient flow, we can write down deformations along the zigzag analogous to formula (5.1) (it is sufficient just to conjugate the maps in 5.1 by a map which shears the original zigzag with vertex angles $\pi/2$ and $3\pi/2$ to a zigzag with angles π/k and $\frac{2k-1}{k}\pi$) and then check that the proof of Lemma 5.3 continues to hold. The rest of the arguments in Section 5 carry over unchanged to the present case. Thus we find a reflexive symmetric zigzag in $\mathcal{Z}_{p,k}$ via the proof of Theorem B. The present theorem then follows. $\square$

Remark. Of course, the arguments in the last two paragraphs of the proof of Theorem C apply equally well to zigzags of arbitrary alternating angles θ and $2\pi-\theta$, so one might well ask why we do not generalize the statement of Theorem C even farther. The answer lies in that while the Teichmüller theory of sections 4 and 5 extends to zigzags with non-orthogonal angles, the discussion in section 3 of the transition from the zigzags to regular minimal surfaces only extends to the zigzags described in the proof of Theorem C. For instance, we of course require a finitely branched cover over a double of the zigzag in order to get a surface of finite genus, so we must restrict our attention at the outset to zigzags with rational angles. However, if the smaller angle should be of the form $\theta=\frac{r}{s}\pi$, we find that an s-sheeted cover of the double of a zigzag would be forced to have an induced metric (2.0) which was not regular at the lifts of the finite vertices.

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