COEND ELEMENTS OF A HOPF ALGEBRA
IN A BRAIDED RIGID MONOIDAL CATEGORY

BY

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DISSERTATION

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Abstract

Let H be a Hopf algebra in a braided rigid monoidal category $\mathcal{V}$ admitting a coend C. We define a "coend element" of H to be a morphism from C to H. We then study certain coend elements of H, which generalize important elements (e.g., pivotal and ribbon elements) of a finite dimensional Hopf algebra over a field. This builds on prior work of Bruguières and Virelizier [BV07; BV12] on elements of Hopf monads and R-matrices of braided Hopf algebras. As a consequence, we provide another description for pivotal and ribbon structures on the category $\mathcal{V}_H$ of H-modules.

To my parents.

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Chapter 1

Introduction

1.1 History

Hopf algebras

Hopf algebras first arose from the investigation of Γ-manifolds (now known as H-spaces) by Hopf in 1941 [Hop41]. The works of Drinfeld [Dri87] and Jimbo [Jim85] on quantum groups in the 1980s, motivated by the study of quantum integrable systems in mathematical physics, further increased interests in these algebraic objects. Initially, to serve the needs of algebraic topology, many restrictions on Hopf algebras were imposed, such as an $\mathbb{N}_0$-grading with finite-dimensional components, graded commutativity, etc. However, as the list of applications grew, Hopf algebras gradually became an independent topic of abstract algebra in its own right, and a clear and consistent set of axioms began to emerge. In brief, a Hopf algebra over a field $\Bbbk$ is a bialgebra H over $\Bbbk$ equipped with an antipode map $S : H \to H$. By a bialgebra, we mean a vector space H that is simultaneously an algebra and a coalgebra, such that these two structures are compatible in a certain sense. Examples of Hopf algebras include the group algebra $\Bbbk G$, the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, and the previously mentioned quantum groups, which are 1-parameter deformations of $U(\mathfrak{g})$ for semisimple complex Lie algebras $\mathfrak{g}$.

In terms of representation theory, Hopf algebras capture an important aspect of the theory of representations of groups and Lie algebras, in that we can form the tensor product of two representations. More precisely, given two modules M and N over a Hopf algebra H, the coproduct of H implements a module structure on the tensor product of M and N. Furthermore, the counit gives rise to a module structure on the trivial vector space $\Bbbk$, and the antipode of H imposes a module structure on the linear dual M^* of M. To keep things simple, we will restrict ourselves to consider only finite dimensional modules over a finite dimensional Hopf algebra H. In this setting, the category $\mathcal{M}_H^{\mathrm{fd}}$ of finite dimensional right H-modules forms a rigid monoidal category, and the forgetful functor $U : \mathcal{M}_H^{\mathrm{fd}} \to \mathsf{Vec}^{\mathrm{fd}}$ is a strict monoidal functor. Therefore, a finite dimensional Hopf algebra H gives rise to a rigid monoidal category $\mathcal{M}_H^{\mathrm{fd}}$ and a strict monoidal functor $\mathcal{M}_H^{\mathrm{fd}} \to \mathsf{Vec}^{\mathrm{fd}}$. In fact, a type of converse statement is true and is part of the broader theory of reconstruction, which we will now explain.

Tannaka-Krein reconstruction

In the 1930s and the 1940s, Tannaka [Tan39] and Krein [Kre49] proved that a compact topological group G can be reconstructed from the category $\Pi(G)$ of finite dimensional representations of G with the use of the forgetful functor $U : \Pi(G) \to \mathsf{Vec}^{\mathrm{fd}}$. The category $\Pi(G)$ is the nonabelian analog of the Pontryagin dual for locally compact abelian groups, and can be thought of as a "dual" of G. The idea of reconstructing an object from its category of representations became known as Tannaka-Krein reconstruction. A version of reconstruction for Hopf algebras is as follows: there is a one-to-one correspondence between (1) finite dimensional Hopf algebras H over a field $\Bbbk$, and (2) pairs $(\mathcal{V}, F)$ consisting of a finite tensor category $\mathcal{V}$, and an exact faithful monoidal functor $F : \mathcal{V} \to \mathsf{Vec}^{\mathrm{fd}}$ [Eti+16, Theorem 5.3.12]. The key to the reconstruction is the algebra isomorphism

$$H \cong \textsc{End}(U_H), h \mapsto (-) \cdot h$$

between H and the space of natural endomorphisms of the forgetful functor $U_H : \mathcal{M}_H^{\mathrm{fd}} \to \mathsf{Vec}^{\mathrm{fd}}$. Using this isomorphism, given a pair $(\mathcal{V}, F)$ as above, we reconstruct H as $\textsc{End}(F)$.

More generally, if H is possibly infinite dimensional, then given a pair $(\mathcal{V}, F)$ consisting of a tensor category $\mathcal{V}$ and an exact faithful functor $F : \mathcal{V} \to \mathsf{Vec}^{\mathrm{fd}}$, one can recover H from a universal object called the coend of F, and $\mathcal{V}$ is now identified with the category of H-comodules. When restricted to the finite-dimensional case, we can consider the dual to the coend of F, the end of F, for which we have the following isomorphisms

$$\mathrm{end}(F) := \int_{X \in \mathcal{V}} F(X)^* \otimes F(X) \cong \int_{X \in \mathcal{V}} \mathrm{End}(F(X)) \cong \textsc{End}(F).$$

Note that the first isomorphism is the linear isomorphism $\mathrm{Hom}(V, W) \cong W^* \otimes V$ and the second is the isomorphism $\int_{X \in \mathcal{V}} \mathrm{Hom}(F(X), G(X)) \cong \textsc{Nat}(F, G)$.

Modular tensor categories and modular Hopf algebras

In 1988, Witten [Wit88] introduced topological quantum field theories (TQFTs), and Atiyah [Ati88] formulated a set of mathematical axioms for TQFTs. Soon afterwards, two constructions of 3-dimensional TQFTs were found: the Reshetikhin-Turaev's surgery construction [RT91] whose input is a modular fusion category, and the Turaev-Viro's state sum construction [TV92] whose input is a spherical fusion category. In 1995, Turaev conjectured that the Turaev-Viro's TQFT derived from a spherical fusion category $\mathcal{C}$ is isomorphic to the Reshetikhin-Turaev's TQFT derived from the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$, which is a modular fusion category. For the purpose of TQFTs, it is therefore more important to consider modular fusion categories.

A *modular fusion category* is a monoidal category equipped with a braiding and a twist, having properties of semisimplicity, linearity, finiteness, and whose braiding is maximally nonsymmetric. We refer to a category with all of these structures and properties, except possibly semisimplicity, as a *modular tensor category*.

Besides its connection to (extended) 3-dimensional TQFTs and quantum invariants of 3-manifolds [Bar+15; Tur16], modular fusion categories also appear in conformal field theories [MS89], vertex operator algebras [Hua05], topological phases of matter and topological quantum computing [Wan10]. Recently, there have been increased interests in the non-semisimple setting, see for example [KL01; CGP13; De +22].

Given their importance, it is therefore desirable to produce constructions of modular fusion categories. In addition to the Drinfeld center of a spherical fusion category [Müg03], there are many general constructions, some of which are modularization [Bru00; Müg00], local modules [Par95; KO02]; see also non-semisimple generalizations in [Shi19; LW22a; LW22b].

Classically, concrete examples of modular fusion categories arise from quantum groups at roots of unity [BK01]; these quantum groups are examples of modular Hopf algebras as defined in [RT91]. From the perspective of Tannaka duality, Hopf algebras are useful in constructing modular categories because there is a correspondence between additional structures on the category $\mathcal{M}_H^{\mathrm{fd}}$ of finite dimensional representations of H, and elements of H. From this observation, R-matrices, ribbon and spherical elements of H are defined in [Dri87; RT90; BW99] and correspond to braidings, ribbon structures, and spherical structures on $\mathcal{M}_H^{\mathrm{fd}}$ respectively. In particular, when H is finite dimensional, the category $\mathcal{M}_H^{\mathrm{fd}}$ of finite dimensional right H-modules is modular if and only if H is equipped with an R-matrix R and a ribbon element t such that (H, R) is factorizable, see e.g. [Tak01]. For instance, the Drinfeld double $D(\Bbbk G)$ of the group algebra of a finite group G has a canonical R-matrix and a ribbon element which makes $\mathcal{M}_{D(\Bbbk G)}$ a modular tensor category [BK01, Section 3.2].

Structures or properties of $\mathcal{M}_H^{\mathrm{fd}}$	Special elements or properties of H
Braiding σ for $\mathcal{M}_H^{\mathrm{fd}}$	R-matrix $R \in H \otimes H$
Pivotal structure ϕ on $\mathcal{M}_H^{\mathrm{fd}}$	Pivotal element $p \in H$
Ribbon structure (twist) θ on $(\mathcal{M}_H^{\mathrm{fd}}, \sigma)$	Ribbon element (twist) t for (H, R)
Nondegeneracy of $(\mathcal{M}_H^{\mathrm{fd}}, \sigma)$	(H, R) is factorizable

Table 1.1: Classical correspondence between structures or properties of $\mathcal{M}_H^{\mathrm{fd}}$ and special elements or properties of H.

In the effort to find more examples of modular tensor categories, one could consider the category of modules over a Hopf algebra in an arbitrary braided finite tensor category $\mathcal{V}$, and generalize the right hand side of Table 1.1.

Hopf algebras in braided monoidal categories

In [Pen71], Penrose showed that morphisms in the symmetric monoidal category of vector spaces can be represented by string diagrams. These were later extended by the works of Joyal, Street, and others to arbitrary monoidal, rigid, braided, pivotal, ribbon and spherical categories, such that morphisms are invariant up to appropriate notions of isotopies, see [Sel11] for an overview. The graphical calculus of string diagrams enables us to replace lengthy algebraic calculations with more natural topological reasoning.

At the same time, algebras and coalgebras can also be defined in any monoidal categories, and bialgebras and Hopf algebras in any braided monoidal categories. We refer to bialgebras or Hopf algebras in a braided monoidal category as *braided bialgebras* and *braided Hopf algebras*, respectively. Since the notions of braided, pivotal and ribbon categories are categorical without references to any linear structures, this raises the following question:

Question 1.1. If H is a Hopf algebra in a braided rigid monoidal category $\mathcal{V}$, is there a way to define braidings, pivotal, ribbon, etc. structures of $\mathcal{V}_H$ in terms of "elements" of H, as in the case when $\mathcal{V} = \mathsf{Vec}^{\mathrm{fd}}$?

Since $\Bbbk$ is the monoidal unit in the monoidal category $\mathsf{Vec}^{\mathrm{fd}}$, and $\mathrm{Hom}(\Bbbk, H) \cong H \cong \mathrm{END}(U_H)$, one could try replacing defining pivotal and ribbon elements of H by morphisms $\mathbb{1} \to H$ for braided Hopf algebras H. For instance, Majid [Maj93] defined an R-matrix as a morphism $\mathbb{1} \to H \otimes H$ satisfying conditions directly generalizing the classical axioms of R-matrices defined in terms of elements. This notion of R-matrix gives rise to a braiding only on a certain subcategory of H-modules, and there is no correspondence between R-matrices and braidings as in the case of Hopf algebras over a field. In [BV12], a new notion of R-matrices is introduced, which recovers this correspondence and is explained below.

Elements of Hopf monads

In [BV07], Bruguières and Virelizier develop the algebraic theory of Hopf monads, partly in order to prove Turaev's conjecture relating the two constructions of TQFTs. Building on the work of Moerdijk [Moe02] on bimonads, they define Hopf monads as monads equipped with additional structures, in the same way that Hopf algebras are algebras with additional structures, so that the category of modules is rigid monoidal. In particular, for any Hopf monad on a rigid monoidal category $\mathcal{C}$, the category $\mathcal{C}_T$ of T-modules is rigid monoidal, and the forgetful functor $U_T : \mathcal{C}_T \to \mathcal{C}$ is strict monoidal. An example of a Hopf monad is $T_H = (-) \otimes H$ where H is a braided Hopf algebra.

Given a Hopf monad T on a rigid monoidal category $\mathcal{C}$, they also define an *element* of T as a natural transformation $1_{\mathcal{C}} \to T$. This terminology is justified by the fact that there is a bijection

$$\mathrm{N\small{AT}}(1_{\mathcal{C}}, T) \cong \mathrm{E\small{ND}}(U_T),$$

generalizing the bijection $\mathrm{Hom}(\Bbbk, H) \cong H \cong \mathrm{E\small{ND}}(U_H)$ when H is a Hopf $\Bbbk$-algebra. Using this bijection and similar correspondences, Bruguières and Virelizier introduce pivotal and ribbon elements of Hopf monads as certain natural transformations $X \to T(X)$, and R-matrices as certain natural transformations $X \otimes Y \to T(Y) \otimes T(X)$.

Quasitriangular braided Hopf algebras

Let $\mathcal{V}$ be a braided rigid monoidal category, and let H be a Hopf algebra in $\mathcal{V}$. In this case, we have a Hopf monad $T_H = (-) \otimes H$, and we can consider pivotal, ribbon elements, and R-matrices for T_H. For instance, an R-matrix for T_H is a natural transformation $X \otimes Y \to Y \otimes H \otimes X \otimes H$, subject to certain axioms. This is a generalization of the notion of R-matrix for $\Bbbk$-Hopf algebras, however, Bruguières and Virelizier show that there is a more direct generalization, provided that the category $\mathcal{V}$ admits a universal object, called the *coend* of $\mathcal{V}$.

The coend of a monoidal functor is an important object in the general theory of reconstruction of Hopf algebras as mentioned above. When the coend C of the identity functor on a braided rigid monoidal category $\mathcal{V}$ exists, we say that $\mathcal{V}$ admits a coend and call C the coend of $\mathcal{V}$. Explicitly, C is equipped with a universal natural transformation $\delta_X : X \to X \otimes C$, satisfying the following factorization property: for any object D equipped with a natural transformation $\alpha_X : X \to X \otimes D$, there is a unique morphism $f : C \to D$ such that $\alpha = (\mathrm{id} \otimes f)\delta$. As a consequence of the braiding on $\mathcal{V}$ and Fubini's theorem for coends, C has an extended factorization property: for any object D equipped with natural transformation

$$\alpha_{X_1,\dots,X_n} : X_1 \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n \otimes D,$$

there exists a unique map $f : C^{\otimes n} \to D$ such that α factors through δ, f and the braiding. Using these factorization properties, Bruguières and Virelizier in [BV12] give a new definition of the R-matrix for a braided Hopf algebra H in $\mathcal{V}$ as a morphism $C \otimes C \to H \otimes H$ satisfying certain axioms. Under this definition, they recover the 1-1 correspondence between R-matrices of H and braidings on $\mathcal{V}_H$. [Ref]

1.2 Summary of main results

Motivated by the definition of the R-matrix in [BV12], we define a particular notion of an "element" of a braided Hopf algebra H in $\mathcal{V}$ as follows.

Definition 1.2. Let H be a Hopf algebra in a braided rigid monoidal category $\mathcal{V}$ admitting a coend C. A *coend element* (or *C-element*) of H is a morphism $h : C \to H$.

For example, when $\mathcal{V} = \mathsf{Vec}^{\mathrm{fd}}$, a $\Bbbk$-element of H is an element of H in the usual sense. We can now state the main results of this paper. Note that $\mathcal{V}_H$ denotes the category of right H-modules in $\mathcal{V}$.

Theorem 1.3 (Theorems 4.16, 4.21)**.** *Let $\mathcal{V}$ be a braided rigid monoidal category admitting a coend C, and let H be a Hopf algebra in $\mathcal{V}$.*

(a) *Consider* C-pivotal elements *of H defined in* Definition 4.14*. There is a canonical bijection between the set* $\mathsf{CPiv}(H)$ *of C-pivotal elements of H and the set* $\mathsf{Piv}(\mathcal{V}_H)$ *of pivotal structures of $\mathcal{V}_H$.*

(b) *Assume that H is quasitriangular with an R-matrix $\mathfrak{R} : C \otimes C \to H \otimes H$, and consider* C-balanced (resp. ribbon) elements *of H defined in* Definition 4.18*.*

   (i) *There is a canonical bijection between the set* $\mathsf{CBal}(H)$ *of C-balanced elements of H and the set* $\mathsf{Bal}(\mathcal{V}_H)$ *of balanced structures of $\mathcal{V}_H$.*

   (ii) *The bijection in (i) further restricts to a bijection between the subset* $\mathsf{CRbn}(H)$ *of C-ribbon elements of H and the subset* $\mathsf{Rbn}(\mathcal{V}_H)$ *of ribbon structures of $\mathcal{V}_H$.*

Thus, we provide an answer to Question 1.1 when $\mathcal{V}$ admits a coend, and consequently obtain a generalization to Table 1.1 as shown in Table 1.2. We also describe a characterization of the nondegeneracy of $\mathcal{V}_H$, which is explained in Section 5.3.

Structures or properties of $\mathcal{V}_H$	Special elements or properties of H
Braiding $\sigma_{M,N} : M \otimes N \to N \otimes M$	R-matrix $\mathfrak{R} : C \otimes C \to H \otimes H$ [BV12]
Pivotal structure $\phi_M : M \to M^{\vee\vee}$	Pivotal element $\mathfrak{p} : C \to H$
Ribbon structure $\theta_M : M \to M$	Ribbon element $\mathfrak{t} : C \to H$
Nondegeneracy	$(H, \mathfrak{R})$ is factorizable

Table 1.2: Generalized correspondence between structures or properties of $\mathcal{V}_H$ and special C-elements or properties of H.

Addtionally, there is a relation between these generalized pivotal and ribbon elements via the Drinfeld element $\mathfrak{u}$ and two other elements $\mathfrak{q}_\mu$ and $\mathfrak{c}_\mu$ as seen in the following theorem, which generalizes well-known results over vector spaces, see e.g., [Dra01] or Section 2.4.

Theorem 1.4 (Theorems 4.24, 4.26)**.** *Let $\mathcal{V}$ be a braided rigid monoidal category admitting a coend C, and let $(H, \mathfrak{R})$ be a quasitriangular Hopf algebra in $\mathcal{V}$ as in [BV12].*

(a) *Consider the C-Drinfeld element $\mathfrak{u}$ as defined in* Definition 4.22*. This element induces a bijection $\mathsf{CBal}(H) \cong \mathsf{CPiv}(H)$ via $\mathfrak{t} \mapsto \mathfrak{t}\mathfrak{u}$.*

(b) *Under the above correspondence, a C-balanced element $\mathfrak{t}$ is ribbon if and only if one of the following two equivalent conditions is satisfied:*

  (i) $\mathfrak{t}^{-2} = \mathfrak{c}$,

  (ii) *the corresponding pivotal element $\mathfrak{p}$ satisfies $\mathfrak{p}^2 = \mathfrak{q}$,*

where $\mathfrak{q} = \mathfrak{q}_\mu$ and $\mathfrak{c} = \mathfrak{c}_\mu$ are certain C-elements defined in Definition 4.25*. The product on* $\mathrm{Hom}(C, H)$ *used in this result is defined in* Figure 4.1(b).

A summary of the relationships between the special elements is given in Figure 1.1.

As mentioned above, to obtain these results, we make use the notion of elements of Hopf monads in [BV07]. A braided Hopf algebra H gives rise to a Hopf monad $T_H = (-) \otimes H$, and therefore we can talk about elements of T_H, i.e. natural transformations $X \to X \otimes H$. We call these natural transformations *monadic elements* of the Hopf algebra H. Using the factorization property of the coend C, these monadic elements translate to morphisms $C \to H$, i.e. coend elements of H. In other words, the monadic elements of H provide an intermediate bridge between the different structures of $\mathcal{V}_H$, and the coend elements of H.

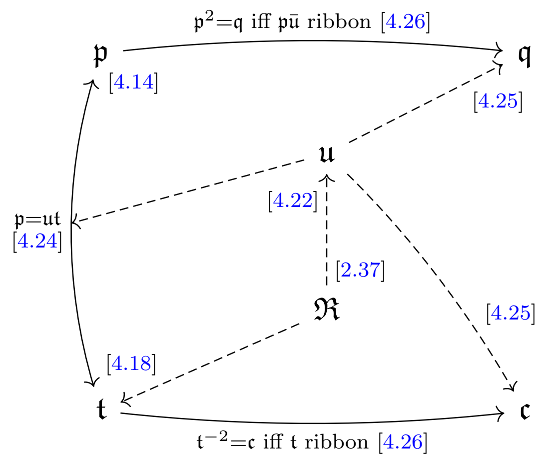


Figure 1.1: Summary of relationships of special coend elements.

There is, however, one small difference betweeen our approach to monadic elements and the one in [BV07]. A general Hopf monad can be defined on any (rigid) monoidal category, not necessarily one equipped with a braiding. As a result of this generality, the definitions of monadic pivotal elements require $\mathcal{V}$ to be a pivotal category. However, when the Hopf monad comes from a Hopf algebra, $\mathcal{V}$ is necessarily braided, and there is a canonical natural isomorphism similar to a pivotal structure, called the Drinfeld isomorphism. Unlike a pivotal structure, the Drinfeld isomorphism is not monoidal, which requires us to slightly modify certain definitions and statements of results compared to [BV07]. Overall, relaxing the pivotal assumption strictly increases the scope of our results, since there are braided tensor categories that are not pivotal, see for instance [Hal21].

Our proofs rely on utilizing the graphical calculus of braided rigid monoidal categories, and the extended factorization property of the coend [Lemma 2.23]. In order to make use of this property of the coend, we often need to transform the graphical calculus diagrams into a specific form, which on some occasions require the ability to untangle certain braidings. For this purpose, we highlight a useful technical result, Lemma 2.27, which allows us to treat a braided monoidal category with coend C almost as a symmetric monoidal category, up to the emergence of certain pairings of C with itself.

1.3 Structure and outline of the thesis

This thesis is organized as follows. In Chapter 2, we introduce the general background material needed for this paper, including monoidal categories, braided Hopf algebras, and coends. In Chapter 3, we summarize the main results of [BV07] concerning elements of a

Hopf monad T in the special context that $T = 1_{\mathcal{V}} \otimes H$ is the Hopf monad induced by a Hopf algebra in a braided rigid monoidal category $\mathcal{V}$. At the end of Chapter 3, we modify the theory slightly to relax the pivotal assumption on $\mathcal{V}$, which is necessary for arbitrary Hopf monads but not for Hopf monads induced by Hopf algebras. This leads us to Chapter 4, where we further assume that $\mathcal{V}$ admits a coend C. Using graphical calculus and the universal property of the coend C, we display the C-element versions of special elements of H in this setting, and prove their generalized properties and relations. Finally, we give some concluding remarks, including a few applications of our work in Chapter 5.

Chapter 2

Preliminaries on monoidal categories, braided Hopf algebras, and coends

The goal of this section is to introduce the general background material, including basic definitions and results, as well as notations and conventions that will be used throughout this paper. In Section 2.1, we define the setting where all of our objects live, which are monoidal categories endowed with various additional structures. In Section 2.2, we introduce the first main object, that of a braided Hopf algebra. We also discuss its generalization in the form of a Hopf monad on a rigid monoidal category. In Section 2.3, we introduce the second main object: a certain universal object called a coend C of a rigid monoidal category. Next, we briefly discuss all classical results about special elements of a Hopf algebra over a field $\Bbbk$ in Section 2.4, which form the basis of our work. Finally, in Section 2.5, we introduce the first result in this direction that forms the inspiration of this work: a generalization of the R-matrix obtained by Bruguières and Virelizier in [BV12].

2.1 Monoidal categories

We review some general facts about monoidal categories, which will be used extensively throughout.

2.1.1 Conventions for categories

We assume that the reader is familiar with standard category concepts as presented in [Mac98; Rie16]. Unless otherwise specified, all categories are small.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The category of functors and natural transformations from $\mathcal{C}$ to $\mathcal{D}$ is denoted by $[\mathcal{C}, \mathcal{D}]$. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, the set of natural transformations from F to G is denoted by $\textsc{Nat}(F,G)$ and the monoid of natural endomorphisms of F is denoted by $\textsc{End}(F)$. Given a category $\mathcal{C}$, the opposite category of $\mathcal{C}$ with all morphisms reversed is denoted by $\mathcal{C}^{\mathrm{op}}$.

2.1.2 Monoidal categories

For a thorough introduction to monoidal categories and graphical calculus, the reader can refer to [TV17], [Eti+16], and [Sel11]. On account of MacLane's coherence theorem, we assume that all monoidal categories are strict. Given a monoidal category $(\mathcal{C},\otimes,\mathbb{1})$, the category $\mathcal{C}^{\mathrm{op}} = (\mathcal{C}^{\mathrm{op}},\otimes,\mathbb{1})$ is also monoidal. We also have a monoidal category $\mathcal{C}^{\otimes\mathrm{op}} = (\mathcal{C},\otimes^{\mathrm{op}},\mathbb{1})$ where $\mathcal{C}$ is equipped with the reversed tensor product $X \otimes^{\mathrm{op}} Y = Y \otimes X$ for all $X,Y \in \mathcal{C}$.

A *monoidal functor* between monoidal categories $(\mathcal{C},\otimes,\mathbb{1})$ and $(\mathcal{D},\underline{\otimes},\underline{\mathbb{1}})$ is a triple $(F,F^{(2)},F^{(0)})$, where $F : \mathcal{C} \to \mathcal{D}$ is a functor, $F^{(2)}$ is a natural transformation

$$F^{(2)}_{X,Y} : F(X) \,\underline{\otimes}\, F(Y) \to F(X \otimes Y),$$

for all $X,Y \in \mathcal{C}$, and

$$F^{(0)} : \underline{\mathbb{1}} \to F(\mathbb{1})$$

is a distinguished morphism, satisfying coherence axioms. A monoidal functor is *strong* if $F^{(2)}$ and $F^{(0)}$ are isomorphisms. A *comonoidal functor* is a monoidal functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

A *monoidal natural transformation* between two monoidal functors $F,G : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\eta : F \to G$ such that

$$\begin{aligned} \eta_{X\otimes Y}\, F^{(2)}_{X,Y} &= G^{(2)}_{X,Y}\,(\eta_X \otimes \eta_Y), \qquad \text{for all } X,Y \in \mathcal{C}, \\ \eta_{\mathbb{1}}\, F^{(0)} &= G^{(0)}. \end{aligned}$$

The set of monoidal natural transformations between two monoidal functors F and G is denoted by $\textsc{Nat}_\otimes(F,G)$, and the set of monoidal natural endomorphisms of a monoidal functor F is denoted by $\textsc{End}_\otimes(F)$.

2.1.3 Rigidity

Let $\mathcal{C}$ be a monoidal category. A *duality pairing* in $\mathcal{C}$ is a quadruple (X,Y,e,c) consisting of objects X,Y of $\mathcal{C}$ and morphisms $e : X \otimes Y \to \mathbb{1}$ and $c : \mathbb{1} \to Y \otimes X$, such that the two compositions

$$X \xrightarrow{X\otimes c} X \otimes Y \otimes X \xrightarrow{e\otimes X} X, \quad Y \xrightarrow{c\otimes Y} Y \otimes X \otimes Y \xrightarrow{Y\otimes e} Y$$

are both identity morphisms. In this case, X is called a *left dual* of Y, Y a *right dual* of X, and e and c are the *evaluation* and *coevaluation* maps respectively. A monoidal category $\mathcal{C}$ is *rigid* if every object $X \in \mathcal{C}$ is part of distinguished duality pairings $({}^\vee X, X, \mathrm{ev}^l_X, \mathrm{coev}^l_X)$ and $(X, X^\vee, \mathrm{ev}^r_X, \mathrm{coev}^r_X)$. The objects ${}^\vee X$ and $X^\vee$ are called the *left dual* and the *right dual* of X, respectively, and ev^l_X, coev^l_X (resp. ev^r_X, coev^r_X) are called the left (resp. right) evaluation and coevaluation maps.

It is well-known that there are unique isomorphisms between any two left (or right) duals of an object X respecting the evaluation and coevaluation maps. In particular, we will abstain from writing the following canonical isomorphisms

$$\begin{aligned} ({}^\vee X)^\vee \cong X \cong {}^\vee(X^\vee), \quad {}^\vee\mathbb{1} \cong \mathbb{1} \cong \mathbb{1}^\vee, \\ {}^\vee(X \otimes Y) \cong {}^\vee Y \otimes {}^\vee X, \quad (X \otimes Y)^\vee \cong Y^\vee \otimes X^\vee. \end{aligned} \tag{2.1}$$

We follow the top-to-bottom convention for the graphical calculus of monoidal categories. For any $X \in \mathcal{C}$, the left and right evaluation and coevaluation maps are depicted as in Figure 2.1.

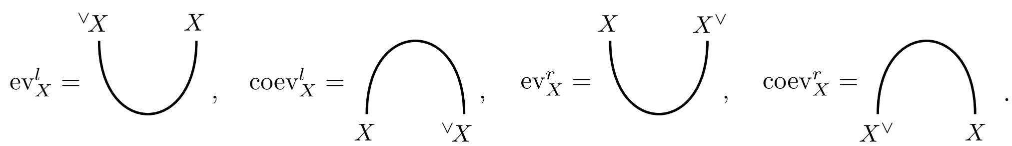


Figure 2.1: Left and right evaluation and coevaluation maps of $X \in \mathcal{C}$.

Furthermore, for any morphism $f : X \to Y$ we define ${}^\vee f$ and $f^\vee$ as in Figure 2.2, such that $(-)^\vee$ and ${}^\vee(-)$ are strong monoidal functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\otimes\mathrm{op}}$.

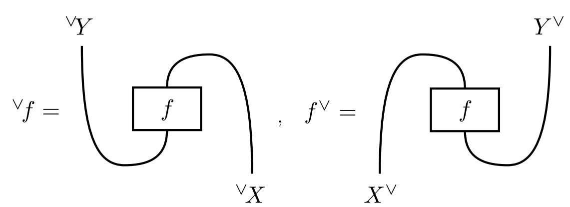


Figure 2.2: Left and right dual of a morphism $f : X \to Y$.

2.1.4 Conjugation by duality functors

Let $\mathcal{C}, \mathcal{D}$ be rigid monoidal categories. For any functor $F : \mathcal{C} \to \mathcal{D}$, define $F^! : \mathcal{C} \to \mathcal{D}$ by

$$F^!(X) = {}^\vee F(X^\vee), \qquad F^!(f) = {}^\vee F(f^\vee).$$

For any natural transformation $\alpha : F \to G$, where F, G are functors $\mathcal{C} \to \mathcal{D}$, define

$$\alpha^!_X = {}^\vee\alpha_{X^\vee} : G^!(X) \to F^!(X).$$

This defines a functor

$$(-)^! : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]^{\mathrm{op}}, \qquad F \mapsto F^!, \quad \alpha \mapsto \alpha^!,$$

which we call the *conjugation by duality functor*. Similarly, one can define a functor ${}^!(-) : [\mathcal{C}, \mathcal{D}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{D}]$ which is quasi-inverse to $(-)^!$. Furthermore, for composable functors F and G between rigid monoidal categories, we have $(G \circ F)^! = G^! \circ F^!$, and for composable natural transformations α and β between functors between rigid monoidal categories, we have $(\beta \circ \alpha)^! = \beta^! \circ \alpha^!$.

Remark 2.2. It follows that for any endofunctor F on a rigid monoidal category, the conjugation by duality functor $(-)^!$ is an antiautomorphism of $\mathrm{End}(F)$ with inverse ${}^!(-)$.

Lemma 2.3 ([BV07, Lemma 3.4])**.** *Let $F, G : \mathcal{C} \to \mathcal{D}$ be strong monoidal functors and let $\alpha : F \to G$ be a monoidal natural transformation. If $\mathcal{C}$ is rigid, then α is an isomorphism with $\alpha^{-1} = \alpha^! = {}^!\alpha$.* □

2.1.5 Pivotal structures

A rigid monoidal category $\mathcal{C}$ is *pivotal* if it is equipped with a *pivotal structure*, i.e., a monoidal natural transformation $\phi_X : X \to X^{\vee\vee}$ such that

$$\begin{aligned} \phi_{X \otimes Y} &= \phi_X \otimes \phi_Y, \qquad \text{for all } X, Y \in \mathcal{C}, \\ \phi_{\mathbb{1}} &= \mathrm{id}_{\mathbb{1}}, \end{aligned}$$

up to the identifications (2.1). The set of pivotal structures of $\mathcal{C}$ is denoted by $\mathsf{Piv}(\mathcal{C})$.

Remark 2.4. By Lemma 2.3, any pivotal structure ϕ on a rigid monoidal category $\mathcal{C}$ is invertible with $\phi_X^{-1} = {}^\vee\phi_{X^\vee} = \phi^\vee_{{}^\vee X}$ for all $X \in \mathcal{C}$. In particular, $\phi^{!!} = \phi$. The invertibility of ϕ also implies that $\phi_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ if we identify $\mathbb{1} = \mathbb{1}^{\vee\vee}$, since $\phi_{\mathbb{1}}^{\otimes 2} = \phi_{\mathbb{1}}$.

2.1.6 Braided categories

Let $\mathcal{C}$ be a monoidal category. A *lax braiding* on $\mathcal{C}$ is a natural transformation $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying

$$\begin{aligned} \sigma_{X,Y\otimes Z} &= (\mathrm{id}_Y \otimes \sigma_{X,Z})(\sigma_{X,Y} \otimes \mathrm{id}_Z), \\ \sigma_{X\otimes Y,Z} &= (\sigma_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \sigma_{Y,Z}), \\ \sigma_{X,\mathbb{1}} &= \mathrm{id}_X = \sigma_{\mathbb{1},X}, \end{aligned}$$

for all $X, Y, Z \in \mathcal{C}$. A *braiding* is a lax braiding that is also invertible. A monoidal category $\mathcal{C}$ is *braided* if it is equipped with a braiding.

If $\mathcal{C} = (\mathcal{C}, \sigma)$ is a braided monoidal category, then the reverse braiding $\bar{\sigma}_{X,Y} = \sigma^{-1}_{Y,X}$ is also a braiding on $\mathcal{C}$. Graphically, the braiding σ and its reverse $\bar{\sigma}$ are represented in Figure 2.3(a).

Remark 2.5. When $\mathcal{C}$ is rigid, every lax braiding σ on $\mathcal{C}$ is invertible, and therefore also a braiding, with $\sigma^{-1}_{X,Y}$ given as in Figure 2.3(b). Furthermore, the invertibility of σ readily implies that $\sigma_{X,\mathbb{1}} = \mathrm{id}_X = \sigma_{\mathbb{1},X}$ for all $X \in \mathcal{C}$.

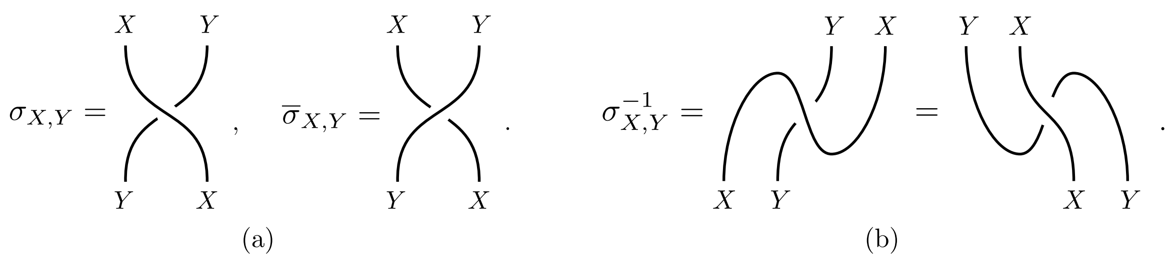


Figure 2.3: Graphical calculus of a braiding on a (rigid) monoidal category.

The Drinfeld center of a monoidal category

Let $\mathcal{C}$ be a monoidal category. The *Drinfeld center* of $\mathcal{C}$, denoted $\mathcal{Z}(\mathcal{C})$, is a braided monoidal category defined as follows. An object of $\mathcal{Z}(\mathcal{C})$ is a pair (M, σ^M), where M is an object of $\mathcal{C}$ and σ^M is a natural isomorphism $\{\sigma^M_X : M \otimes X \to X \otimes M\}_{X\in\mathcal{C}}$, called a *half-braiding*, such that

$$\sigma^M_{X\otimes Y} = (\mathrm{id}_X \otimes \sigma^M_Y)(\sigma^M_X \otimes \mathrm{id}_Y),$$

for all $Y \in \mathcal{C}$. A morphism between (M, σ^M) and (N, σ^N) is a morphism $f : M \to N$ in $\mathcal{C}$ that commutes with the half-braidings, i.e. f satisfies

$$(\mathrm{id}_X \otimes f)\, \sigma^M_X = \sigma^N_X (f \otimes \mathrm{id}_X)$$

for all $X \in \mathcal{C}$. The category $\mathcal{Z}(\mathcal{C})$ has a canonical braiding, which is given for any two objects (M, σ^M) and (N, σ^N) by σ^M_N. If $\mathcal{C}$ is rigid or pivotal, so is $\mathcal{Z}(\mathcal{C})$, see [TV17, Section 5.2.1].

2.1.7 Ribbon categories

A *balanced category* is a braided monoidal category $(\mathcal{C}, \sigma)$ equipped with a *balanced structure*, which is defined as a natural endomorphism $\theta \in \mathrm{E}\textsc{nd}(1_{\mathcal{C}})$ satisfying

$$\begin{aligned} \theta_{X\otimes Y} &= (\theta_X \otimes \theta_Y)\, \sigma_{Y,X}\, \sigma_{X,Y}, \qquad \text{for all } X, Y \in \mathcal{C}, \\ \theta_{\mathbb{1}} &= \mathrm{id}_{\mathbb{1}}. \end{aligned}$$

A balanced structure on $(\mathcal{C}, \sigma)$ is sometimes also called a *twist*. If $\mathcal{C}$ is furthermore rigid, then the twist θ is said to be *self-dual* if it satisfies

$$\theta = \theta^!.$$

In this case, we say that θ is a *ribbon structure* and $\mathcal{C}$ is a *ribbon category*. The set of balanced (resp. ribbon) structures on $\mathcal{C}$ will be denoted by $\mathsf{Bal}(\mathcal{C})$ (resp. $\mathsf{Rbn}(\mathcal{C})$).

2.1.8 Drinfeld morphisms

In this section, we fix a braided rigid monoidal category $(\mathcal{C}, \sigma)$. Using graphical calculus, the results in this section are straightforward to verify; see also [HPT16, Appendix A.2] for similar results on balanced and pivotal structures of $\mathcal{C}$.

Definition 2.6. Define natural transformations $\nu, \bar{\nu}, \nu^!, \bar{\nu}^!$ as in Figure 2.4. We refer to these morphisms collectively as *Drinfeld morphisms*, with ν called the (*right*) *Drinfeld morphism*.

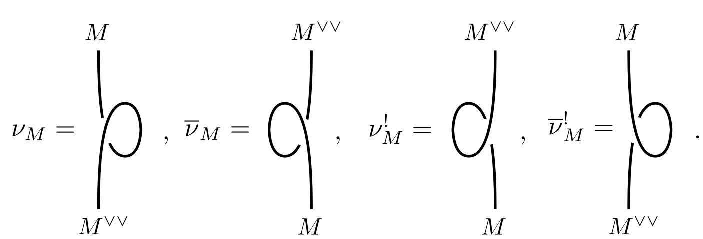


Figure 2.4: Drinfeld morphisms in a braided rigid monoidal category.

Lemma 2.7. *The morphism ν is invertible with inverse $\nu^{-1} = \bar{\nu}$. Moreover, there are relations*

$$\begin{aligned}
\nu_{M\otimes N} &= (\nu_M \otimes \nu_N)\,\bar{\sigma}_{N,M}\,\bar{\sigma}_{M,N},\\
\bar{\nu}^!_{M\otimes N} &= (\bar{\nu}^!_M \otimes \bar{\nu}^!_N)\,\sigma_{N,M}\,\sigma_{M,N},\\
\nu^!_{M\otimes N} &= \bar{\sigma}_{N,M}\,\bar{\sigma}_{M,N}(\nu^!_M \otimes \nu^!_N),\\
\bar{\nu}_{M\otimes N} &= \sigma_{N,M}\,\sigma_{M,N}(\bar{\nu}_M \otimes \bar{\nu}_N).
\end{aligned} \tag{2.8}$$

for all $M, N \in \mathcal{C}$. The morphism $\nu^!$ is the conjugation of ν by duality functors, with inverse $\bar{\nu}^!$, such that $\nu^{!!} = \nu$. $\square$

Lemma 2.9. *The Drinfeld morphism ν induces a bijection*

$$\mathsf{Bal}(\mathcal{C}) \cong \mathsf{Piv}(\mathcal{C}), \quad \theta \mapsto \nu \circ \theta \tag{2.10}$$

between balanced and pivotal structures of $\mathcal{C}$. $\square$

Remark 2.11. Since any pivotal structure ϕ is invertible with $\phi^{-1} = \phi^!$, and the Drinfeld isomorphism ν is invertible, it follows that any balanced/ribbon structure θ on a rigid monoidal category is invertible as well: if $\theta = \bar{\nu}\phi$ then $\theta^{-1} = \phi^{-1}\nu$. Furthermore, in this case the condition $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ is automatic, since $\theta_{\mathbb{1}}^{\otimes 2} = \theta_{\mathbb{1}}$.

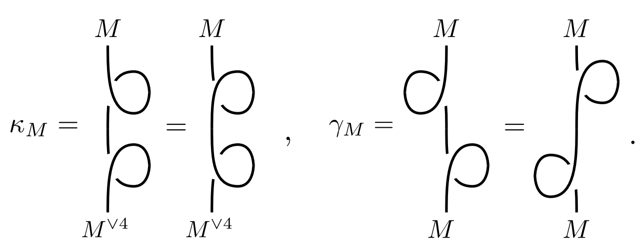


Figure 2.5: The morphisms κ and γ.

Definition 2.12. Define two natural transformations κ and γ by

$$\kappa_M = \nu_{M^{\vee\vee}}\bar{\nu}^!_M = \bar{\nu}^!_{M^{\vee\vee}}\nu_M, \qquad \gamma_M = (\nu\nu^!)_{^{\vee\vee}M} = (\nu^!\nu)_M,$$

graphically displayed in Figure 2.5. The equalities follow from the fact that the Drinfeld morphisms satisfy $(-)^{!!} = \mathrm{id}$, see Lemma 2.7.

Lemma 2.13. *Let $\mathcal{C}$ be a braided rigid monoidal category, and let κ and γ be defined as in* Figure 2.5*. Under the correspondence between pivotal and balanced structures* (2.10)*, a* *pivotal structure ϕ corresponds to a ribbon structure θ if and only if one of the following equivalent conditions hold:*

(i) $\phi^2 = \kappa$.

(ii) $\theta^{-2} = \gamma$. □

Definition 2.14. Following Drabant in [Dra01], we call a pivotal structure ϕ such that the corresponding balanced structure θ is ribbon a *strong pivotal structure*. The set of ribbon pivotal structures of $\mathcal{C}$ is denoted by $\mathsf{SPiv}(\mathcal{C})$ and we have a commutative diagram

$$\begin{tikzcd} \mathsf{SPiv}(\mathcal{C}) \arrow[r, hook] \arrow[d, "\cong"] & \mathsf{Piv}(\mathcal{C}) \arrow[d, "\cong"] \\ \mathsf{Rbn}(\mathcal{C}) \arrow[r, hook] & \mathsf{Bal}(\mathcal{C}) \end{tikzcd}$$

where the vertical arrows are the bijection $\nu \circ (-)$ in Lemma 2.9.

2.1.9 Symmetric categories

Let $\mathcal{C}$ be a braided monoidal category with braiding σ. The *symmetric center* $\mathcal{C}'$ of $\mathcal{C}$ is defined as the full subcategory consisting of objects X such that

$$\sigma_{Y,X}\, \sigma_{X,Y} = \mathrm{id}_{X \otimes Y}$$

for all $Y \in \mathcal{C}$. A *symmetric category* is a braided monoidal category $\mathcal{C}$ such that $\mathcal{C}' = \mathcal{C}$.

A symmetric category $\mathcal{C}$ has a canonical ribbon structure $\theta = 1_{\mathcal{C}}$, hence a canonical pivotal structure which is given by the Drinfeld morphism ν. A braided rigid monoidal category $\mathcal{C}$ admitting a coend C is symmetric if and only if the canonical Hopf pairing ω of C is equal to $\epsilon \otimes \epsilon$, see Section 2.3 for the definition of C and ω. This is the case for instance when $C = \mathbb{1}$ [BV12, Remark 8.1].

2.1.10 Modular tensor categories

Let $\Bbbk$ be a field. A $\Bbbk$-*linear abelian category* is an abelian category with a compatible enrichment over the category Vec of vector spaces. A $\Bbbk$-linear abelian category $\mathcal{A}$ is *finite* if $\mathcal{A}$ is equivalent to the category $\mathcal{M}_A^{\mathrm{fd}}$ of finite dimensional modules over some finite dimensional $\Bbbk$-algebra A; see [Eti+16, Definition 1.8.6] for an equivalent definition.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\Bbbk$-linear abelian categories. The *Deligne tensor product* of $\mathcal{A}$ and $\mathcal{B}$ is a $\Bbbk$-linear abelian category $\mathcal{A} \boxtimes \mathcal{B}$ equipped with a functor $\boxtimes : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \boxtimes \mathcal{B}$ that is $\Bbbk$-linear and right exact in each variable and is universal among such functors out of $\mathcal{A} \times \mathcal{B}$. When $\mathcal{A} \simeq \mathcal{M}_A^{\mathrm{fd}}$ and $\mathcal{B} \simeq \mathcal{M}_B^{\mathrm{fd}}$ are finite, their Deligne tensor product exists and is given by $\mathcal{M}^{\mathrm{fd}}_{A \otimes_{\Bbbk} B}$, see [Eti+16, Proposition 1.11.2] for a slightly more general result.

A *finite tensor category* (over $\Bbbk$) is a rigid monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ such that (1) $\mathcal{C}$ is a finite linear category over $\Bbbk$, (2) $\otimes$ is $\Bbbk$-bilinear, and (3) $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$. A *fusion category* is a semisimple finite tensor category.

Theorem 2.15 ([Shi19, Theorem 1.1])**.** *For a braided finite tensor category $(\mathcal{C}, \sigma)$ over an algebraically closed field, the following assertions are equivalent:*

(i) *The symmetric center of $\mathcal{C}$ is trivial.*

(ii) *The functor $\mathcal{C} \boxtimes \mathcal{C} \xrightarrow{T_+ \boxtimes T_-} \mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C}) \xrightarrow{\otimes} \mathcal{Z}(\mathcal{C})$ is an equivalence, where $T_+(X) = (X, \sigma_{X,?})$ and $T_-(X) = (X, \sigma^{-1}_{?,X})$.*

(iii) *The canonical Hopf pairing ω of the coend C of $\mathcal{C}$ is nondegenerate, see* Definition 2.16 *and* Lemma 2.23*.*

(iv) *The linear map* $\mathrm{Hom}(C, \mathbb{1}) \to \mathrm{Hom}(\mathbb{1}, C)$*, $f \mapsto (f \otimes \mathrm{id})\,\omega$ is bijective.*

If one of the equivalent conditions of Theorem 2.15 is satisfied for a braided finite tensor category $\mathcal{C}$, we say that $\mathcal{C}$ is *nondegenerate*. A *modular tensor category* is a nondegenerate braided finite tensor category equipped with a ribbon structure. In literature, a modular tensor category is also often assumed to be semisimple, i.e. a nondegenerate braided ribbon fusion category.

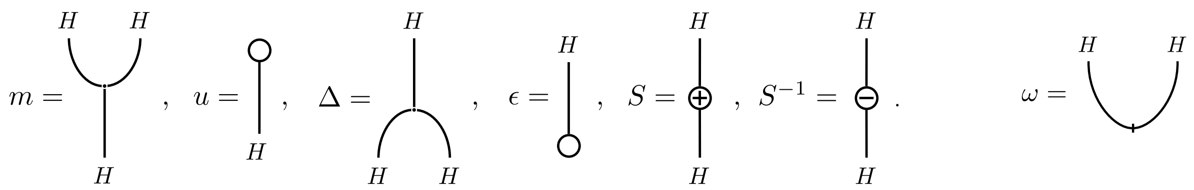
In this paper we will not assume that categories are linear or abelian, unless explicitly mentioned otherwise.

2.2 Braided Hopf algebras and Hopf monads

2.2.1 Braided Hopf algebras

For the definitions of algebras, coalgebras, bialgebras and Hopf algebras in a braided monoidal category $\mathcal{V}$, see [TV17, Chapter 6]. For a Hopf algebra H, we represent its product m, unit u, coproduct Δ, counit ϵ, antipode S and its inverse S^{-1} graphically as in Figure 2.6(a).

A right H-module is a pair (M, r) where $M \in \mathcal{V}$ and $r : M \otimes H \to M$ is a morphism in $\mathcal{V}$ satisfying the usual associativity and unital axioms. It is well-known that the category $\mathcal{V}_H$ of right H-modules is a rigid monoidal category; the H-action r, the monoidal structure and the dual actions are displayed in Figure 2.7.



(a) (b)

Figure 2.6: Graphical calculus for structure morphisms of H.

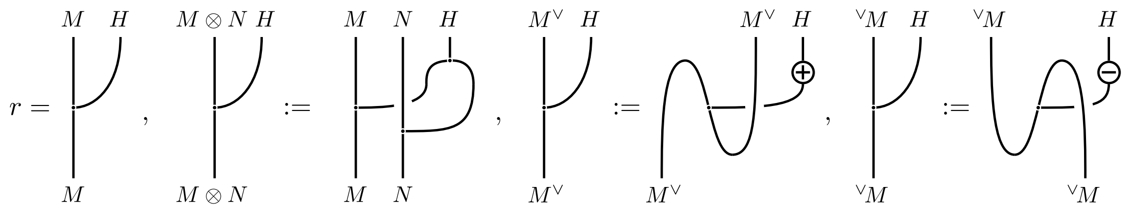


Figure 2.7: The monoidal structure of $\mathcal{V}_H$.

Definition 2.16 ([TV17, Section 6.2.3])**.** Let $H = (H, m, u, \Delta, \epsilon)$ be a bialgebra in a braided monoidal category $\mathcal{V}$. A *bialgebra pairing* for H is a morphism $\omega : H \otimes H \to \mathbb{1}$ in $\mathcal{V}$ such that

$$\omega(m \otimes \mathrm{id}_H) = \omega(\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_H)(\mathrm{id}_{H\otimes H} \otimes \Delta), \qquad \omega(u \otimes \mathrm{id}_H) = \epsilon,$$
$$\omega(\mathrm{id}_H \otimes m) = \omega(\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_H)(\Delta \otimes \mathrm{id}_{H\otimes H}), \qquad \omega(\mathrm{id}_H \otimes u) = \epsilon.$$

We depict a bialgebra pairing ω graphically as in Figure 2.6(b). Bialgebra pairings for Hopf algebras are called *Hopf pairings*.

A bialgebra pairing ω is *non-degenerate* if there exists a *copairing*, i.e., a morphism $\Omega : \mathbb{1} \to H \otimes H$ satisfying the snake equations

$$(\mathrm{id}_H \otimes \omega)(\Omega \otimes \mathrm{id}_H) = \mathrm{id}_H = (\mathrm{id}_H \otimes \Omega)(\omega \otimes \mathrm{id}_H).$$

Equivalently, a bialgebra pairing ω is a morphism whose left and right adjoint maps $H \to H^\vee$ and $H \to {}^\vee H$ are algebra homomorphisms, and it is nondegenerate if these maps are also isomorphisms.

2.2.2 Hopf monads

Let $\mathcal{C}$ be a category. A *monad* on $\mathcal{C}$ is an algebra (T, μ, η) in the strict monoidal category $(\mathrm{End}(\mathcal{C}), \circ, 1_{\mathcal{C}})$ of endofunctors of $\mathcal{C}$. Given a monad (T, μ, η) on $\mathcal{C}$, a *T-module* is a pair (M, r) consisting of an object $M \in \mathcal{C}$ and a morphism $r : T(M) \to M$ in $\mathcal{C}$ such that $r\,T(r) = r\,\mu_M$ and $r\,\eta_M = \mathrm{id}_M$. Given two T-modules (M, r) and (N, s), a *T-module morphism* $f : (M, r) \to (N, s)$ is a morphism $f : M \to N$ of underlying objects in $\mathcal{C}$ such that $fr = s\,T(f)$. The collection of T-modules and T-module morphisms forms a category, denoted by $\mathcal{C}_T$.

A monad (T, μ, η) on a monoidal category $\mathcal{C}$ is a *bimonad* if $T = (T, T^{(2)}, T^{(0)})$ is a comonoidal functor, and μ and η are comonoidal natural transformations. If $\mathcal{C}$ is a rigid monoidal category and T is a bimonad, we say that T is a *Hopf monad* if it is equipped with two natural transformations $s^l_X : T({}^\vee T(X)) \to {}^\vee X$ and $s^r_X : T(T(X)^\vee) \to X^\vee$ for $X \in \mathcal{C}$, satisfying [BV07, Equations (20)-(23)]. They are called the *left antipode* and the *right antipode* for T, respectively.

Remark 2.17. In [BLV11], Bruguières, Lack and Virelizier define Hopf monads on any monoidal categories, not necessarily rigid, using the notion of fusion operators. When restricting to rigid monoidal categories, the two notions of Hopf monads coincide.

Example 2.18. Let $\mathcal{V}$ be a braided rigid monoidal category and let H be a Hopf algebra in $\mathcal{V}$. The endofunctor $T = 1_{\mathcal{V}} \otimes H$ is a Hopf monad with structure morphisms presented in Figure 2.8 [BV12, Example 2.4]. Moreover, the category of T-modules coincides with the category of right H-modules in $\mathcal{V}$.

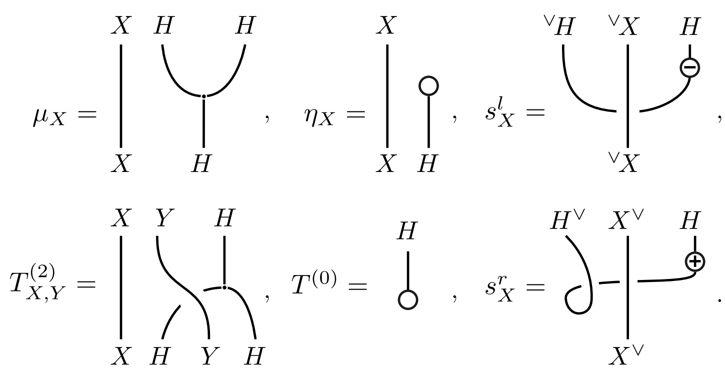


Figure 2.8: Structure morphisms of the Hopf monad $T = 1_{\mathcal{V}} \otimes H$.

2.3 Coends

The references for this section are [Mac98, Sections IX.5 - IX.8] and [TV17, Sections 6.4 - 6.6].

Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to D$ be a functor. For an object D in $\mathcal{D}$, a *dinatural transformation* from F to D is a family $d = \{d_X : F(X, X) \to D\}_{X \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(Y,X) \arrow[r, "F(f,\mathrm{id}_X)"] \arrow[d, "F(\mathrm{id}_Y,f)"'] & F(X,X) \arrow[d, "d_X"] \\ F(Y,Y) \arrow[r, "d_Y"'] & D \end{tikzcd}$$

commutes. Let $\textsc{Dinat}(F, D)$ denote the set of dinatural transformations from F to D.

A *coend* of a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a pair (C, ι) where C is an object of $\mathcal{D}$ and $\iota \in \textsc{Dinat}(F, C)$ such that every dinatural transformation d from F to an object D of $\mathcal{D}$ factors as $d = f \circ \iota$ for a unique morphism $f : C \to D$. In other words, we have a bijection

$$\mathrm{Hom}_{\mathcal{D}}(C, D) \to \textsc{Dinat}(F, D), \qquad f \mapsto f \circ \iota. \tag{2.19}$$

2.3.1 The coend of a rigid monoidal category

Let $\mathcal{V}$ be a rigid monoidal category, and let $F : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ denote the functor defined on objects by $(X, Y) \mapsto {}^{\vee}X \otimes Y$. We say that $\mathcal{V}$ *admits a coend* if a coend of the functor F exists. We often denote this coend by (C, ι) or simply C, with C an object of $\mathcal{V}$, and ι the dinatural transformation with X-component $\iota_X : {}^{\vee}X \otimes X \to C$ for $X \in \mathcal{V}$.

Using the snake identity, the proof of the following lemma is straightforward.

Lemma 2.20. *Let $\mathcal{V}$ be a rigid monoidal category, and let F be the functor defined as above. For any object $D \in \mathcal{V}$, there is a bijection*

$$\textsc{Dinat}(F, D) \to \textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes D), \quad \phi \mapsto \tilde{\phi}_X := (\mathrm{id}_X \otimes \phi_X)(\mathrm{coev}^l_X \otimes \mathrm{id}_X) \tag{2.21}$$

with inverse $\psi \mapsto \bar{\psi}_X = (\mathrm{ev}^l_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \psi_X)$. □

By using the above lemma, in particular by composing the two bijections (2.19) and (2.21), we see that equivalently, if (C, ι) is a coend of $\mathcal{V}$ and $\delta = \tilde{\iota}$, then for any $D \in \mathcal{V}$ we have a bijection

$$\Sigma^{(1)}_D : \mathrm{Hom}(C, D) \to \textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes D), \qquad f \mapsto (\mathrm{id}_X \otimes f)\, \delta_X. \tag{2.22}$$

The natural transformation $\delta_X : X \to X \otimes C$ is called the *universal coaction* associated to the coend C. Even though ι and δ carry the same data, working with δ is often slightly more convenient. In terms of graphical calculus, we color C-strands gray, with the universal coaction $\delta_X : X \to X \otimes C$ graphically depicted as in Figure 2.9.

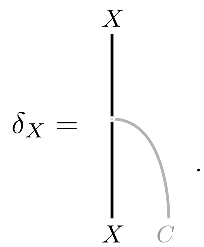


Figure 2.9: The universal coaction δ for the coend C.

2.3.2 Extended factorization property of the coend of a braided rigid monoidal category

In this section, we further assume that the rigid monoidal category $\mathcal{V}$ has a braiding σ.

Lemma 2.23. *Let $(\mathcal{V}, \sigma)$ be a braided rigid monoidal category that admits a coend (C, δ). For all $n \geq 1$, if there is an object D with a natural transformation*

$$\alpha = \{\alpha_{X_1,\dots,X_n} : X_1 \otimes \cdots \otimes X_n \to X_1 \otimes \cdots \otimes X_n \otimes D\}_{(X_1,\dots,X_n)\in\mathcal{V}^n}$$

then there exists a unique map $f : C^{\otimes n} \to D$ such that the following diagram

$$\begin{tikzcd}
X_1 \otimes \cdots \otimes X_n \arrow[r, "\alpha_{X_1,\dots,X_n}"] \arrow[d, "\delta_{X_1}\otimes\cdots\otimes\delta_{X_n}"'] & X_1 \otimes \cdots \otimes X_n \otimes D \\
X_1 \otimes C \otimes \cdots \otimes X_n \otimes C \arrow[r, "\beta"] & X_1 \otimes \cdots \otimes X_n \otimes C^{\otimes n} \arrow[u, "\mathrm{id}\otimes f"']
\end{tikzcd}$$

commutes. Here β is obtained by repeatedly applying the braiding σ of $\mathcal{V}$. In other words, for all $n \geq 1$ and all $D \in \mathcal{V}$, there is a bijection

$$\begin{aligned} \Sigma_D^{(n)} : \mathrm{Hom}(C^{\otimes n}, D) &\to \mathrm{N\scriptstyle{AT}}(\otimes^{(n-1)}, \otimes^{(n-1)} \otimes D) \\ f &\mapsto (\mathrm{id} \otimes f)\, \beta\, (\otimes^{(n-1)}\delta^{\times n}), \end{aligned} \tag{2.24}$$

where $\otimes^{(n-1)}$ is the functor $(X_1, \dots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n$.

Proof. The lemma follows from induction on n, the Fubini Theorem for coends [Mac98, Section IX.8], and the naturality of the braiding. See [BV12, Lemma 5.4] for the proof of a more general result. □

As a consequence of the (extended) factorization property of coends, we obtain:

Corollary 2.25 ([TV17, Sections 6.4, 6.5])**.** *Let $(\mathcal{V}, \sigma)$ be a rigid braided category that admits a coend C. Then C has a canonical Hopf algebra structure $(C, m, u, \Delta, \epsilon, S)$ and a canonical*

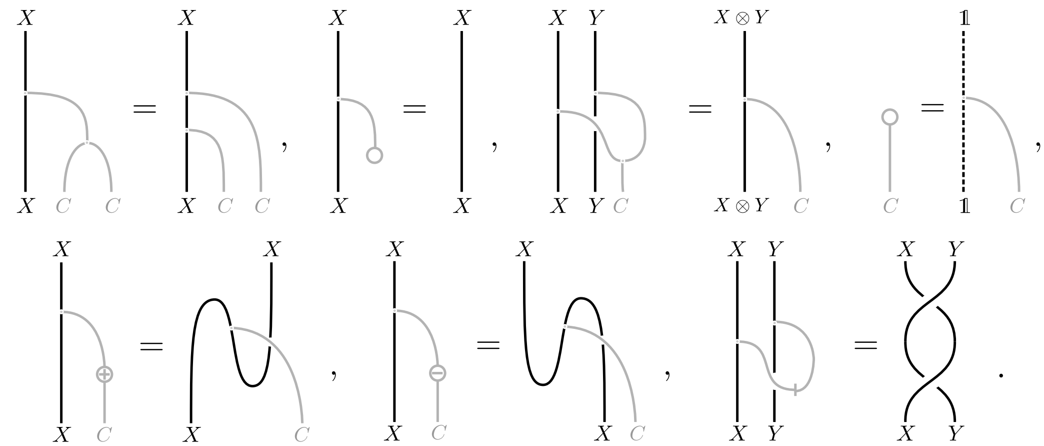
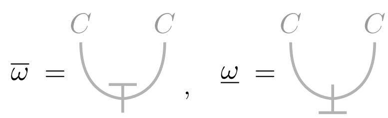


Figure 2.10: Defining properties of the canonical Hopf algebra structure on the coend C.

Hopf pairing $\omega : C \otimes C \to \mathbb{1}$ *compatible with the universal coaction* δ *and the braiding* σ *as illustrated in* Figure 2.10. $\square$

Remark 2.26 ([BV12, Remark 8.2])**.** The universal coaction of C on itself can be expressed in terms of its Hopf algebra structure by $\delta_C = (\mathrm{id} \otimes m)(\sigma \otimes \mathrm{id})(S \otimes \Delta)\Delta$.

We also define two variants $\overline{\omega}$ and $\underline{\omega}$ of ω, depicted graphically by



which will allow us to freely switch between the braiding σ and its reverse braiding $\overline{\sigma}$, as follows. The pairing $\underline{\omega}$ is the convolution inverse of ω in the sense that

$$(\omega \otimes \underline{\omega})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta \otimes \Delta) = \epsilon \otimes \epsilon = (\underline{\omega} \otimes \omega)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta \otimes \Delta).$$

The pairing $\overline{\omega}$ is the canonical Hopf pairing for C when $\mathcal{V}$ is braided with the reverse braiding $\overline{\sigma}$.

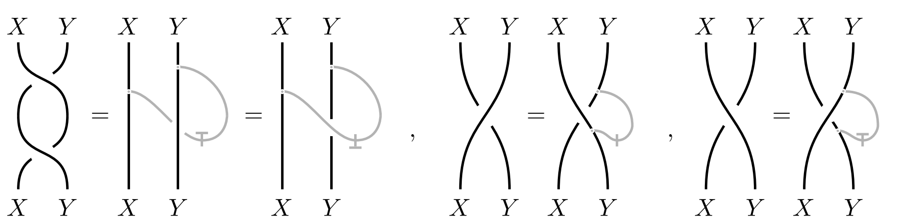


Figure 2.11: The variants $\overline{\omega}$ and $\underline{\omega}$ of ω.

Lemma 2.27. *Let* ω *be the canonical Hopf pairing of the coend* C*.*

(a) *The diagrams in* Figure 2.11 *hold.*

(b) $\overline{\omega} = \omega(S \otimes \mathrm{id}) = \omega(\mathrm{id} \otimes S^{-1})\overline{\sigma}, \quad \underline{\omega} = \omega(S^{-1} \otimes \mathrm{id}) = \omega(S \otimes \mathrm{id})\sigma.$

Proof. Part (a) is immediate from the definitions of $\overline{\omega}$ and $\underline{\omega}$. For part (b), the key is that the braiding σ can be expressed using σ^{-1} as shown in Figure 2.3(b). The proof for one of the formulas of $\underline{\omega}$ is shown in Figure 2.12, and the proofs of the other formulas are completely analogous. □

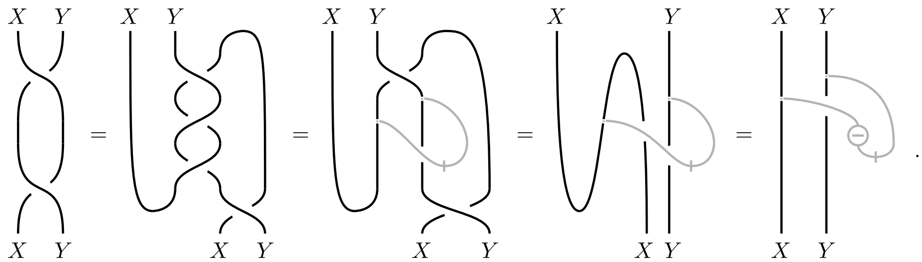


Figure 2.12: Proof that $\underline{\omega} = \omega(S^{-1} \otimes \mathrm{id})$.

Remark 2.28. For some graphical calculus diagrams, we rotate certain morphisms such as the Hopf pairing ω and the mulitplication map m_C following a C-coaction by 90 degrees counterclockwise, see for instance the first diagram in Figure 2.13 below.

2.3.3 Examples of coends

Example 2.29. Let $\mathcal{C}$ be a finite tensor $\Bbbk$-category, i.e., $\mathcal{C}$ is equivalent to the category $\mathcal{M}_A$ of finite dimensional modules over a finite dimensional $\Bbbk$-algebra A, such that the bifunctor $\otimes$ is bilinear on morphisms, and $\mathrm{End}(\mathbb{1}) \cong \Bbbk$. In this case, it is known that the coend of $\mathcal{C}$ exists, see [KL01, Section 5.1.3] or [Shi17a, Theorem 3.34]. If $\mathcal{C}$ is addtionally semisimple so that $\mathcal{C}$ is a fusion category, we can exhibit the coend as

$$C = \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} {}^{\vee}i \otimes i,$$

where i runs over representatives of simple objects of $\mathcal{C}$, see [TV17, Section 6.4.4]. When $\mathcal{C} = \mathsf{Vec}^{\mathrm{fd}}$, the coend is $\Bbbk$ with the universal dinatural transformation given by the evaluation maps.

Example 2.30 ([BV12, Section 6.3])**.** Let $\mathcal{V}$ be a braided rigid monoidal category admitting a coend C with universal coaction δ, and let H be a Hopf algebra in $\mathcal{V}$. Then the coend of $\mathcal{B} = \mathcal{V}_H$ exists and is given by $Z(H) = {}^{\vee}H \otimes C$. This is a right H-module with H-action $z : Z(H) \otimes H \to Z(H)$ and universal coaction $\tilde{\delta}$ given by Figure 2.13.

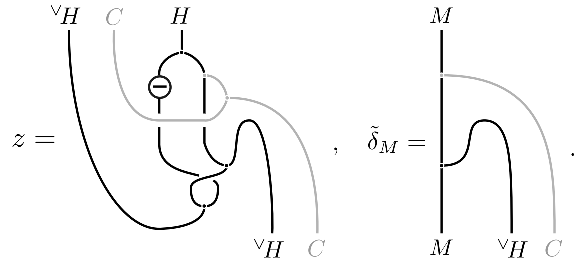


Figure 2.13: The H-action and the universal coaction for the coend of $\mathcal{V}_H$.

2.4 Special elements of Hopf algebras over a field

We summarize certain results on special elements of bialgebras and Hopf algebras over a field $\Bbbk$ that we generalize in this paper, see e.g. [Dra01; Rad12]. In this section, $\otimes = \otimes_\Bbbk$, and we denote the symmetric braiding in $\mathsf{Vec}^{\mathrm{fd}}$ by τ. We emphasize that since we work with right modules rather than left modules, certain definitions may appear different from the literature.

Definition 2.31. Let H be a finite dimensional bialgebra over $\Bbbk$.

(i) An *R-matrix* for H is an element $R = R_i \otimes R^i \in H \otimes H$ satisfying

- (R1) $R\Delta(h) = \tau_{H,H}\Delta(h)R$ for all $h \in H$.
- (R2) $(\mathrm{id} \otimes \Delta)(R) = R_i R_j \otimes R^j \otimes R^i$.
- (R3) $(\Delta \otimes \mathrm{id})(R) = R_i \otimes R_j \otimes R^i R^j$.
- (R4) $(\mathrm{id} \otimes \epsilon)(R) = (\epsilon \otimes \mathrm{id})(R) = 1$.

(ii) A *grouplike element* of H is an element $g \in H$ such that

- (G1) $\Delta(g) = g \otimes g$.
- (G2) $\epsilon(g) = 1$.

(iii) If H is furthermore a Hopf algebra, we define a *pivotal element* of H to be an element $p \in H$ such that

- (P1) p is a grouplike element, i.e., p satisfies (G1) and (G2).
- (P2) $S^2(h) = p^{-1}hp$ for all $h \in H$.

(iv) Let R be an R-matrix for H. A *balanced element* (or *twist*) of H with respect to R is an element $t \in H$ such that

- (T1) t is central: $th = ht$ for all $h \in H$.

(T2) $\epsilon(t) = 1$.

(T3) $\Delta(t) = (t \otimes t)(R_{21}R)$, where $R_{21} = \tau_{H,H}R$.

A balanced element is called a *ribbon element* if it further satisfies

(T4) $S(t) = t$.

These elements are defined from the perspective of Tannaka duality, motivated by certain additional structures on the category of modules, as explained in the following proposition.

Proposition 2.32. *Let H be a finite dimensional bialgebra over $\Bbbk$.*

(i) *There is a bijective correspondence between R-matrices for H and lax braidings on $\mathcal{M}_H^{\mathrm{fd}}$, given by $R \mapsto (-) \cdot R_i \otimes (-) \cdot R^i$, and $\sigma \mapsto \sigma_{H,H}(1_H \otimes 1_H)$.*

(ii) *If H is furthermore a Hopf algebra, then there is a bijective correspondence between pivotal elements of H and pivotal structures on $\mathcal{M}_H^{\mathrm{fd}}$, given by $p \mapsto (-) \cdot p$, and $\phi \mapsto \phi_H(1_H)$, where we have identified $M^{\vee\vee} \cong M$ as vector spaces for any H-module M.*

(iii) *Suppose R is an R-matrix for H. Then there is a bijective correspondence between balanced elements of H and balanced structures on $\mathcal{M}_H^{\mathrm{fd}}$, given by $t \mapsto (-) \cdot t$, and $\theta \mapsto \theta_H(1_H)$. Furthermore, a balanced element t is a ribbon element if and only if the corresponding balanced structure θ is ribbon.* □

Remark 2.33. Let H be a finite dimensional Hopf algebra over $\Bbbk$.

(a) By Remark 2.5 and Proposition 2.32(a), any R-matrix R for H is invertible, with $R^{-1} = (\mathrm{id} \otimes S^{-1})(R)$. As a result, the axiom (R4) is automatically satisfied. Furthermore, if σ is the braid on $\mathcal{M}_H^{\mathrm{fd}}$ corresponding to R, then the reverse braid $\overline{\sigma}$ is given by $\overline{R} = \tau_{H,H}R^{-1}$.

(b) Similarly, any grouplike element $g \in H$ is invertible with $g^{-1} = S(g)$. As a result, the axiom (G2) is automatically satisfied.

(c) If R is an R-matrix for H, then any twist for H with respect to R is invertible by Remark 2.11 and Proposition 2.32(c). As a result, the axiom (T2) is automatically satisfied.

Definition 2.34. Let H be a finite dimensional Hopf algebra over $\Bbbk$ with an R-matrix $R = R_i \otimes R^i$. We define $u = R_i S(R^i)$, $q = uS(u)^{-1}$, $c = uS(u)$, and call u the (*right*) *Drinfeld element*.

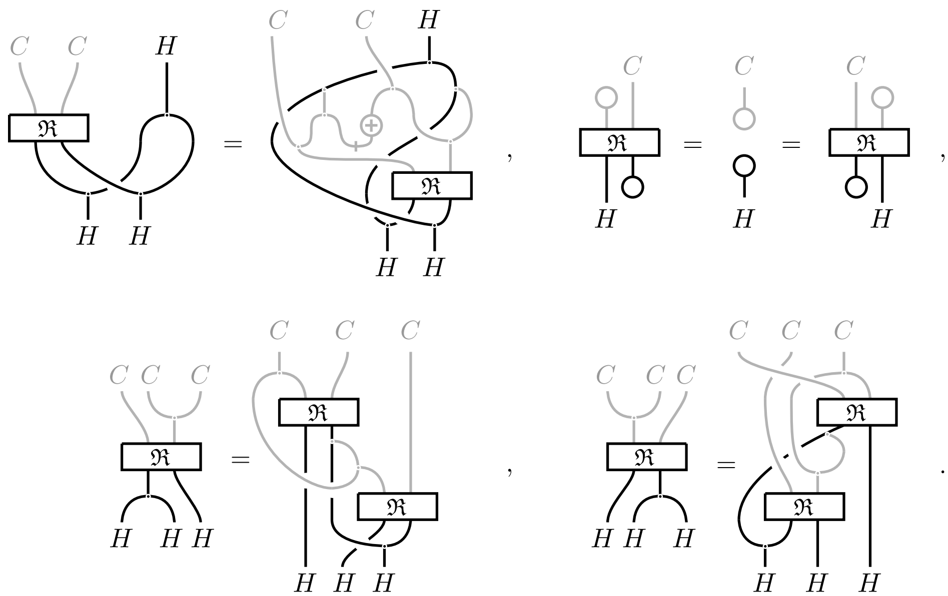


Figure 2.14: Axioms of an R-matrix for a braided bialgebra or Hopf algebra.

Remark 2.35. If H has an R-matrix R, the morphisms ν, κ and γ on the braided category $\mathcal{M}_H$ of finite dimensional H-modules (see Section 2.1.8) are given by the right actions of u, q and c, respectively, if we identify $M \cong M^{\vee\vee} \cong M^{\vee\vee\vee\vee}$ as vector spaces. As a result, the elements u, q, c are invertible, with u^{-1} corresponding to $\bar{\nu}$, and therefore given by $u^{-1} = S^2(R_i)R^i$.

The following corollary is a straightforward consequence of Lemma 2.9, Lemma 2.13, and Remark 2.35.

Corollary 2.36. *Let (H, R) be a finite dimensional quasitriangular Hopf algebra over $\Bbbk$. Then there is a one-to-one correspondence between the pivotal elements of H and the balanced elements of H, given by $p \mapsto u^{-1}p$ and $t \mapsto ut$. Further, under the above correspondence, a pivotal element p maps to a balanced element t that is ribbon if and only if one of the following two equivalent conditions are satisfied:*

(i) $p^2 = q$.

(ii) $t^{-2} = c$. □

2.5 Quasitriangular braided Hopf algebras

Let $\mathcal{V}$ be a braided rigid monoidal category admitting a coend C, and let H be a bialgebra in $\mathcal{V}$. We recall the notion of a quasitriangular bialgebra as defined in [BV12].

Definition 2.37 ([BV12, Section 8.6])**.** A morphism $\mathfrak{R} : C \otimes C \to H \otimes H$ satisfying the four diagrammatic equations of Figure 2.14 is called an *R-matrix* for H. A *quasitriangular bialgebra* or *quasitriangular Hopf algebra* is a bialgebra or Hopf algebra equipped with an R-matrix.

$\sigma^{\mathfrak{R}}_{M,N} =$

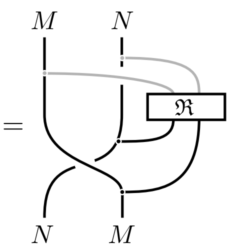


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Figure 2.15: The lax braiding on $\mathcal{V}_H$ induced by an R-matrix $\mathfrak{R}$.

The key property of R-matrices defined in this way is that there is a bijective correspondence between them and lax braidings on the category of modules as follows.

Theorem 2.38 ([BV12, Section 8.6])**.** *Given an R-matrix $\mathfrak{R} : C \otimes C \to H \otimes H$, let $\sigma = \sigma^{\mathfrak{R}}$ be defined as in* Figure 2.15 *for any two right H-modules M and N. Then σ is a lax braiding on $\mathcal{V}_H$ and the assignment $\mathfrak{R} \mapsto \sigma^{\mathfrak{R}}$ is a bijection between R-matrices for H and lax braidings on $\mathcal{V}_H$.* □

We note that when H is a Hopf algebra, the lax braiding induced by an R-matrix for $\mathfrak{R}$ is invertible, see Remark 4.17.

Chapter 3

Monadic elements of braided Hopf algebras

This section is a summary of the relevant results of [BV07], with the exception of the new materials in Section 3.6, in the special context of a Hopf monad T_H induced by a Hopf algebra H in a braided rigid monoidal category $\mathcal{V}$. See Example 2.18 for the structure morphisms of this Hopf monad. We note that while the monadic perspective is more general, all the results in this section can be obtained by direct computation using graphical calculus.

3.1 The monoid $\mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ of monadic elements

Let $\mathcal{V}$ be a monoidal category and let $H = (H, m, u)$ be an algebra in $\mathcal{V}$. The category of modules over the monad $T_H = 1_{\mathcal{V}} \otimes H$ coincides with the category of right modules over H and hence is denoted by $\mathcal{V}_H$. Let $U_H : \mathcal{V}_H \to \mathcal{V}$ and $F_H : \mathcal{V} \to \mathcal{V}_H$ denote the forgetful functor and the free module functor, respectively. Recall that (F_H, U_H) is an adjunction with unit and counit given by

$$\eta : 1_{\mathcal{V}} \to U_H F_H, \qquad \eta_X = \mathrm{id}_X \otimes u,$$
$$\epsilon : F_H U_H \to 1_{\mathcal{V}_H}, \qquad \epsilon_{(M,r)} = r,$$

respectively, such that $T_H = U_H F_H$. As a result, we can use graphical calculus to obtain the following lemma:

Lemma 3.1 ([BV07, Lemma 1.3]). *Let $\mathcal{D}$ be any category and let $F, G : \mathcal{V} \to \mathcal{D}$ be functors.*

We have mutually inverse bijections

$$\begin{aligned}(-)^{\sharp(F,G)} : \textsc{Nat}(F, GT_H) &\leftrightarrows \textsc{Nat}(FU_H, GU_H) : (-)^{\flat(F,G)}\\ g &\mapsto g^{\sharp(F,G)}_{(M,r)} = G(r)g_M\\ f_{XH}F(\eta_X) = f^{\flat(F,G)}_X &\leftarrow\!\shortmid f.\end{aligned}$$

□

3.1.1 Convolution product

As a corollary of Lemma 3.1 when $\mathcal{D} = \mathcal{V}$ and $F = G = 1_{\mathcal{V}}$, we obtain a bijection $\textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H) \cong \textsc{End}(U_H)$. In this case, $\textsc{End}(U_H)$ is also a monoid with function composition as multiplication. Transporting this monoid structure on $\textsc{End}(U_H)$ to $\textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$, we obtain:

Definition 3.2. For $h, k \in \textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$, define the *convolution product* $h * k$ as displayed in Figure 3.1. Note the second equality follows from the naturality of h.

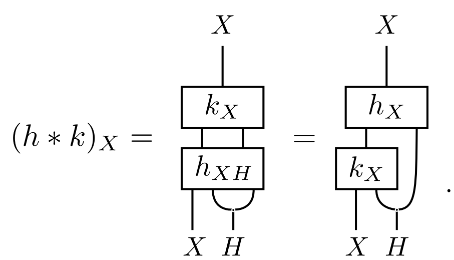


Figure 3.1: Convolution product of $\textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$

Lemma 3.3. *Let $\mathcal{V}$ be a monoidal category, and let H be an algebra in $\mathcal{V}$. The convolution product $*$ gives $\textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ a monoid structure with unit $1_{\mathcal{V}} \otimes u$, such that the mutually inverse bijections*

$$\begin{aligned}(-)^{\sharp} : \textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H) &\leftrightarrows \textsc{End}(U_H) : (-)^{\flat}\\ h &\mapsto h^{\sharp}_{(M,r)} = rh_M\\ f_{XH}\eta_X = f^{\flat}_X &\leftarrow\!\shortmid f\end{aligned} \tag{3.4}$$

are isomorphisms of monoids. □

Remark 3.5. When $\mathcal{V} = \mathsf{Vec}^{\mathrm{fd}}$, we have the more familiar mutually inverse bijections

$$\begin{aligned}H &\leftrightarrows \textsc{End}(U_H)\\ h &\mapsto (-) \cdot h\\ f_{(H,m)}(1_H) &\leftarrow\!\shortmid f\end{aligned} \tag{3.6}$$

which is part of the theory of reconstructing H from its category of representations using the forgetful functor. In an abstract monoidal category we cannot speak of elements of H, therefore the left hand side of (3.6) does not make sense. However, we always have the bijection (3.4). Thus, the elements of $\textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ can be seen as a substitute for the elements of H in this more general setting. For this reason, we will call these elements of $\textsc{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ the *monadic elements* of H.

The investigation of certain special (pivotal, ribbon, etc.) monadic elements of H is the main focus of this section. The maps $(-)^{\sharp}$ and $(-)^{\flat}$, which replace the classical bijections in (3.6), will play a key role.

A correspondence in higher dimensions

To prove results for R-matrices and essential properties of ribbon and pivotal elements, we need a further result for the Cartesian product of H, which is also a corollary of Lemma 3.1 as follows. Let $\times$ denote the Cartesian product of categories and functors. If $\mathcal{V}$ is a monoidal category and H is an algebra in $\mathcal{V}$, then $\mathcal{V}^{\times n}$ is monoidal and $H^{\times n}$ is an algebra in $\mathcal{V}^{\times n}$ in a natural way, such that $(\mathcal{V}^{\times n})_{A^{\times n}} = \mathcal{V}_H^{\times n}$, the forgetful functor $U_{H^{\times n}} = U_H^{\times n}$, and the free module functor $F_{H^{\times n}} = F_H^{\times n}$. Applying Lemma 3.1 to the algebra $H^{\times n}$, we obtain the following result:

Lemma 3.7. *Let $n \geq 1$ be a positive integer, and let $F, G : \mathcal{V}^{\times n} \to \mathcal{V}$ be functors. We have a bijection*

$$(-)^{\sharp(F,G,n)} : \textsc{Nat}(F, GT_H^{\times n}) \leftrightarrows \textsc{Nat}(FU_H^{\times n}, GU_H^{\times n}) : (-)^{\flat(F,G,n)}$$

defined as follows: for any $f \in \textsc{Nat}(F, GT_H^{\times n})$,

$$f^{\sharp(F,G,n)}_{(M_1,r_1),\dots,(M_n,r_n)} = G(r_1, \dots, r_n) f_{(M_1,\dots,M_n)},$$

and for any $g \in \textsc{Nat}(FU_H^{\times n}, GU_H^{\times n})$,

$$g^{\flat(F,G,n)}_{(X_1,\dots,X_n)} = g_{(T(X_1),\dots,T(X_n))} F(\eta_{X_1}, \dots, \eta_{X_n}).$$

$\square$

In particular, let $F = G = \otimes^{(n-1)} : \mathcal{V}^{\times n} \to \mathcal{V}$ be repeated applications of the monoidal functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$. We have the following result which will be useful later on:

Corollary 3.8. *Let $n \geq 1$ be a positive integer. We have a bijection*

$$(-)^{\sharp(n)} : \textsc{Nat}(\otimes^{(n-1)}, \otimes^{(n-1)} T_H^{\times n}) \leftrightarrows \textsc{End}(\otimes^{(n-1)} U_H^{\times n}) : (-)^{\flat(n)}$$

defined as follows: for any $f_{(X_1,\dots,X_n)} : X_1 \otimes \cdots \otimes X_n \to (X_1 \otimes H) \otimes \cdots \otimes (X_n \otimes H)$*,*

$$f^{\sharp(n)}_{((M_1,r_1),\dots,(M_n,r_n))} = (r_1 \otimes \cdots \otimes r_n) f_{(M_1,\dots,M_n)},$$

and for any $g_{((M_1,r_1),\dots,(M_n,r_n))} \in \mathrm{End}(M_1 \otimes \cdots \otimes M_n)$*,*

$$g^{\flat(n)}_{(X_1,\dots,X_n)} = g_{(X_1 \otimes H,\cdots,X_n \otimes H)}(\eta_{X_1} \otimes \cdots \otimes \eta_{X_n})$$

□

3.1.2 Central elements

Definition 3.9. For a monadic element $h \in \mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$, define the maps $L_h, R_h \in \mathrm{END}(1_{\mathcal{V}} \otimes H)$ as in Figure 3.2. If $L_h = R_h$ then we say that h is *central*.

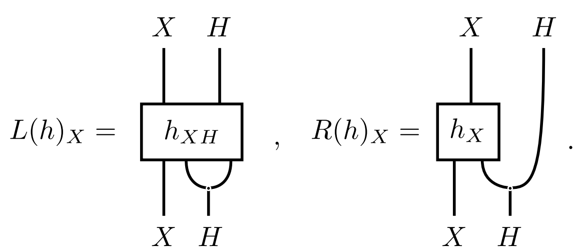


Figure 3.2: Left and right multiplication in $\mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$.

For all $k \in \mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$, we have $L_h \circ k = h * k = R_k \circ h$, and hence L_h and R_h stand for *left* and *right multiplication* by h, respectively. In particular, if h is central then h commutes with all elements in the monoid $\mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$.

When $\mathcal{V} = \mathsf{Vec}$, the central elements of H are the ones whose multiplication maps are H-linear on all H-modules. We have an analogous result for the monadic central elements:

Lemma 3.10 ([BV07, Lemma 1.5])**.** *For* $h \in \mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$*, the following are equivalent:*

(i) *h is central, i.e.,* $L_h = R_h$*,*

(ii) $h^{\sharp}_{(M,r)}$ *is H-linear for all* $(M, r) \in \mathcal{V}_H$*, or equivalently,* $h^{\sharp} \in \mathrm{END}(U_H)$ *lifts to an element in* $\mathrm{END}(1_{\mathcal{V}_H})$*.* □

Remark 3.11. The proof of the preceding lemma follows from Lemma 3.1 with $F = T_H$ and $G = 1_{\mathcal{V}}$.

3.1.3 Monadic grouplike elements

Definition 3.12. Let H be a bialgebra in a braided monoidal category $\mathcal{V}$. A monadic element $g \in \text{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H)$ is *grouplike* if g satisfies the two diagrams in Figure 3.3. The set of monadic grouplike elements of H is denoted by $\mathsf{MGrp}(H)$.

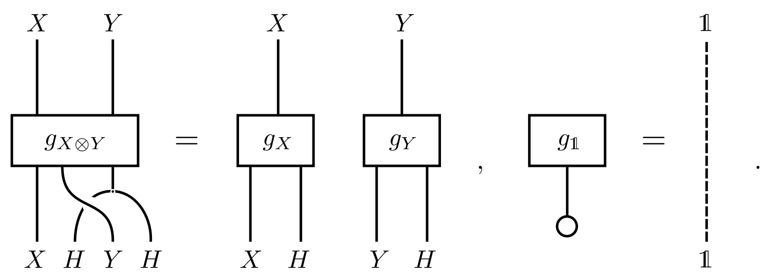


Figure 3.3: Axioms for a monadic grouplike element g.

The following result is again a generalization of a well-known property of grouplike elements of a $\Bbbk$-bialgebra H.

Lemma 3.13 ([BV07, Lemma 3.20])**.** *The isomorphism* (3.4) *restricts to an isomorphism* $\mathsf{MGrp}(H) \cong \text{END}_\otimes(U_H)$*, i.e., a monadic element g is grouplike if and only if $g^\sharp$ is a monoidal natural endomorphism.* □

Remark 3.14. By Lemmas 2.3 and 3.13, when $\mathcal{V}$ is rigid and H is a Hopf algebra, any monadic grouplike element g is convolution invertible with $\bar{g} = S(g) = S^{-1}(g)$. In particular, the second axiom $\epsilon_H g_{\mathbb{1}} = \text{id}_{\mathbb{1}}$ in Figure 3.3 is automatically satisfied.

3.1.4 Antipodes

Let $\mathcal{V}$ be a braided rigid monoidal category. We have seen that there is an isomorphism of monoids $\text{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \cong \text{END}(U_H)$. When H is a Hopf algebra, U_H is an endofunctor on the rigid monoidal category $\mathcal{V}_H$, so there is an antiautomorphism $(-)^!$ on $\text{END}(U_H)$ as in Remark 2.2. We can further transport $(-)^!$ to $\text{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H)$ to obtain the following definition.

Definition 3.15. Define maps $S, S^{-1} : \text{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \to \text{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H)$ by $S(f) = ({}^!(f^\sharp))^\flat$ and $S^{-1}(f) = ((f^\sharp)^!)^\flat$.

Remark 3.16. As mentioned in the paragraph preceding [BV07, Lemma 3.18], explicit formulas for S and S^{-1} are given by Figure 3.4.

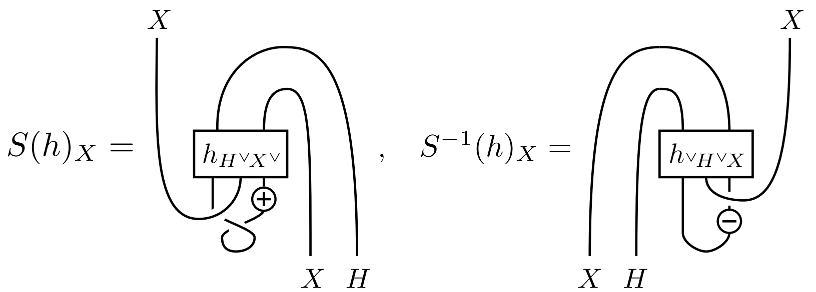


Figure 3.4: Formulas for $S^{\pm 1}(h)$ for $h \in \mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$.

3.2 Monadic pivotal elements

In this section, we will also assume that $\mathcal{V}$ is a braided pivotal category with pivotal structure ψ, or equivalently by Lemma 2.9, a balanced structure $\theta = \bar{\nu}\psi$, and H is a Hopf algebra in $\mathcal{V}$. We present the theory of monadic pivotal elements as first introduced in [BV07], however, we will later present a slightly different view in Section 3.6.

3.2.1 The square of the antipode

Recall that a pivotal element of a $\Bbbk$-Hopf algebra H is a grouplike element g such that $S^2(h) = ghg^{-1}$ for all $h \in H$. We can rewrite this as $L_g(h) = R_g S^2(h)$, where L_g and R_g are left and right multiplication maps by g, respectively. The correct monadic generalization of the endomorphism S^2 is given by the following definition.

Definition 3.17. The *square of the antipode* $\mathcal{S}^2_\psi \in \mathrm{END}(1_{\mathcal{V}} \otimes H)$ *with respect to* ψ is defined as in Figure 3.5.

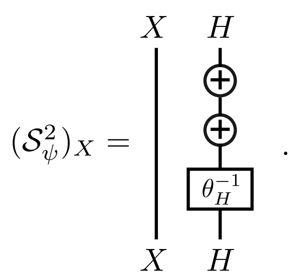


Figure 3.5: The square of the antipode, $\mathcal{S}^2_\psi$

Lemma 3.18 ([BV07, Lemma 7.5])**.** *For all $g \in \mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$, the following are equivalent:*

(i) $L_g = R_g \mathcal{S}^2_\psi$,

(ii) $(\psi g^\sharp)_M : M \to M^{\vee\vee}$ *is H-linear for all $M \in \mathcal{V}_H$, i.e., $\psi g^\sharp$ lifts to an element of* $\mathrm{NAT}(1_{\mathcal{V}_H}, (-)^{\vee\vee}_{\mathcal{V}_H})$. ☐

3.2.2 Monadic pivotal elements

Definition 3.19. A *monadic ψ-pivotal element* of a braided Hopf algebra H is a monadic grouplike element $p \in \mathsf{MGrp}(H)$ such that $L_p = R_p \mathcal{S}_\psi^2$. The set of monadic ψ-pivotal elements of H is denoted by $\mathsf{MPiv}_\psi(H)$.

As a consequence of Lemma 3.13 and Lemma 3.18, we obtain:

Theorem 3.20 ([BV07, Proposition 7.6]). *Let $\mathcal{V}$ be a braided pivotal category with pivotal structure ψ, and let H be a Hopf algebra in $\mathcal{V}$. We have mutually inverse bijections*

$$\psi(-)^\sharp : \mathsf{MPiv}_\psi(H) \leftrightarrows \mathsf{Piv}(\mathcal{V}_H) : (\psi^{-1}(-))^\flat$$

between the set $\mathsf{MPiv}_\psi(H)$ of monadic ψ-pivotal elements and the set $\mathsf{Piv}(\mathcal{V}_H)$ of pivotal structures of $\mathcal{V}_H$.

Proof. A pivotal structure ϕ on $\mathcal{V}_H$ is a natural collection of morphisms $M \to M^{\vee\vee}$ on the underlying object M of each H-module (M, r) that is (1) monoidal, and (2) H-linear. By composing with ψ^{-1}, we see that ϕ satisfies (1) if and only if $(\psi^{-1}\phi)^\flat$ is monadic grouplike by Lemma 3.13. Similarly, ϕ satisfies (2) if and only if $L_p = R_p S_\psi^2$ by Lemma 3.18. $\square$

Remark 3.21. If $1_\mathcal{V} \otimes u_H$ is a monadic pivotal element of H, then H is said to be *involutory* [BV07, Section 7.5]. In this case, the corresponding pivotal structure on $\mathcal{V}_H$ is a lift of ψ, i.e., the forgetful functor U_H is a pivotal functor.

3.3 Monadic R-matrices

Definition 3.22. Let $\mathcal{V}$ be a braided monoidal category, and let H be a bialgebra in $\mathcal{V}$. A *monadic R-matrix* R for H is a natural transformation

$$\{R_{X,Y} : X \otimes Y \to Y \otimes H \otimes X \otimes H\}_{X,Y \in \mathcal{V}}$$

satisfying the axioms in Figure 3.6. A *monadic quasitriangular bialgebra* (resp. *Hopf algebra*) is a braided bialgebra (resp. Hopf algebra) H in $\mathcal{V}$ equipped with a monadic R-matrix.

Theorem 3.23 ([BV07, Theorem 8.5]). *Let $\mathcal{V}$ be a braided monoidal category, and let H be a bialgebra in $\mathcal{V}$. Given a monadic R-matrix R for H, we obtain a lax braiding σ^R on $\mathcal{V}_H$ by $\sigma^R_{(M,r),(N,s)} = (s \otimes r)R_{M,N}$. Conversely, given a lax braiding σ on $\mathcal{V}_H$, we obtain a monadic R-matrix R^σ by $R^\sigma_{X,Y} = \sigma_{XH,YH}(\mathrm{id} \otimes u_H \otimes \mathrm{id} \otimes u_H)$. The correspondences $R \mapsto \sigma^R$ and $\sigma \mapsto R^\sigma$ are mutually inverse operations.* $\square$

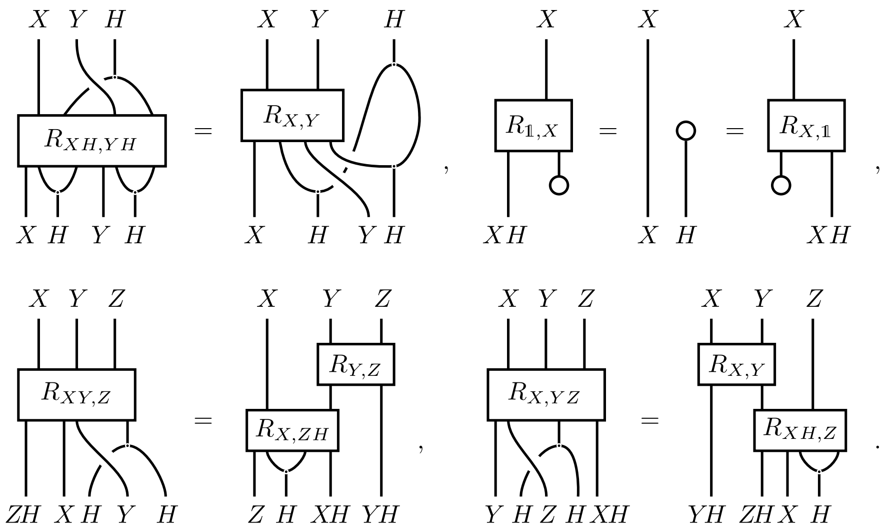


Figure 3.6: The axioms of a monadic R-matrix R.

Remark 3.24. When $\mathcal{V}$ is rigid and H is a Hopf algebra, any R-matrix R gives rise to an (invertible lax) braiding σ. As a consequence, R is convolution invertible, and its convolution inverse $\overline{R}$ corresponds to the reverse braiding $\overline{\sigma}$. Using Figure 2.3, we can express $\overline{R}$ in terms of R as in Figure 3.7 below, see also [BV07, Corollary 8.7]. Furthermore, in this case the axiom represented by the second diagram in Figure 3.6, which corresponds to the axiom $\sigma_{X,\mathbb{1}} = \mathrm{id}_{\mathbb{1}} = \sigma_{\mathbb{1},X}$ for all X, is no longer needed, see Remark 2.5.

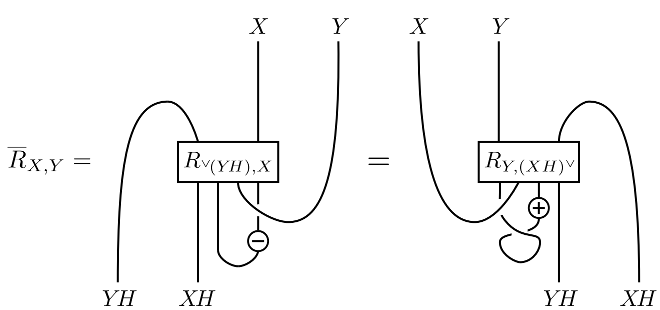


Figure 3.7: The monadic reverse R-matrix $\overline{R}$ corresponding to $\overline{\sigma}$.

3.4 Monadic ribbon elements

Definition 3.25. Let (H, R) be a monadic quasitriangular bialgebra in a braided monoidal category $\mathcal{V}$. A *monadic balanced element* (or *monadic twist*) of H is a monadic element $t \in \mathrm{Nat}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ satisfying the following properties:

(MT1) t is central (in the sense of Definition 3.9).

(MT2) $\epsilon_H t_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.

(MT3) t satisfies the diagram in Figure 3.8.

Furthermore, when $\mathcal{V}$ is rigid and H is a Hopf algebra, we say that t is *ribbon* if t further satisfies

(MT4) $S(t) = t$.

The set of monadic balanced (resp. ribbon) elements of H is denoted by $\mathsf{MBal}(H)$ (resp. $\mathsf{MRbn}(H)$).

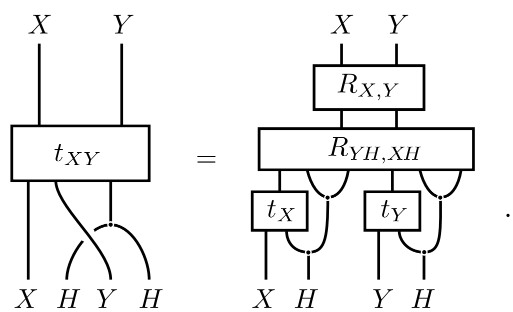


Figure 3.8: The axiom (MT3) of a monadic balanced element.

Theorem 3.26 ([BV07, Theorem 8.13])**.** *Let $\mathcal{V}$ be a braided monoidal category, and let (H, R) be a quasitriangular bialgebra in $\mathcal{V}$. We have mutually inverse bijections*

$$\mathsf{MBal}(H) \cong \mathsf{Bal}(\mathcal{V}_H), \quad t \mapsto t^\sharp, \quad \theta^\flat \leftarrow\!\shortmid \theta,$$

between the set $\mathsf{MBal}(H)$ of monadic balanced elements of H and the set $\mathsf{Bal}(\mathcal{V}_H)$ of balanced structures of $\mathcal{V}_H$, which further restricts to a bijection $\mathsf{MRbn}(H) \cong \mathsf{Rbn}(\mathcal{V}_H)$ when $\mathcal{V}$ is rigid and H is a Hopf algebra in $\mathcal{V}$. □

Remark 3.27. When $\mathcal{V}$ is rigid and H is a Hopf algebra in $\mathcal{V}$, any monadic balanced element is convolution invertible and the axiom (MT2) can be omitted, see Remark 2.11.

3.5 The monadic Drinfeld element

Let $\mathcal{V}$ be a braided pivotal category with pivotal structure ψ, and let H be a monadic quasitriangular Hopf algebra in $\mathcal{V}$ with monadic R-matrix R. We present the theory of the monadic Drinfeld element as first introduced in [BV07], however, we will later present some modification to the theory in Section 3.6.

By Proposition 3.23, the category $\mathcal{V}_H$ is a braided rigid monoidal category with braiding $\sigma = \sigma^R$. Hence we can consider the right Drinfeld morphism ν in this category, see Section 2.1.8. To utilize the bijection $\mathrm{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \cong \mathrm{END}(U_H)$, we compose the Drinfeld morphism ν with ψ^{-1} to obtain an endomorphism $\psi^{-1}\nu \in \mathrm{END}(U_H)$.

Definition 3.28. Define the monadic ψ-Drinfeld element u_ψ as in Figure 3.9.

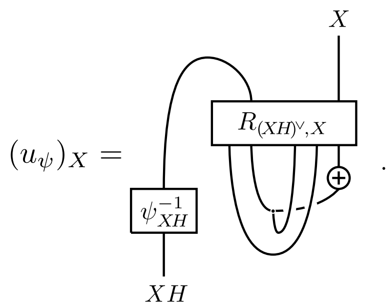


Figure 3.9: Monadic ψ-Drinfeld element u_ψ

Lemma 3.29 ([BV07, Theorem 8.10])**.** *The monadic ψ-Drinfeld element u_ψ corresponds to $\psi^{-1}\nu$ under the bijection $\mathrm{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \cong \mathrm{END}(U_H)$ in (3.4).* □

Theorem 3.30 ([BV07, Theorem 8.14])**.** *Let (H, R) be a monadic quasitriangular Hopf algebra on a braided pivotal category $\mathcal{V}$. The monadic ψ-Drinfeld element u_ψ induces a bijection*

$$u_\psi * (-) : \mathsf{MBal}(H) \cong \mathsf{MPiv}_\psi(H) \tag{3.31}$$

between the set of monadic balanced elements of H and the set of monadic ψ-pivotal elements of H.

Proof. By design, the diagram

$$\begin{tikzcd}
\mathsf{Bal}(\mathcal{V}_H) \arrow[r, "\nu\circ(-)", "\cong"'] & \mathsf{Piv}(\mathcal{V}_H) \arrow[d, "\cong"', "(\psi^{-1}(-))^\flat"] \\
\mathsf{MBal}(H) \arrow[u, "(-)^\sharp", "\cong"'] \arrow[r, "u_\psi*(-)"'] & \mathsf{MPiv}_\psi(H),
\end{tikzcd}$$

commutes, and three out of four arrows are bijections by Lemma 2.9, Theorem 3.20, and Theorem 3.26. □

Definition 3.32. Define monadic elements $q_\psi := u_\psi * S(\overline{u}_\psi)$ and $c := u_\psi * S(u_\psi)$, where $\overline{u}_\psi$ is the convolution inverse of u_ψ in $\mathrm{NAT}(1_\mathcal{V}, 1_\mathcal{V} \otimes H)$.

Remark 3.33. The element c does not depend on ψ, in fact ψ no longer appears in c since it is cancelled by ψ^{-1}.

Theorem 3.34 ([BV07, Theorem 8.14, Corollary 8.15])**.** *Under the bijection* (3.31)*, a monadic balanced element t is ribbon if and only if one of the following equivalent conditions hold:*

(i) *The corresponding ψ-pivotal element p satisfies $p^2 = q_\psi$.*

(ii) $t^{-2} = c$*.*

Proof. Under the bijection $\textsc{Nat}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \cong \textsc{End}(U_H)$, q_ψ and c correspond to $\psi^{-2}\kappa$ and γ respectively, see Definition 2.12. The result now follows from Lemma 2.13. $\square$

3.6 Relaxing the pivotal assumption

Having described the theory of monadic elements as presented in [BV07], we introduce some small modifications to the theory. A general Hopf monad can be defined on any rigid monoidal category, not necessarily one equipped with a braiding. Therefore, the definitions of monadic pivotal elements and the monadic Drinfeld element require a pivotal structure ψ on $\mathcal{V}$, in order to utilize the bijection (3.4). However, when the Hopf monad comes from a Hopf algebra, $\mathcal{V}$ is necessarily braided, and we have a canonical isomorphism $X \to X^{\vee\vee}$, namely the Drinfeld isomorphism. By utilizing the Drinfeld isomorphism, we can remove the pivotal assumption on $\mathcal{V}$. Unlike a pivotal structure, the Drinfeld isomorphism is not monoidal, which requires us to slightly modify certain definitions and results. However, relaxing the pivotal assumption strictly increases the scope of our results, since it is known that there are braided tensor categories that are not pivotal, see for example [Hal21].

Remark 3.35. We use μ to denote the Drinfeld morphism in $\mathcal{V}$ and reserve ν for the Drinfeld morphism in $\mathcal{V}_H$ coming from an R-matrix for H.

3.6.1 Monadic twisted grouplike elements

Definition 3.36. Let H be a bialgebra in a braided monoidal category $(\mathcal{V}, \sigma)$. A monadic element $g \in \textsc{Nat}(1_\mathcal{V}, 1_\mathcal{V} \otimes H)$ is *twisted grouplike* if g satisfies the two diagrams in Figure 3.10.

Lemma 3.37. *Under the isomorphism $(-)^\sharp : \textsc{Nat}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \cong \textsc{End}(U_H)$, a monadic element g is twisted grouplike if and only if the corresponding natural endomorphism $g^\sharp$ satisfies $g^\sharp_{M\otimes N} = (g^\sharp_M \otimes g^\sharp_N)\,\sigma_{N,M}\sigma_{M,N}$ for all H-modules M and N, and $g^\sharp_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$.*

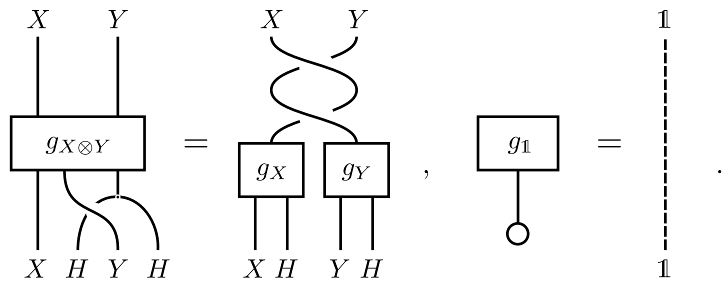


Figure 3.10: Axioms for a monadic twisted grouplike element g.

Proof. Similar to the proof of Lemma 3.13, the only difference being the introduction of the double braiding. □

3.6.2 The square of the antipode

Definition 3.38. The *square of the antipode* $\mathcal{S}^2 \in \text{END}(1_{\mathcal{V}} \otimes H)$ is defined as in Figure 3.11.

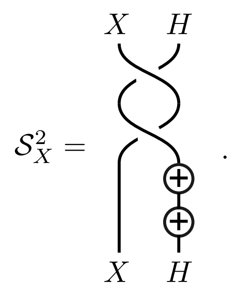


Figure 3.11: The square of the antipode, $\mathcal{S}^2$

Lemma 3.39. *For all $g \in \text{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$, the following are equivalent:*

(i) $L_g = R_g \mathcal{S}^2$,

(ii) $(\mu g^\sharp)_M : M \to M^{\vee\vee}$ *is H-linear for all $M \in \mathcal{V}_H$, i.e., $\mu g^\sharp$ lifts to an element of* $\text{NAT}(1_{\mathcal{V}_H}, (-)^{\vee\vee}_{\mathcal{V}_H})$.

Proof. Similar to Lemma 3.18. Note that the double braiding comes from the fact that $\mu_X \otimes \mu_H = \mu_{XH}\sigma_{H,X}\sigma_{X,H}$, see (2.8), and the fact that the H-action on $M^{\vee\vee}$ can be described in terms of the double dual of the H-action on M, the Drinfeld morphism, and the antipode of H squared. □

3.6.3 Monadic pivotal elements

Definition 3.40. A *monadic pivotal element* of a braided Hopf algebra H is a monadic twisted grouplike element p such that $L_p = R_p \mathcal{S}^2$. The set of monadic pivotal elements of H is denoted by $\mathsf{MPiv}(H)$.

As a consequence of Lemma 3.37 and Lemma 3.39, we obtain:

Theorem 3.41. *Let $\mathcal{V}$ be a braided rigid monoidal category, and let H be a Hopf algebra in $\mathcal{V}$. We have mutually inverse bijections*

$$\mu(-)^\sharp : \mathsf{MPiv}(H) \leftrightarrows \mathsf{Piv}(\mathcal{V}_H) : (\bar{\mu}(-))^\flat$$

between the set $\mathsf{MPiv}(H)$ of monadic pivotal elements and the set $\mathsf{Piv}(\mathcal{V}_H)$ of pivotal structures of $\mathcal{V}_H$. □

Proof. Same as the proof of Theorem 3.20, however, since we compose a monoidal natural isomorphism with $\bar{\mu}$, a double braiding is introduced, and therefore the corresponding monadic element is no longer grouplike but twisted grouplike. □

3.6.4 The monadic Drinfeld element

Let (H, R) be a monadic quasitriangular Hopf algebra in a braided rigid monoidal category $\mathcal{V}$, and let μ and ν denote the Drinfeld morphism in $\mathcal{V}$ and $\mathcal{V}_H$, respectively. We compose ν with $\bar{\mu}$ to obtain an endomorphism $\bar{\mu}\nu \in \textsc{End}(U_H)$, then use the isomorphism (3.4) to obtain the following definition.

Definition 3.42. Define the monadic Drinfeld element u as in Figure 3.12.

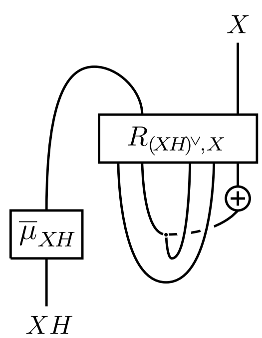
$u_X =$  .

Figure 3.12: The monadic Drinfeld element u

Since the Drinfeld morphism is composed with $\bar{\mu}$ instead of ψ^{-1}, the formula for u is essentially the same as that of u_ψ but with $\bar{\mu}_{XH}$ replacing ψ^{-1}_{XH}, and therefore the proof of the following lemma is completely analogous to that of Lemma 3.29.

Lemma 3.43. *The monadic Drinfeld element u corresponds to $\bar{\mu}\nu$ under the bijection* $\textsc{Nat}(1_\mathcal{V}, 1_\mathcal{V} \otimes H) \cong \textsc{End}(U_H)$ *in* (3.4). □

Theorem 3.44. *Let (H, R) be a monadic quasitriangular Hopf algebra on a braided rigid monoidal category $\mathcal{V}$. The monadic Drinfeld element u induces a bijection*

$$u * (-) : \mathsf{MBal}(H) \cong \mathsf{MPiv}(H) \tag{3.45}$$

between the set of monadic balanced elements of H and the set of monadic pivotal elements of H. $\square$

Proof. By design, the diagram

$$\begin{tikzcd}
\mathsf{Bal}(\mathcal{V}_H) \arrow[r, "\nu\circ(-)", "\cong"'] & \mathsf{Piv}(\mathcal{V}_H) \arrow[d, "\cong"', "(\bar{\mu}(-))^\flat"] \\
\mathsf{MBal}(H) \arrow[u, "(-)^\sharp", "\cong"'] \arrow[r, "u*(-)"'] & \mathsf{MPiv}(H),
\end{tikzcd}$$

commutes, and three out of four arrows are bijections by Lemma 2.9, Theorem 3.26, and Theorem 3.41. $\square$

Definition 3.46. Let c_0 denote the automorphism $\mu^! \mu$, and let $\bar{c}_0$ denote its inverse, i.e., $\bar{c}_0 = \bar{\mu}^! \bar{\mu}$. Also define two monadic elements $q_\mu = (u * S(\bar{u}))\, \bar{c}_0$ and $c_\mu = (u * S(u))\, c_0$.

Theorem 3.47. *Under the bijection (3.45), a monadic balanced element t is ribbon if and only if one of the following equivalent conditions hold:*

(i) *The corresponding pivotal element p satisfies $p^2 = q_\mu$.*

(ii) $t^{-2} = c_\mu$. $\square$

Proof. Under the bijection (3.45), p^2 corresponds to $\bar{\mu}^2\phi^2$, while $u * S(\bar{u})$ corresponds to $\bar{\mu}\mu^!\kappa$. When t is ribbon, we have $\phi^2 = \kappa$ by Lemma 2.13, hence the difference between p^2 and $u * S(\bar{u})$ is given by $\bar{c}_0$. Similarly, the difference between t^{-2} and $u * S(u)$ is given by c_0. $\square$

Chapter 4

Coend elements of braided Hopf algebras

In this section, we fix a braided rigid monoidal category $\mathcal{V}$, which we further assume to admit a coend C, see Section 2.3. We also fix a bialgebra H in $\mathcal{V}$. Our goal is to study certain (pivotal, ribbon, etc.) structures on the category $\mathcal{V}_H$ of right H-modules in terms of certain morphisms $C \to H$.

4.1 The monoid $\mathrm{Hom}(C, H)$ of coend elements of H

In this section, it suffices for H to be an algebra. Recall the factorization property of C, in particular the bijection in (2.22),

$$\begin{aligned} \Sigma = \Sigma_H^{(1)} : \mathrm{Hom}(C, H) \cong \mathrm{N{\scriptsize AT}}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H) \\ \mathfrak{a} \mapsto \Sigma(\mathfrak{a})_X := (\mathrm{id}_X \otimes \mathfrak{a})\, \delta_X \end{aligned} \tag{4.1}$$

where $\delta_X : X \to X \otimes C$ is the universal coaction. The map Σ is displayed in Fig 4.1(a).

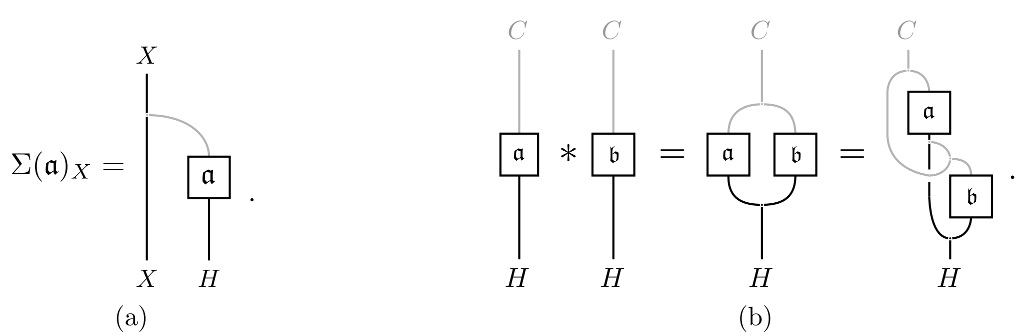


Figure 4.1: The map Σ and the convolution product of $\mathrm{Hom}(C, H)$.

Definition 4.2. The set $\mathrm{Hom}(C, H)$ is called the set of *coend elements* (or simply *C-elements*) of H.

The bijection Σ is therefore a correspondence between the set of coend elements and the set of monadic elements described in Section 3.1.

Since the set $\mathrm{N\small{AT}}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ of monadic elements is in fact a monoid (Lemma 3.3), we can use Σ to transport the opposite monoid structure of $\mathrm{N\small{AT}}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ to obtain two formulas for the product $*$ on $\mathrm{Hom}(C, H)$ as shown in Figure 4.1(b), with unit $\epsilon_C u_H$. In Hopf algebra literature, the first formula is the well-known convolution product of $\mathrm{Hom}(C, H)$. As a consequence of our setup, we obtain the following lemma.

Lemma 4.3. *The map Σ in (4.1) is anti-multiplicative and unital. Hence,* $\mathrm{Hom}(C, H)$ *with the product defined in* Figure 4.1(b) *with unit $\epsilon_C u_H$ is a monoid, and Σ is an antiisomorphism of monoids.* □

Remark 4.4. From now on, we will simply use concatenation for the convolution product on $\mathrm{Hom}(C, H)$.

4.1.1 Central elements

Definition 4.5. A C-element $\mathfrak{a} : C \to H$ is *central* if $\mathfrak{a}$ satisfies the diagram in Figure 4.2.

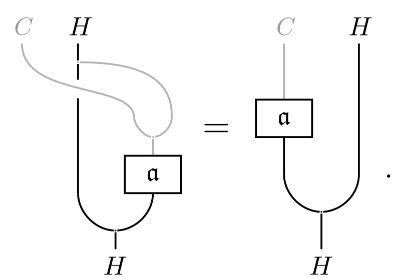


Figure 4.2: Axiom of a central element in $\mathrm{Hom}(C, H)$.

Lemma 4.6. *A C-element $\mathfrak{a} : C \to H$ is central if and only if the corresponding monadic element $a = \Sigma(\mathfrak{a})$ is central, see* Definition 3.9.

Proof. Let $\mathfrak{a} : C \to H$ be a C-element, and let $a = \Sigma(\mathfrak{a})$ be the corresponding monadic element. Recall that a is central if and only if $L_a = R_a$, where L_a and R_a are defined as in Figure 3.2. We can bend the incoming H-strand on both sides of the equation $L_{\Sigma(\mathfrak{a})} = R_{\Sigma(\mathfrak{a})}$ downwards using the coevaluation morphism coev^l_H, in order to apply Σ^{-1}. Finally, by using ev^l_H, we obtain Figure 4.2. □

4.1.2 Coend grouplike elements

Recall from Definition 3.12 the notion of a monadic grouplike element. The corresponding coend version of these elements is as follows.

Definition 4.7. A C-element $\mathfrak{g} : C \to H$ is *grouplike* if it satisfies the axioms presented in Figure 4.3. Note that the first diagram uses the universal coaction of C on itself, for which there is a formula as shown in Remark 2.26.

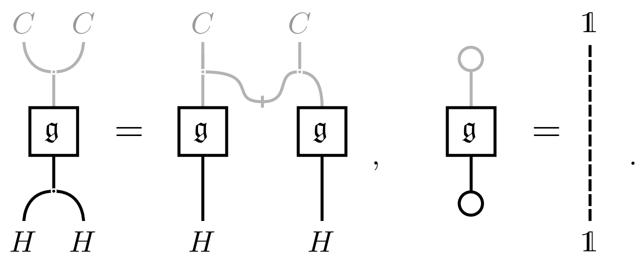


Figure 4.3: Axioms of a grouplike element in $\mathrm{Hom}(C, H)$.

Lemma 4.8. *A C-element $\mathfrak{g} : C \to H$ is grouplike if and only if the corresponding monadic element $g = \Sigma(\mathfrak{g})$ is monadic grouplike.*

Proof. Recall that the monadic element $g = \Sigma(\mathfrak{g})$ is grouplike if it satisfies the diagrams in Figure 3.3. Clearly, the second diagram in Figure 4.3 corresponds to the second diagram in Figure 3.3. It remains to show that the first diagrams in the two figures correspond to each other. The proof for this equivalence is given in Figure 4.4. Note that the second equality comes from Lemma 2.27. ☐

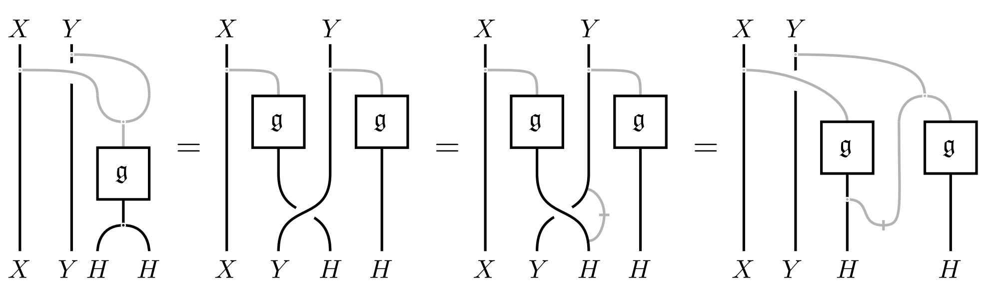


Figure 4.4: Proof of the first defining property of coend grouplike elements.

4.1.3 Coend twisted grouplike elements

Definition 4.9. A C-element $\mathfrak{g} : C \to H$ is *twisted grouplike* if it satisfies the axioms presented in Figure 4.5.

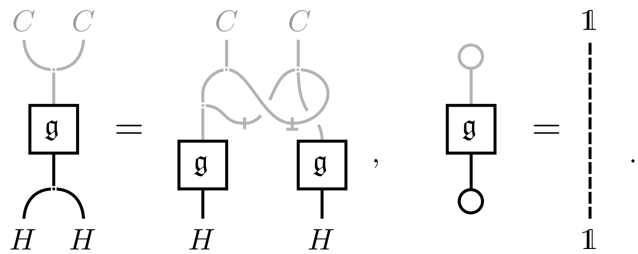


Figure 4.5: Axioms of a twisted grouplike element in $\mathrm{Hom}(C, H)$.

Since the definition of a monadic twisted grouplike element (Definition 3.36) only differs from that of a monadic grouplike element by a double braiding, the proof of the following result is a straightforward extension of the proof of Lemma 4.8.

Lemma 4.10. *A C-element $\mathfrak{g} : C \to H$ is twisted grouplike if and only if the corresponding monadic element $g = \Sigma(\mathfrak{g})$ is twisted grouplike.* □

Remark 4.11. If H is a Hopf algebra and not merely a bialgebra, then the second axiom $\epsilon_H \mathfrak{g} u_C = \mathrm{id}_{\mathbb{1}}$ in Figure 4.3 and Figure 4.5 is not needed, see Remark 3.14.

4.1.4 The antipode S on $\mathrm{Hom}(C, H)$

In this section H is always a Hopf algebra. Recall from Definition 3.15 the antipode map S on $\mathrm{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$. By using the bijection Σ, we transport this map to $\mathrm{Hom}(C, H)$ with the following definition.

Definition 4.12. Define an endomorphism S on $\mathrm{Hom}(C, H)$ as in Figure 4.6.

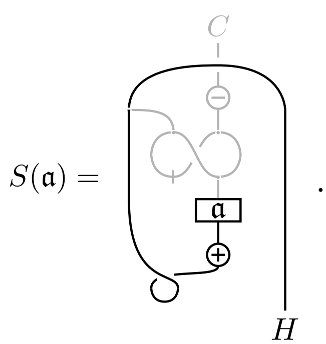


Figure 4.6: The definition of the endomorphism S on $\mathrm{Hom}(C, H)$.

Lemma 4.13. *For all $\mathfrak{a} \in \mathrm{Hom}(C, H)$, we have $\Sigma(S(\mathfrak{a})) = S(\Sigma(\mathfrak{a}))$, where $S(\mathfrak{a})$ is defined as in* Figure 4.6*, and $S(a)$ is defined as in* Figure 3.4 *for $a = \Sigma(\mathfrak{a})$. As a consequence, S is an antiautomorphism on* $\mathrm{Hom}(C, H)$.

Proof. See Figure 4.7. The first equality follows from the definition of $S(\Sigma(\mathfrak{a}))$. The second equality follows from Lemma 2.27. Finally, we use the naturality of the braiding and the properties of C as in presented in Figure 2.10 of Corollary 2.23. The second statement is a clear consequence of the first, since S is an antiautomorphism on $\text{NAT}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ and Σ is anti-multiplicative. $\square$

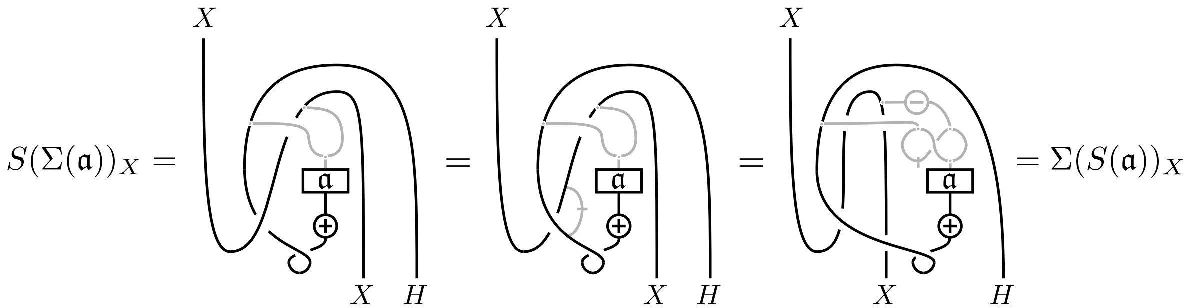


Figure 4.7: Proof that $\Sigma \circ S(\mathfrak{a}) = S \circ \Sigma(\mathfrak{a})$.

4.2 Coend pivotal elements

In this section, H is a Hopf algebra in a braided rigid monoidal category $\mathcal{V}$. Recall from Definition 3.40 the notion of a monadic pivotal element.

Definition 4.14. A C*-pivotal element* of H is a C-twisted grouplike element $\mathfrak{p}$ (Definition 4.9) that satisfies the axiom in Figure 4.8. The set of C-pivotal elements of H is denoted by $\mathsf{CPiv}(H)$.

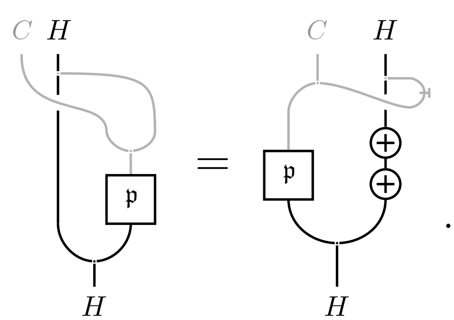


Figure 4.8: The main axiom of a pivotal element in $\text{Hom}(C, H)$.

Lemma 4.15. *The bijection Σ in (4.1) between coend and monadic elements of H restricts to a bijection $\mathsf{MPiv}(H) \cong \mathsf{CPiv}(H)$ between the respective pivotal elements.*

Proof. We have already seen in Lemma 4.10 that a C-twisted grouplike element $\mathfrak{p}$ corresponds to a monadic twisted grouplike element p. It remains to show that the condition $L_p = R_p\mathcal{S}^2$ translates to Figure 4.8 for $p = \Sigma(\mathfrak{p})$, which is straightforward. $\square$

Theorem 4.16. *There is a bijection* $\mathsf{CPiv}(H) \cong \mathsf{Piv}(\mathcal{V}_H)$ *between C-pivotal elements of H and pivotal structures of $\mathcal{V}_H$.*

Proof. Compose the two bijections $\mathsf{CPiv}(H) \xrightarrow{\Sigma} \mathsf{MPiv}(H)$ in Lemma 4.15 and $\mathsf{MPiv}(H) \xrightarrow{\mu(-)^\sharp} \mathsf{Piv}(\mathcal{V}_H)$ in Proposition 3.41. $\square$

4.3 Coend R-matrices

We have seen that the results on coend R-matrices were already obtained by Bruguières and Virelizier in [BV12], see Section 2.5. In particular, Theorem 2.38 is a consequence of the correspondence between monadic and coend R-matrices [BV12, Section 8.6].

Remark 4.17. By Remark 3.24, when H is a Hopf algebra, any monadic R-matrix R for H is convolution invertible, and the axiom corresponding to the last diagram in Figure 2.14 is automatically satisfied. Furthermore, the extended factorization property of C implies that there exists a coend R-matrix $\overline{\mathfrak{R}}$ corresponding to the convolution inverse $\overline{R}$ of R. Using the first formula for $\overline{R}$ in Figure 3.7, we can express $\overline{\mathfrak{R}}$ in terms of $\mathfrak{R}$ as in Figure 4.9 below.

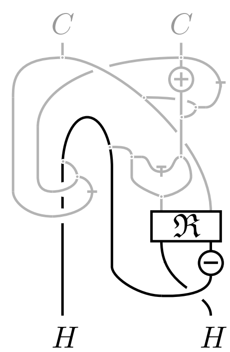
$\overline{\mathfrak{R}} =$  .

Figure 4.9: The reverse coend R-matrix $\overline{R}$ in terms of $\mathfrak{R}$.

4.4 Coend ribbon elements

Definition 4.18. Let $(H, \mathfrak{R})$ be a quasitriangular bialgebra in a braided rigid monoidal category $\mathcal{V}$ admitting a coend C. A *C-balanced element* (or *C-twist*) is a C-element $\mathfrak{t} : C \to H$ satisfying the following properties:

(CT1) $\mathfrak{t}$ is central (in the sense of Definition 4.5).

(CT2) $\epsilon_H \mathfrak{t} u_C = \mathrm{id}_{\mathbb{1}}$.

(CT3) $\mathfrak{t}$ satisfies the axiom in Figure 4.10.

Furthermore, when H is a Hopf algebra, $\mathfrak{t}$ is *ribbon* if it also satisfies

(CT4) $S(\mathfrak{t}) = \mathfrak{t}$.

The set of coend pivotal and ribbon elements are denoted by $\mathsf{CBal}(H)$ and $\mathsf{CRbn}(H)$, respectively.

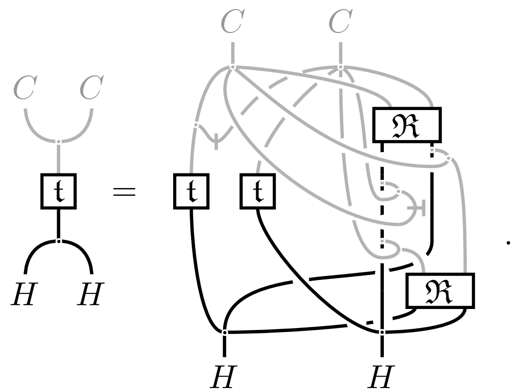


Figure 4.10: The axiom (CT3) of a coend ribbon element $\mathfrak{t}: C \to H$.

Lemma 4.19. *Under the bijection Σ in (4.1), the sets $\mathsf{MBal}(H)$ and $\mathsf{MRbn}(H)$ correspond to the sets $\mathsf{CBal}(H)$ and $\mathsf{CRbn}(H)$, respectively.*

Proof. Let $t = \Sigma(\mathfrak{t})$ be the corresponding monadic element. The equivalences of the pairs (CT1) and (MT1), (CT2) and (MT2), and (CT4) and (MT4) of axioms for $\mathfrak{t}$ and t are clear. It remains to show that (CT3) is equivalent to (MT3). The calculations are displayed in Figure 4.11. In more detail, the first equality is obtained by applying the reverse braiding to the middle two strands in (MT3). For the second equality, the monadic elements R and t are expressed in terms of their coend counterparts $\mathfrak{R}$ and $\mathfrak{t}$. In the third equality, we use the naturality of the braiding to rearrange the morphisms. The last equality uses Lemma 2.27. $\square$

Remark 4.20. When H is a Hopf algebra, any coend balanced element $\mathfrak{t}$ is convolution invertible, and the axiom (CT2) is automatically satisfied, see Remark 3.27.

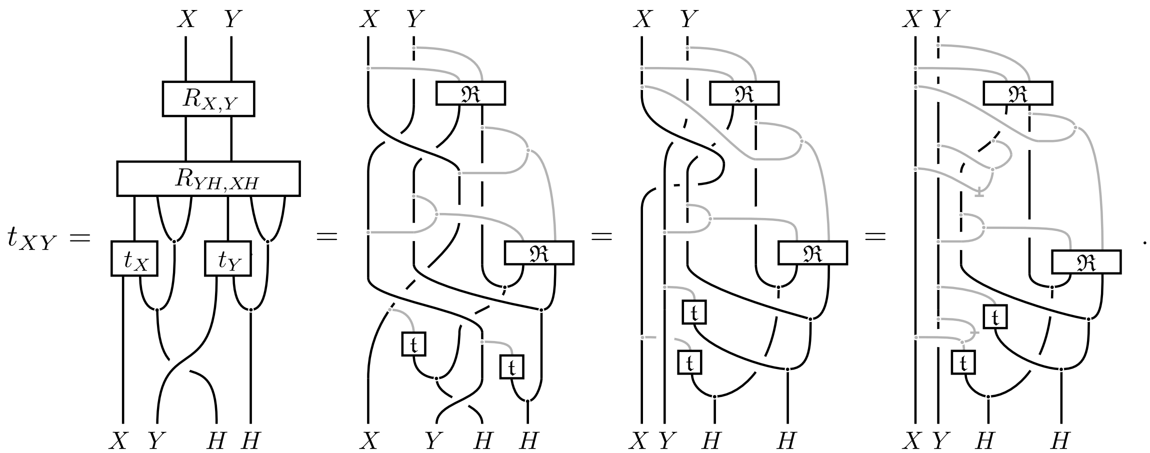


Figure 4.11: Graphical proof of the axiom (CT3).

Theorem 4.21. *Let $\mathcal{V}$ be a braided rigid monoidal category admitting a coend C, and let $(H, \mathfrak{R})$ be a quasitriangular bialgebra in $\mathcal{V}$.*

(a) *Balanced structures θ on $\mathcal{V}_H$ correspond bijectively to balanced C-elements $\mathfrak{t} : C \to H$.*

(b) *When H is a Hopf algebra, a balanced structure θ is ribbon if and only if the corresponding C-balanced element $\mathfrak{t}$ is ribbon.*

Proof. The result follows from composing the bijections Σ and $(-)^\sharp$ in Theorem 3.26 and Lemma 4.19. □

4.5 The coend Drinfeld element

In this section, we assume that $(H, \mathfrak{R})$ is a quasitriangular Hopf algebra. Recall that in this case, we denote by μ the Drinfeld morphism in $\mathcal{V}$, and the monadic Drinfeld element u corresponding to the Drinfeld morphism in $\mathcal{V}_H$ is defined as in Definition 3.42.

Definition 4.22. Let $\mathfrak{u}$ be the coend element corresponding to the monadic Drinfeld element u under the bijection Σ. We call $\mathfrak{u}$ the *coend Drinfeld element* of H.

Remark 4.23. The Drinfeld element $\mathfrak{u}$ can be expressed using the R-matrix $\mathfrak{R}$ as in Figure 4.12.

Theorem 4.24. *Let $(H, \mathfrak{R})$ be a quasitriangular Hopf algebra in a braided rigid monoidal category $\mathcal{V}$ admitting a coend C. Via the coend Drinfeld element $\mathfrak{u}$, there is a bijection* $\mathsf{CBal}(H) \cong \mathsf{CPiv}(H)$*, given by* $\mathfrak{t} \mapsto \mathfrak{t}\mathfrak{u}$*.*

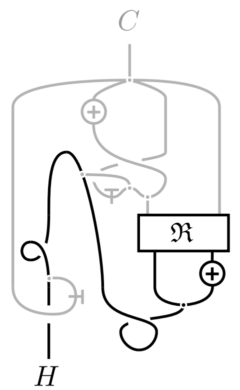
$\mathfrak{u} =$  .

Figure 4.12: The C-Drinfeld element $\mathfrak{u}$.

Proof. By design, the diagram

$$\begin{tikzcd}
\mathsf{MBal}(H) \arrow[r, "u*(-)", "\cong"'] & \mathsf{MPiv}(H) \\
\mathsf{CBal}(H) \arrow[u, "\Sigma", "\cong"'] \arrow[r, "(-)\cdot\mathfrak{u}"'] & \mathsf{CPiv}(H) \arrow[u, "\cong", "\Sigma"']
\end{tikzcd}$$

commutes, and three out of four arrows are bijections by Proposition 3.44 and Lemmas 4.15, 4.19. ☐

Recall the definitions of c_0, q_μ and c_μ in Definition 3.46. We translate these monadic elements to their respective coend counterparts as follows.

Definition 4.25. Define $\mathfrak{q}_\mu = (\mathfrak{u}S(\bar{\mathfrak{u}}) \otimes \bar{\mathfrak{c}}_0)\Delta_C$ and $\mathfrak{c}_\mu = (\mathfrak{u}S(\mathfrak{u}) \otimes \mathfrak{c}_0)\Delta_C$, where $\bar{\mathfrak{u}}$ is the convolution inverse of $\mathfrak{u}$, $\mathfrak{c}_0 : C \to \mathbb{1}$ is the coend element of $\mathbb{1}$ corresponding to c_0 under the bijection $\Sigma^{(1)}_{\mathbb{1}}$ in (2.22), and $\bar{\mathfrak{c}}_0$ is the convolution inverse of $\mathfrak{c}_0$, see Definition 3.46.

Theorem 4.26. *Under the correspondence $\mathfrak{t} \mapsto \mathfrak{p} = \mathfrak{u}\mathfrak{t}$ between balanced and pivotal elements, $\mathfrak{t}$ is ribbon if and only if $\mathfrak{p}^2 = \mathfrak{q}_\mu$, or $\mathfrak{t}^{-2} = \mathfrak{c}_\mu$.*

Proof. Clearly, $\mathfrak{q}_\mu$ and $\mathfrak{c}_\mu$ are defined to correspond to the monadic elements q_μ and c_μ in Definition 3.46. Consider the elements $\mathfrak{t} \in \mathsf{CBal}(H)$, $\mathfrak{p} = \mathfrak{u}\mathfrak{t}$, $p = \Sigma(\mathfrak{p})$, and $t = \Sigma(\mathfrak{t})$. Since the monadic elements p and t satisfy $p^2 = q_\mu$ and $t^{-2} = c_\mu$ by Theorem 3.47, and Σ is anti-multiplicative, their coend counterparts $\mathfrak{p}$ and $\mathfrak{t}$ also satisfy the analogous properties. ☐

Remark 4.27. If $\mathcal{V}$ has a pivotal structure ψ, then one can develop the theory of coend ψ-pivotal elements, the coend ψ-Drinfeld element $\mathfrak{u}_\psi$, and the elements $\mathfrak{q}_\psi = \mathfrak{u}_\psi S(\bar{\mathfrak{u}}_\psi)$ and $\mathfrak{c}_\psi = \mathfrak{u}_\psi S(\mathfrak{u}_\psi)$ in a completely analogous way, from the theory of the corresponding monadic elements in Sections 3.2 and 3.5.

Chapter 5

Applications and further directions

5.1 Special cases of H

Let H be a Hopf algebra in a braided rigid monoidal category $\mathcal{V}$ admitting a coend C. Two special cases of the Hopf algebra H are worth mentioning: $H = \mathbb{1}$, or $H = C$. When $H = \mathbb{1}$, the C-elements are morphisms $C \to \mathbb{1}$ and correspond to endomorphisms of the identity functor on $\mathcal{V}_H = \mathcal{V}$. Note that the R-matrix corresponding to the default braiding σ on $\mathcal{V}$ is simply $\epsilon_C \otimes \epsilon_C \otimes u_H \otimes u_H$. In this case, the set of monadic and coend pivotal elements coincide with the set of monadic and coend balanced elements,

$$\mathsf{MBal}(\mathcal{V}) = \mathsf{MPiv}(\mathcal{V}), \qquad \mathsf{CBal}(\mathcal{V}) = \mathsf{CPiv}(\mathcal{V}),$$

while the monadic and coend Drinfeld morphisms are the units for the convolution product in $\mathrm{N{\scriptsize AT}}(1_{\mathcal{V}}, 1_{\mathcal{V}} \otimes H)$ and $\mathrm{Hom}(C, H)$ respectively. By making these simplifications, we obtain another description for the set of balanced (pivotal) and ribbon structures on any braided rigid monoidal category admitting a coend, see Section 4.2 and Section 4.4.

When $H = C$, there is a canonical R-matrix $C \otimes C \to C \otimes C$ which is given by $u_C \otimes \epsilon_C \otimes \mathrm{id}_C$, such that we have a braided isomorphism $\mathcal{V}_C \cong \mathcal{Z}(\mathcal{V})$ [BV13, Example 2.9]. Therefore, we can study structures on $\mathcal{Z}(\mathcal{V})$ in terms of endomorphisms of C. We remark that when $\mathcal{V}$ is not necessarily braided, there is still a description of $\mathcal{Z}(\mathcal{V})$ as a category of modules over a quasitriangular Hopf monad, see [TV17, Theorem 9.3].

5.2 Linear braided Hopf algebras and the Radford isomorphism

In this thesis, we have not assumed that the rigid monoidal category $\mathcal{V}$ is abelian or $\Bbbk$-linear, which is often the case in applications. With these assumptions added, $\mathcal{V}$ is a braided finite tensor category, and we may call a Hopf algebra H in such a category a *linear braided Hopf algebra*. In this case, the category $\mathcal{V}_H$ is a finite tensor category. The theory of finite tensor categories is much richer than the theory of monoidal categories, see [Eti+16]. For instance, for any finite tensor category $\mathcal{C}$, there is an object D in $\mathcal{C}$, called the *distinguished invertible object*, together with an isomorphism

$$\rho_X : D \otimes X \otimes D^{-1} \xrightarrow{\cong} {}^{\vee\vee\vee\vee}X,$$

called the *Radford isomorphism*. When $\mathcal{V} = \mathsf{Vec}^{\mathrm{fd}}$, i.e. when H is a finite dimensional Hopf algebra over $\Bbbk$, D and ρ are given by the distinguished grouplike elements α_H and g_H of H^* and H respectively, and the H-linearity of ρ translates to the Radford's S^4 formula. For a general braided finite tensor category $\mathcal{V}$ and a Hopf algebra H in $\mathcal{V}$, Shimizu has obtained a description of the distinguished invertible object D of $\mathcal{V}_H$ in [Shi17b, Theorem 5.2] in terms of the distinguished modular function $\alpha_H : H \to \mathbb{1}$.

Question 5.1. What is the Radford isomorphism ρ for the finite tensor category $\mathcal{V}_H$?

While it can be shown that the isomorphism ρ in $\mathcal{V}_H$ must be implemented by a grouplike coend element $\mathfrak{g} : C \to H$, it is not clear how it is related to the distinguished multiplicative functional $g : \mathbb{1} \to H$, or the generalized Radford's S^4 formula as described in [Bes+00].

We remark that an answer to Question 5.1 is a step toward describing a nonsemisimple spherical structure on $\mathcal{V}_H$ as defined in [DSS20], which relies on the Radford isomorphism. By [Shi21, Theorem 5.11], a nonsemisimple spherical structure on a finite tensor category $\mathcal{C}$ gives rise to a ribbon structure on $\mathcal{Z}(\mathcal{C})$, which turns $\mathcal{Z}(\mathcal{C})$ into a modular tensor category. This is a generalization of the classical result that the Drinfeld center of a spherical fusion category is a modular fusion category. Therefore, a non-semisimple structure on $\mathcal{V}_H$ would produce a modular tensor category, namely $\mathcal{Z}(\mathcal{V}_H)$.

5.3 Factorizable linear braided Hopf algebras and ribbon structures of the Drinfeld center

Let us consider a quasitriangular linear braided Hopf algebra, i.e. a quastriangular Hopf algebra $(H, \mathfrak{R})$ in a braided finite tensor category $\mathcal{V}$. By [Shi19], the nondegeneracy of $\mathcal{V}_H$ is equivalent to the nondegeneracy of the canonical Hopf pairing of the coend of $\mathcal{V}_H$. We have seen that the coend of $\mathcal{V}_H$ is ${}^{\vee}H \otimes C$ from Example 2.30. Its structure morphisms are displayed in [BV12, Section 8.5].

Definition 5.2. Let $(H, \mathfrak{R})$ be a quasitriangular Hopf algebra in a braided finite tensor category $\mathcal{V}$. We say that $(H, \mathfrak{R})$ is *factorizable* if the canonical Hopf pairing ω for ${}^{\vee}H \otimes C$ is nondegenerate.

Remark 5.3. We can now conclude that if H is a Hopf algebra in a braided finite tensor category $\mathcal{V}$, then $\mathcal{V}_H$ is a (nonsemisimple) modular tensor category precisely when there is an R-matrix $\mathfrak{R}$ and a ribbon element $\mathfrak{t}$ for H such that $(H, \mathfrak{R})$ is factorizable.

A construction that frequently gives rise to a modular tensor category is the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a finite tensor category $\mathcal{C}$. By [Shi19, Theorem 1.1] and [Eti+16, Proposition 8.6.3], $\mathcal{Z}(\mathcal{C})$ is a nondegenerate braided finite tensor category, and hence $\mathcal{Z}(\mathcal{C})$ is modular if and only if $\mathcal{Z}(\mathcal{C})$ has a ribbon structure. As a result, the set of ribbon structures of $\mathcal{Z}(\mathcal{C})$ is a subject of great interest. For instance, when $\mathcal{C}$ is nonsemisimple spherical, $\mathcal{Z}(\mathcal{C})$ is ribbon as mentioned above.

Note that when $\mathcal{C} = \mathcal{M}_H^{\mathrm{fd}}$ for a finite dimensional Hopf algebra H over $\Bbbk$, there is a braided isomorphism $\mathcal{Z}(\mathcal{C}) \cong \mathcal{M}_{D(H)}^{\mathrm{fd}}$, where $D(H)$ is the Drinfeld double of H. Thus, the modularity of $\mathcal{Z}(\mathcal{M}_H^{\mathrm{fd}})$ now translates to the set of ribbon elements of $D(H)$. In 1993, Kauffman and Radford obtains the following description for the set of the ribbon elements of the Drinfeld double $D(H)$:

Theorem 5.4 ([KR93, Theorem 3])**.** *Let H be a finite-dimensional Hopf algebra over $\Bbbk$, and let $D(H)$ be the Drinfeld double of H. The set of ribbon elements of $D(H)$ corresponds bijectively to the set*

$$\sqrt{(\alpha_H, g_H)} = \{(\beta, h) \in G(H^*) \times G(H) \mid \beta^2 = \alpha_H, h^2 = g_H,$$
$$S^2(x) = h(\beta \rightharpoonup x \leftharpoonup \beta^{-1})h^{-1} \text{ for all } x \in H\}.$$

where g_H and α_H are the distinguished grouplike elements of H and H^* respectively.

In other words, the ribbon structures of $\mathcal{Z}(\mathcal{M}_H^{\mathrm{fd}})$ can be described in terms of square roots of the distinguished grouplike elements of H and H^*, satisfying the square root of the Radford's S^4 formula. The main result of [Shi21], which describes the set of ribbon structures of $\mathcal{Z}(\mathcal{C})$ in terms of square roots of the distinguished invertible object of $\mathcal{C}$ that satisfy the square root of the Radford isomorphism, can be seen as a tensor categorical generalization of this result.

In [BV12], Bruguières and Virelizier constructed the Drinfeld double $D(H)$ of a Hopf algebra H in a braided rigid monoidal category $\mathcal{V}$, such that $D(H) = H \otimes {}^\vee H \otimes C$ is a quasitriangular Hopf algebra in $\mathcal{V}$ and there is an isomorphism $\mathcal{Z}(\mathcal{V}_H) \cong \mathcal{V}_{D(H)}$ of braided monoidal categories. For instance, when $H = \mathbb{1}$, we have that $D(H)$ is the quasitriangular Hopf algebra C described in Section 5.1 above. We now have two more descriptions for the set of ribbon structures of $\mathcal{Z}(\mathcal{C})$ when $\mathcal{C} = \mathcal{V}_H$: as ribbon C-elements of $D(H)$ (Theorem 4.21), and as pivotal C-elements of $D(H)$ that are square roots of $\mathfrak{q}_\mu$ (Theorem 4.26). It would be interesting to see whether a third description exists in a form that directly generalizes the Kauffman-Radford theorem.

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