Quantum Symmetry ...and more

Chelsea Walton

University of Illinois Urbana-Champaign

AWM Symposium
April 6, 2019

Symmetry

Figure: Mosque in Al Ain, UAE; and Skyscraper in Guangzhou, China

Definition

Given an object X, a symmetry of X is an invertible, property-preserving transformation from X to itself.

The collection of symmetries of X (say of a special type $\mathcal{T}$) is a group $\mathrm{Sym}(X) := \mathrm{Sym}^{\mathcal{T}}(X)$ under composition.

Symmetry

Figure: Mosque in Al Ain, UAE; and Skyscraper in Guangzhou, China

Example

$\mathrm{Sym}(\text{Mosque}) = \mathbb{Z}_2$

$\mathrm{Sym}(\text{Skyscraper}) = D_6 \cong S_3$

$\mathrm{Sym}^{\text{rotations}}(\text{Skyscraper}) = \mathbb{Z}_3$

The collection of symmetries of X (say of a special type $\mathcal{T}$) is a group $\mathrm{Sym}(X) := \mathrm{Sym}^{\mathcal{T}}(X)$ under composition.

Symmetry: more examples

Definition

Given an object X, a symmetry of X is an invertible, property-preserving transformation from X to itself.

The collection of symmetries of X (say of a special type $\mathcal{T}$) is a group $\mathrm{Sym}(X) := \mathrm{Sym}^{\mathcal{T}}(X)$ under composition.

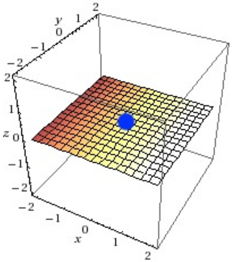
Example (Symmetries of the complex plane)

$\mathrm{Sym}^{\text{affine}}(\mathbb{C}^2) = \mathbb{C}^2 \rtimes \mathrm{GL}_2(\mathbb{C})$ [translations, rotations, dilations]

$\mathrm{Sym}^{\text{linear}}(\mathbb{C}^2) = \mathrm{GL}_2(\mathbb{C})$ [affine w/ origin fixed: rotations and dilations]

$\mathrm{Sym}^{\text{rotations}}(\mathbb{C}^2) = \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ [rotations only]

Symmetries of $\mathbb{C}$-vector spaces



Example (Symmetries of $\mathbb{C}^2$ as a $\mathbb{C}$-vector space)

$\mathrm{Sym}^{\mathrm{linear}}(\mathbb{C}^2) = \mathrm{GL}_2(\mathbb{C})$ [affine w/ origin fixed: rotations and dilations]

Symmetries of $\mathbb{C}$-vector spaces & group actions

For a $\mathbb{C}$-vector space V, we get that $\mathsf{Sym}(V) = \mathsf{GL}(V)$, the group of invertible $\mathbb{C}$-linear transformations $V \to V$.

If $\dim_{\mathbb{C}}(V) = n < \infty$, then $V \cong \mathbb{C}^n$ and $\mathsf{Sym}(V) = \mathsf{GL}_n(\mathbb{C})$.

Definition

A group G acts on a $\mathbb{C}$-vs V, $G \curvearrowright V$, if $\exists$ a group homom:

$$\phi : G \to \mathsf{GL}(V), \quad g \mapsto [\phi_g : V \to V].$$

Equiv., G acts on V if $\exists$ map $G \times V \to V$, $(g, v) \mapsto g \bullet v$, so that

$$(g_2 g_1) \bullet v = g_2 \bullet (g_1 \bullet v), \quad e_G \bullet v = v, \qquad g_1, g_2 \in G,\ v \in V.$$

Symmetries of vector spaces are encoded by G-rep'ns $\equiv$ G-modules

Symmetries of $\mathbb{C}$-algebras

$$\boxed{\mathbb{C}\text{-algebra} = \mathbb{C}\text{-vector space} + \text{unital ring}}$$

Definition

A $\mathbb{C}$-algebra is a triple (A, m_A, u_A), where

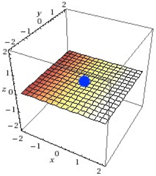
* A is a $\mathbb{C}$-vector space,
* $m_A : A \otimes A \to A$, $a \otimes b \mapsto ab$, is a $\mathbb{C}$-linear map (multip'n),
* $u_A : \mathbb{C} \to A$, $1_{\mathbb{C}} \mapsto 1_A$, is a $\mathbb{C}$-linear map (unit),

where m_A is assoc., and u_A is comp. w/ m_A so that diag. commutes:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m_A \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \otimes m_A"'] & A \otimes A \arrow[d, "m_A"] \\
A \otimes A \arrow[r, "m_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[d, "m_A"] & \\
\mathbb{C} \otimes A = \arrow[ur, "u_A \otimes \mathrm{id}_A"] & A & = A \otimes \mathbb{C} \arrow[ul, "\mathrm{id}_A \otimes u_A"']
\end{tikzcd}$$

$\forall a, b, c \in A$: $\quad (ab)c = a(bc) \qquad\qquad 1_A a = a = a 1_A$

Symmetries of $\mathbb{C}$-algebras



Example (Symmetries of polynomial algebras)

Take $V = \mathbb{C}x \oplus \mathbb{C}y$.

Have that $A = S(V) = \mathbb{C}[x, y]$ is a $\mathbb{C}$-algebra...

...which is the coordinate algebra $\mathcal{O}(\mathbb{C}^2)$ of the complex plane.

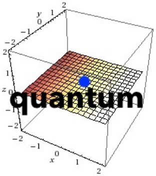
Recall: $\mathrm{Sym}^{\text{linear}}(\mathbb{C}^2) = \mathrm{GL}_2(\mathbb{C})$.

In fact:

$$\boxed{\mathrm{Sym}^{\text{linear}}(\mathbb{C}^2) = \mathrm{GL}_2(\mathbb{C}) \cong \mathrm{Aut}^{\text{graded}}(\mathbb{C}[x, y])}$$

graded = degree-preserving

Symmetries of $\mathbb{C}$-algebras



Example (Symmetries of quantum poly'l algs)

One coordinate alg of a quantum cpx plane $\mathcal{O}(\mathbb{C}^2_{\mathsf{q}})$:

$$A_q = \mathbb{C}_q[x, y] = \mathbb{C}\langle x, y\rangle/(xy - qyx), \text{ for } q \in \mathbb{C}^\times$$

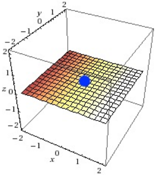
$\mathbb{C}^2_{\mathsf{q}}$ cannot be visualized, nor can $\mathrm{Sym}^{\mathrm{linear}}(\mathbb{C}^2_{\mathsf{q}})$

Work algebraically to get that when $q \neq \pm 1$:

$$\boxed{\text{“}\mathrm{Sym}^{\mathrm{linear}}(\mathbb{C}^2_{\mathsf{q}})\text{”} = \mathrm{Aut}^{\mathrm{graded}}(\mathbb{C}_q[x, y]) = \mathbb{C}^\times \times \mathbb{C}^\times}$$

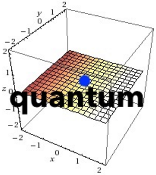
graded = degree-preserving

Symmetries of $\mathbb{C}$-algebras



$$\text{“Sym}^{\text{linear}}(\mathbb{C}^2)\text{”} = \mathrm{Aut}^{\text{graded}}(\mathbb{C}[x,y]) = \mathrm{GL}_2(\mathbb{C})$$

deform ~~deform~~



$$\text{“Sym}^{\text{linear}}(\mathbb{C}^2_{\mathsf{q}})\text{”} = \mathrm{Aut}^{\text{graded}}(\mathbb{C}_{q\neq\pm1}[x,y]) = \mathbb{C}^\times \times \mathbb{C}^\times$$

Need a better framework of symmetry, especially for noncommutative (quantum) algebras

Symmetries of $\mathbb{C}$-algebras: A New Framework

Recall that a $\mathbb{C}$-algebra is a triple (A, m_A, u_A), where

* A is a $\mathbb{C}$-vector space,
* $m_A : A \otimes A \to A$ is a $\mathbb{C}$-linear map (multiplication),
* $u_A : \mathbb{C} \to A$ is a $\mathbb{C}$-linear map (unit),

Need category $\mathcal{C}$ (collection of objects & maps btw objects) where

* A, $A \otimes A$, $\mathbb{C}$ are objects of $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1})$ is "monoidal",
* m_A, u_A are maps in $\mathcal{C}$.

Been using Rep(G): $G \curvearrowright (A \otimes A)$ by $g \bullet (a \otimes b) = (g \bullet a) \otimes (g \bullet b)$.

Upgrade to Rep(H) for H a "Hopf alg." (structure that deforms).

Hopf algebras

A Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ is an assoc. algebra $(H,\ m_H,\ u_H)$, a coassociative coalgebra $(H,\ \Delta,\ \varepsilon)$, with antipode map S, satisfying compatibility conditions.

Take $\tau : H \otimes H \to H \otimes H$, with $h \otimes \ell \mapsto \ell \otimes h$.

H is cocommutative if $\tau \circ \Delta = \Delta$.

Classical Examples:

* group algebra $\mathbb{C}G$: we have for $g \in G$

  $$m \checkmark \quad u \checkmark \quad \Delta(g) = g \otimes g \quad \varepsilon(g) = 1_{\mathbb{C}}, \quad S(g) = g^{-1}.$$

* universal enveloping algebra of a Lie algebra $U(\mathfrak{g})$: for $x \in \mathfrak{g}$

  $$m \checkmark \quad u \checkmark \quad \Delta(x) = 1_H \otimes x + x \otimes 1_H, \quad \varepsilon(x) = 0_{\mathbb{C}} \quad S(x) = -x.$$

* $\mathbb{C}G$ and $U(\mathfrak{g})$ are cocommutative.

Quantum Symmetries of $\mathbb{C}$-algebras via Hopf actions

Definition

A Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ acts on an algebra $A = (A, m_A, u_A)$, $H \curvearrowright A$ if A is an H-module algebra:

$A \in \mathrm{Rep}(H)$ (inducing $A \otimes A \in \mathrm{Rep}(H)$), and m_A, u_A are H-maps.

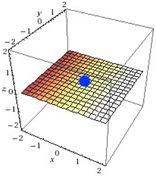
I.e., for any $a, b \in A$ and $h \in H$, with $\Delta(h) := \sum h_1 \otimes h_2$, get:

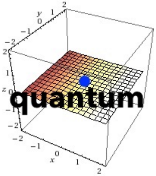
$$\begin{tikzcd}
H \otimes A \otimes A \arrow[r, "\mathrm{id}_H \otimes m_A"] \arrow[d, "(A\otimes A)\in\mathrm{Rep}(H)"'] & H \otimes A \arrow[d, "A\in\mathrm{Rep}(H)"] \\
A \otimes A \arrow[r, "m_A"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
h \otimes a \otimes b \arrow[r, mapsto] \arrow[d, mapsto] & h \otimes ab \arrow[d, mapsto] \\
\sum (h_1 \bullet a) \otimes (h_2 \bullet b) \arrow[r, mapsto] & \boxed{h \bullet (ab) = \sum (h_1 \bullet a)(h_2 \bullet b)}
\end{tikzcd}$$

(similarly, $h \bullet 1_A = \varepsilon(h) 1_A$)

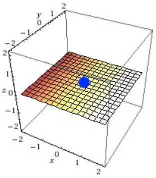
Quantum Symmetries of $\mathbb{C}$-algebras via Hopf actions

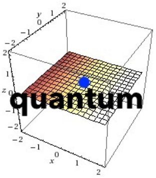




$$
\begin{tikzcd}
\mathrm{Aut}^{\text{graded}}_{\text{Rep(group)}}(\mathbb{C}[x,y]) \arrow[d, "\text{deform}"'] & = & \mathrm{GL}_2(\mathbb{C}) \arrow[d, "\cancel{\text{deform}}"] \\
\mathrm{Aut}^{\text{graded}}_{\text{Rep(group)}}(\mathbb{C}_{q\neq\pm1}[x,y]) & = & \mathbb{C}^\times \times \mathbb{C}^\times
\end{tikzcd}
$$

Quantum Symmetries of $\mathbb{C}$-algebras via Hopf actions





$$\begin{tikzcd}
\mathrm{Aut}^{\text{graded}}_{\mathrm{Rep(Hopf)}}(\mathbb{C}[x,y]) \arrow[d, "\text{deform}"'] & = & \mathcal{O}(\mathrm{GL}_2(\mathbb{C})) \arrow[d, "\text{deform}"] \\
\mathrm{Aut}^{\text{graded}}_{\mathrm{Rep(Hopf)}}(\mathbb{C}_q[x,y]) & = & \mathcal{O}_q(\mathrm{GL}_2(\mathbb{C}))
\end{tikzcd}$$

(Technically, should say CoRep(Hopf))

No Quantum Symmetry

Classical Symmetry = Actions of groups G (of $\mathbb{C}G$)
and of Lie algebras $\mathfrak{g}$ (of $U(\mathfrak{g})$)

$\mathbb{C}G$ and $U(\mathfrak{g})$ are <u>cocommutative Hopf algebras</u>:
$\Delta = \tau \circ \Delta$, where $\tau(h \otimes \ell) = \ell \otimes h$, for $h, \ell \in H$.

Theorem (Cartier-Kostant-Milnor-Moore)

If H is a <u>cocommutative</u> Hopf algebra over $\mathbb{C}$, then

$$H \cong U(\mathfrak{g})\#\mathbb{C}G, \text{ for some } G \curvearrowright \mathfrak{g}.$$

Further, if H is finite-dim'l, then $H \cong \mathbb{C}G$, for some group G.

No Quantum Symmetry

Given an action of a Hopf algebra H on an algebra A,
we say there is **No Quantum Symmetry**
$H \curvearrowright A$ must factor through
the action of a cocommutative Hopf algebra.

$\exists$ Hopf ideal I of H so that H/I is cocommutative and $H/I \curvearrowright A$.

Theorem (Cartier-Kostant-Milnor-Moore)

If H is a cocommutative Hopf algebra over $\mathbb{C}$, then

$$H \cong U(\mathfrak{g})\#\mathbb{C}G, \text{ for some } G \curvearrowright \mathfrak{g}.$$

Further, if H is finite-dim'l, then $H \cong \mathbb{C}G$, for some group G.

No Quantum Symmetry : Results

On commutative domains

Etingof and W.. Semisimple Hopf actions on commutative domains. Advances in Mathematics 251C (2014), pp. 47-61.

Skryabin. Finiteness of the number of coideal subalgebras. Proceedings of the American Mathematical Society 145 (2017), 2859-2869.

** A Hopf algebra is semisimple if each of its modules is a $\oplus$ of simple modules.
Ex. Hopf algebras built from groups: $\mathbb{C}G$, $(\mathbb{C}G)^*$ [duals], $(\mathbb{C}G)_\sigma$ [twists]...

No Quantum Symmetry : Results

On quantizations of commutative domains

Chan, W., Wang, Zhang. Hopf actions on filtered regular algebras. Journal of Algebra 397, no. 1 (2014), pp. 68-90.

Cuadra, Etingof, W.. Semisimple Hopf actions on Weyl algebras. Advances in Mathematics 282 (2015), pp. 47-55.

Cuadra, Etingof, W.. Finite dimensional Hopf actions on Weyl algebras. Advances in Mathematics 302 (2016), pp. 25-39.

Etingof and W.. Finite dimensional Hopf actions on algebraic quantizations. Algebra and Number Theory 10, no. 10, (2016), 2287-2310.

Etingof and W.. Finite dimensional Hopf actions on deformation quantizations. Proc. AMS 145 (2017), pp. 1917-1925.

Genuine Quantum Symmetry

Given an action of a Hopf algebra H on an algebra A,

we say there is **Genuine Quantum Symmetry**

when $H \curvearrowright A$ does *not* factor through

the action of a cocommutative Hopf algebra.

Genuine Quantum Symmetry : Results

On commutative domains: fields

Etingof and W.. Pointed Hopf actions on fields, I. Transformation Groups 20, no. 4 (2015), pp. 985-1013.

Etingof and W.. Pointed Hopf actions on fields, II. Journal of Algebra 460 (2016), pp. 253-283.

** A Hopf algebra is pointed if each of its simple comodules is 1-dim'l vs.

Ex. Hopf algebras built from Lie algebras: $U(\mathfrak{g}), \quad U_q(\mathfrak{g}), \quad u_q(\mathfrak{g})\ldots$

Genuine Quantum Symmetry : Results

On commutative domains: fields

Such Hopf algebras are called Galois-theoretical. Examples below, all of which are fin.-dim'l, noncom., noncocom., non-ss, pointed.

H	"Cartan type"
Taft algebras $T_\zeta(n)$	A_1
Nichols Hopf algebras $E(n)$	$\mathsf{A}_1^{\times n}$
the book algebra $\mathbf{h}(\zeta, 1)$	$\mathsf{A}_1 \times \mathsf{A}_1$
the Hopf algebra H_{81} of dimension 81	A_2
$u_q(\mathfrak{sl}_2)$	$\mathsf{A}_1 \times \mathsf{A}_1$
$u_q(\mathfrak{gl}_2)$	$\mathsf{A}_1 \times \mathsf{A}_1$
Twists $u_q(\mathfrak{gl}_n)^{J^+}$, $u_q(\mathfrak{gl}_n)^{J^-}$ for $n \geq 2$	$\mathsf{A}_{n-1} \times \mathsf{A}_{n-1}$
Twists $u_q(\mathfrak{sl}_n)^{J^+}$, $u_q(\mathfrak{sl}_n)^{J^-}$ for $n \geq 2$	$\mathsf{A}_{n-1} \times \mathsf{A}_{n-1}$
Twists $u_q^{\geq 0}(\mathfrak{g})^J$ for $2^{\mathrm{rank}(\mathfrak{g})-1}$ of such J	same type as $\mathfrak{g}$

Here, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra.

(Quantum) Symmetry of Path Algebras

Path algebra = algebra on directed graph (“quiver”)

Figure: (undirected) Graphs in real life

Definition

A path algebra $\mathbb{C}Q$ of a quiver (directed graph) Q (over $\mathbb{C}$) has

$$\begin{aligned} \mathbb{C}\text{-vector space basis} &= \text{paths of quiver} \\ \text{Multiplication} &= \text{concatenation of paths, 0 elsewhere} \end{aligned}$$

(Quantum) Symmetry of Path Algebras

Example

Q:

$$\begin{tikzcd} e_1 \arrow[r, bend left, "a"] & e_2 \arrow[l, bend left, "b"] \end{tikzcd}$$

E.g., $e_1, e_2, a, b, ab, aba, abab, \dots$ are basis elements of $\mathbb{C}Q$,

$$e_1 a = a \text{ in } \mathbb{C}Q, \quad a^2 = 0 \text{ in } \mathbb{C}Q.$$

Definition

A path algebra $\mathbb{C}Q$ of a quiver (directed graph) (over $\mathbb{C}$) has

$\mathbb{C}$-vector space basis = paths of quiver

Multiplication = concatenation of paths, 0 elsewhere

(Quantum) Symmetry of Path Algebras

Example

Q:

$$\begin{tikzcd} e_1 \arrow[r, bend left, "a"] & e_2 \arrow[l, bend left, "b"] \end{tikzcd}$$

E.g., $e_1, e_2, a, b, ab, aba, abab, \ldots$ are basis elements of $\mathbb{C}Q$,

$$e_1 a = a \text{ in } \mathbb{C}Q, \quad a^2 = 0 \text{ in } \mathbb{C}Q.$$

Note that $\mathbb{Z}_2 = \langle g : g^2 = 1 \rangle$ acts on $\mathbb{C}Q$:

$$\boxed{g \cdot e_1 = e_2, \qquad g \cdot e_2 = e_1, \qquad g \cdot a = b, \qquad g \cdot b = a}$$

So, $\mathbb{C}Q$ admits **classical symmetry**...

(Quantum) Symmetry of Path Algebras

Example (continued)

$$\begin{tikzcd} e_1 \arrow[r, bend left, "a"] & e_2 \arrow[l, bend left, "b"] \end{tikzcd}$$

$\mathbb{Z}_2 = \langle g \rangle$ acts on $\mathbb{C}Q$: $\boxed{g \cdot e_1 = e_2, \quad g \cdot e_2 = e_1, \quad g \cdot a = b, \quad g \cdot b = a}$

... $\mathbb{C}Q$ also admits **genuine quantum symmetry**

Extend to action of the Sweedler Hopf algebra:

$$H = \langle g, x \ : g^2 = 1,\ x^2 = 0,\ gx + xg = 0 \rangle \text{ with}$$
$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g$$
$$\Delta(x) = 1 \otimes x + x \otimes g, \quad \epsilon(x) = 0, \quad S(x) = -xg$$

$$\boxed{\begin{array}{ll} x \cdot e_1 = -\gamma(e_1 + e_2), & x \cdot e_2 = \gamma(e_1 + e_2) \\ x \cdot a = \gamma(a - b) + \lambda e_1, & x \cdot b = \gamma(a - b) - \lambda e_2, \ \text{for} \ \gamma, \lambda \in \mathbb{C} \end{array}}$$

Quantum Symmetry of Path Algebras

This example and classification results on Quantum Symmetry of Path Algebras are available here:

Kinser and W.. Actions of some pointed Hopf algebras on path algebras of quivers. Algebra and Number Theory 10, no. 1 (2016) pp. 117-154.

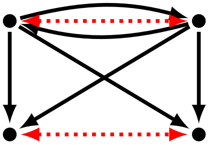
H: certain finite-dimensional, pointed Hopf algebras

Q: finitely many vertices and arrows, loopless, no parallel arrows

$H \curvearrowright \mathbb{C}Q$: preserves ascending filtration by path length

E.g., Sweedler Hopf algebra actions on the path algebra of Q:

The action of $\mathbb{Z}_2$ is given by 



Categorification: Algebras in monoidal categories

Figure: Yes, Chelsea, come to the light!

Categorification: Algebras in monoidal categories

Figure: No, Chelsea, it's too fancy!

Categorification: Algebras in monoidal categories

Previous work on actions of Hopf algebras H over $\mathbb{C}$ on $\mathbb{C}$-algebras A.

Studied H-module algebras A

Equiv to studying algebras A in the monoidal category

$$(\mathrm{Rep}(H), \otimes_{\mathbb{C}}, \mathbb{C})$$

Next: Understand algebras in general monoidal categories

$$(\mathcal{C}, \otimes, \mathbf{1})$$

Figure: The slow climb

Categorification: Algebras in monoidal categories

... projects to get started:

1. Work in preparation (with Etingof, Kinser):

Tensor algebras in finite tensor categories.

Builds on prev. work w/ Kinser

$$\mathbb{C}Q = T_{\mathbb{C}Q_0}(\mathbb{C}Q_1)$$

Q_0, Q_1 = vertex, arrow set of Q

2. Current WINART2 project.

3. Other dreams.

Figure: The slow climb

Thanks for listening to this portion of the talk!

Any questions?

Photos from unsplash.com: azhrjl [mosque]; denys nevozhai [skyscraper]; clint-adair, samuel zeller [graphs]; sam schooler, max larochelle [clouds]; luke brugger [snail]