

AN INTRODUCTION TO TENSOR CATEGORIES

HOUR 1: ON THE ROAD TO FUSION

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MOTIVATION

TO MODEL PHYSICAL PHENOMENA

MOTIVATION

TO MODEL PHYSICAL PHENOMENA
WITH THE FRAMEWORKS OF
REPRESENTATION THEORY
&
CATEGORY THEORY

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THE STUDY OF
MATRIX SOLUTIONS
TO EQUATIONS

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& MAPS BETWEEN THEM

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THE MORE SOPHISTICATED THE PHYSICAL SYSTEM—
THE MORE SOPHISTICATED THE CATEGORY OF REPRESENTATIONS

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CATEGORIES OF REPRESENTATIONS

REPIN OF $S_{\uparrow}$
STRUCTURE

CATEGORIES OF REPRESENTATIONS

REPIN OF $S_{\uparrow}$ $\equiv$ REALIZATION AS $\text{Sym}(\overset{\text{GADGET}}{\downarrow} U)$
STRUCTURE
A COLLECTION
OF SYMMETRIES

CATEGORIES OF REPRESENTATIONS

REPIN OF $\mathcal{S}_\uparrow$ $\equiv$ REALIZATION AS $\text{Sym}(\overset{\text{GADGET}}{U})$
STRUCTURE
A COLLECTION
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NEEDS TO HAVE THE
SAME STRUCTURE AS $\mathcal{S}$

CATEGORIES OF REPRESENTATIONS

REPRESENTATION OF $\mathcal{S} \equiv$

$$\pi: \mathcal{S} \xrightarrow{\text{STRUCTURE MAP}} \text{End}^*(U)$$

POSSIBLE CONDITIONS

REPIN OF $\mathcal{S}_{\uparrow}$ $\equiv$ REALIZATION AS $\text{Sym}(U)$

STRUCTURE

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NEEDS TO HAVE THE
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POSSIBLE CONDITIONS

TAKE $U = \mathbb{K}$ -VECTOR SPACE V

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$$\begin{array}{ccc} \text{REP/N OF } G & \equiv & \pi : G \longrightarrow GL(V) \\ \text{GROUP} & & \text{GROUP HOMOM.} \end{array}$$

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$$\begin{array}{ccc} \text{REP/N OF } A & \equiv & \pi : A \longrightarrow \text{End}(V) \\ \mathbb{K}\text{-ALGEBRA} & & \text{ALG. HOMOM.} \end{array}$$

CATEGORIES OF REPRESENTATIONS

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POSSIBLE CONDITIONS

TAKE $U = \mathbb{K}$ -VECTOR SPACE V OF DIM n

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CATEGORIES OF REPRESENTATIONS

$$\begin{array}{ccc} \text{REP/N OF } A & \equiv & \pi : A \longrightarrow \text{End}(V) \cong \text{Mat}_n(\mathbb{R}) \\ \mathbb{R}\text{-ALGEBRA} & & \text{ALG. HOMOM.} \end{array}$$

CATEGORIES OF REPRESENTATIONS

$$\otimes := \otimes_{\mathbb{K}}$$

(LEFT) MODULE OVER $A \equiv D : A \otimes V \longrightarrow V$ ACTION MAP



REPACKAGING

REP/N OF A $\equiv \pi : A \longrightarrow \text{End}(V) \cong \text{Mat}_n(\mathbb{K})$
 $\mathbb{K}$ -ALGEBRA ALG. HOMOM.

CATEGORIES OF REPRESENTATIONS

$$\otimes := \otimes_{\mathbb{K}}$$

(LEFT) MODULE OVER $A \equiv \begin{array}{l} \triangleright : A \otimes V \longrightarrow V \\ a \otimes v \longmapsto \pi(a)(v) \end{array} \quad \begin{array}{l} \text{ACTION MAP} \end{array}$



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MODULES / REP'S OF A FORM A CATEGORY

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$$\phi : (V, \triangleright) \longrightarrow (V', \triangleright') \equiv \\ \mathbb{K}\text{-LINEAR MAP } V \xrightarrow{\phi} V' \exists.$$

$$\begin{array}{ccc} A \otimes V & \xrightarrow{\text{id} \otimes \phi} & A \otimes V' \\ \triangleright \downarrow & \cong & \downarrow \triangleright' \\ V & \xrightarrow{\phi} & V' \end{array}$$

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$\equiv A\text{-Mod}$

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JUST A COLLECTION, NOT A SET

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$$\equiv: \text{Hom}_{A\text{-Mod}}(V, V')$$

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A $\mathbb{K}$ -VECTOR SPACE

HOMOLOGICAL CONDITIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

HOMOLOGICAL CONDITIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

PREADDITIVE

$\text{Hom}_{\mathcal{C}}(X, Y)$ $\forall X, Y \in \mathcal{C}$
IS AN ABELIAN GROUP

HOMOLOGICAL CONDITIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

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$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
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$(R-)$ LINEAR

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
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FEATURE OF

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IS A R -VECTOR SPACE

LOCALLY
FINITE

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
IS A F.D. R -VECTOR SPACE

OBJECTS HAVE
FINITE LENGTH

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(k -)LINEAR

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
IS A k -VECTOR SPACE

LOCALLY
FINITE

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
IS A F.D. k -VECTOR SPACE

OBJECTS HAVE
FINITE LENGTH

FINITE

ENOUGH PROJECTIVES

FINITELY MANY
SIMPLE OBJECTS
UP TO ISOM.

FEATURE OF
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LOCALLY
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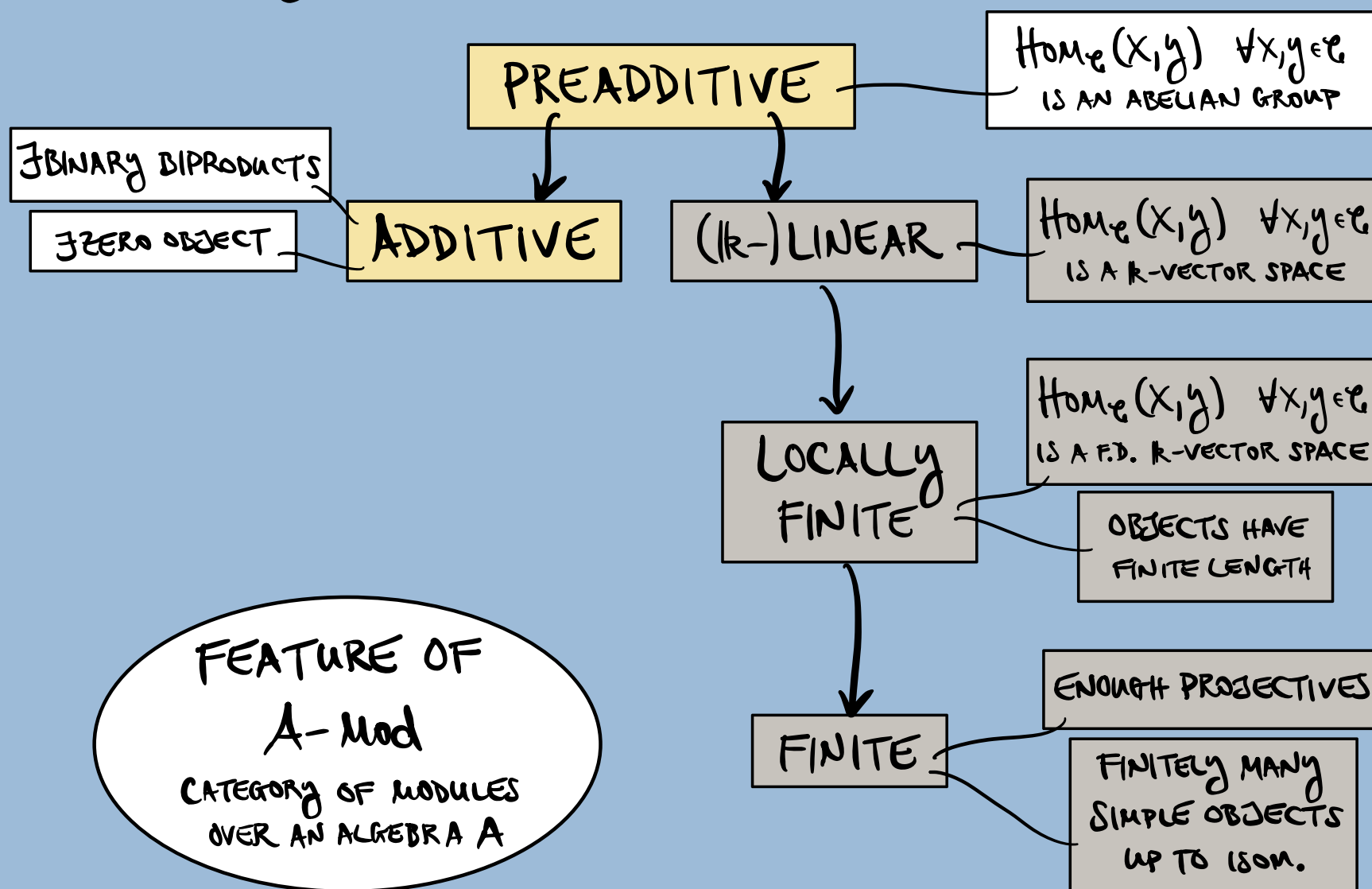
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FEATURE OF
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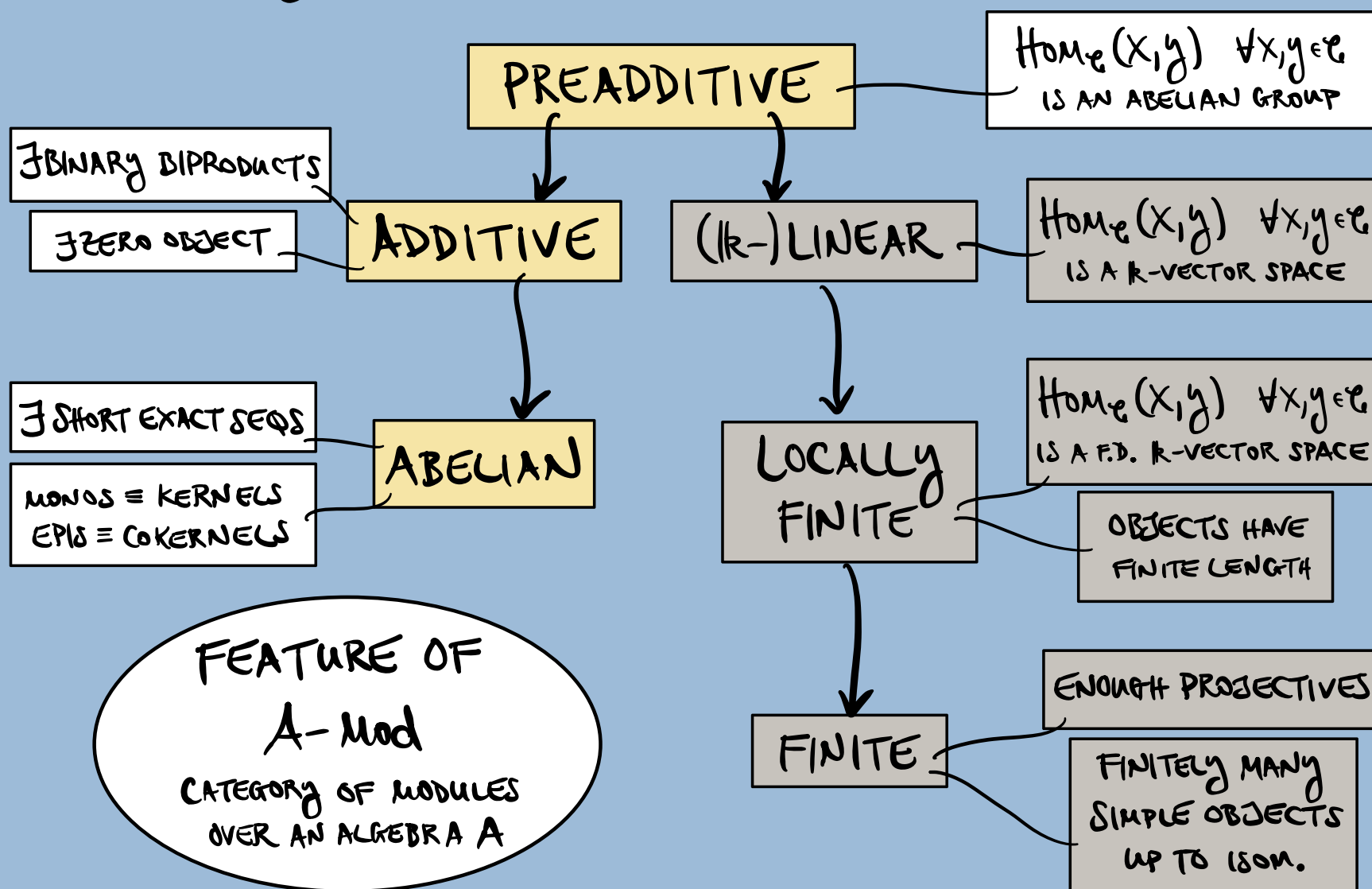
HOMOLOGICAL CONDITIONS ON CATEGORIES

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HOMOLOGICAL CONDITIONS ON CATEGORIES

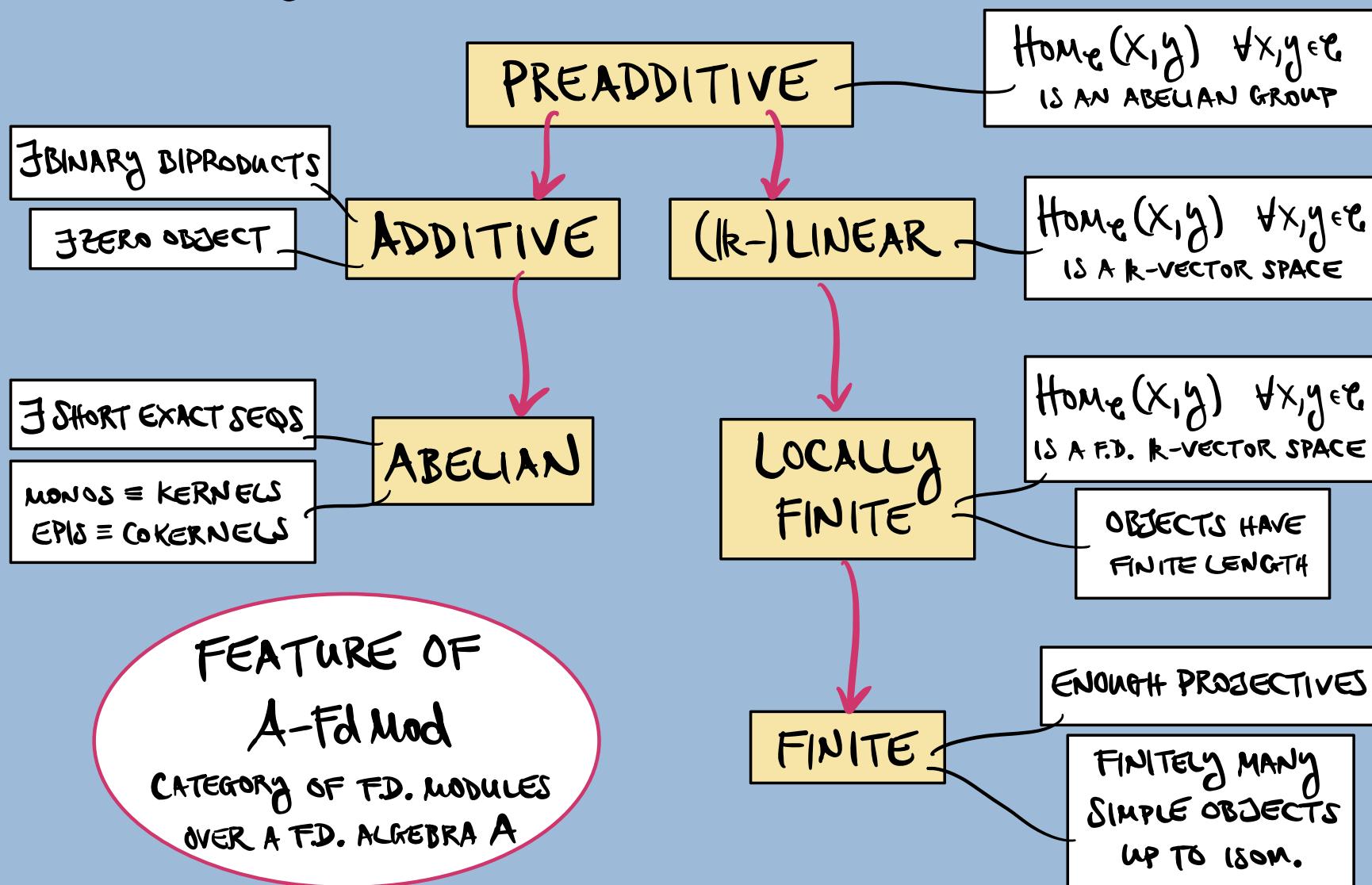
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HOMOLOGICAL CONDITIONS ON CATEGORIES

≡ ON THE ROAD
TO FUSION ≡

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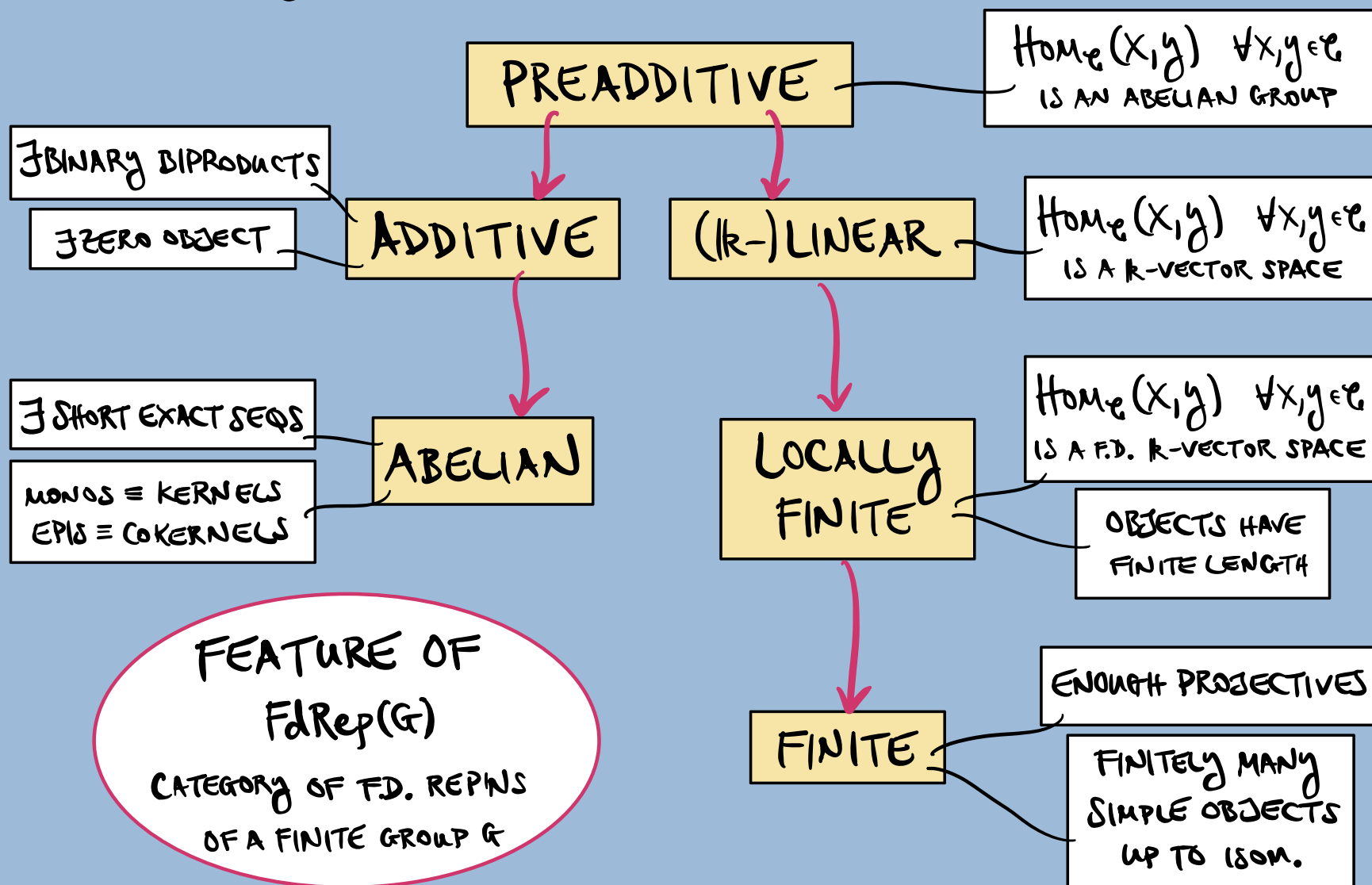


FEATURE OF
 $A\text{-fd Mod}$
CATEGORY OF F.D. MODULES
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HOMOLOGICAL CONDITIONS ON CATEGORIES

≡ ON THE ROAD
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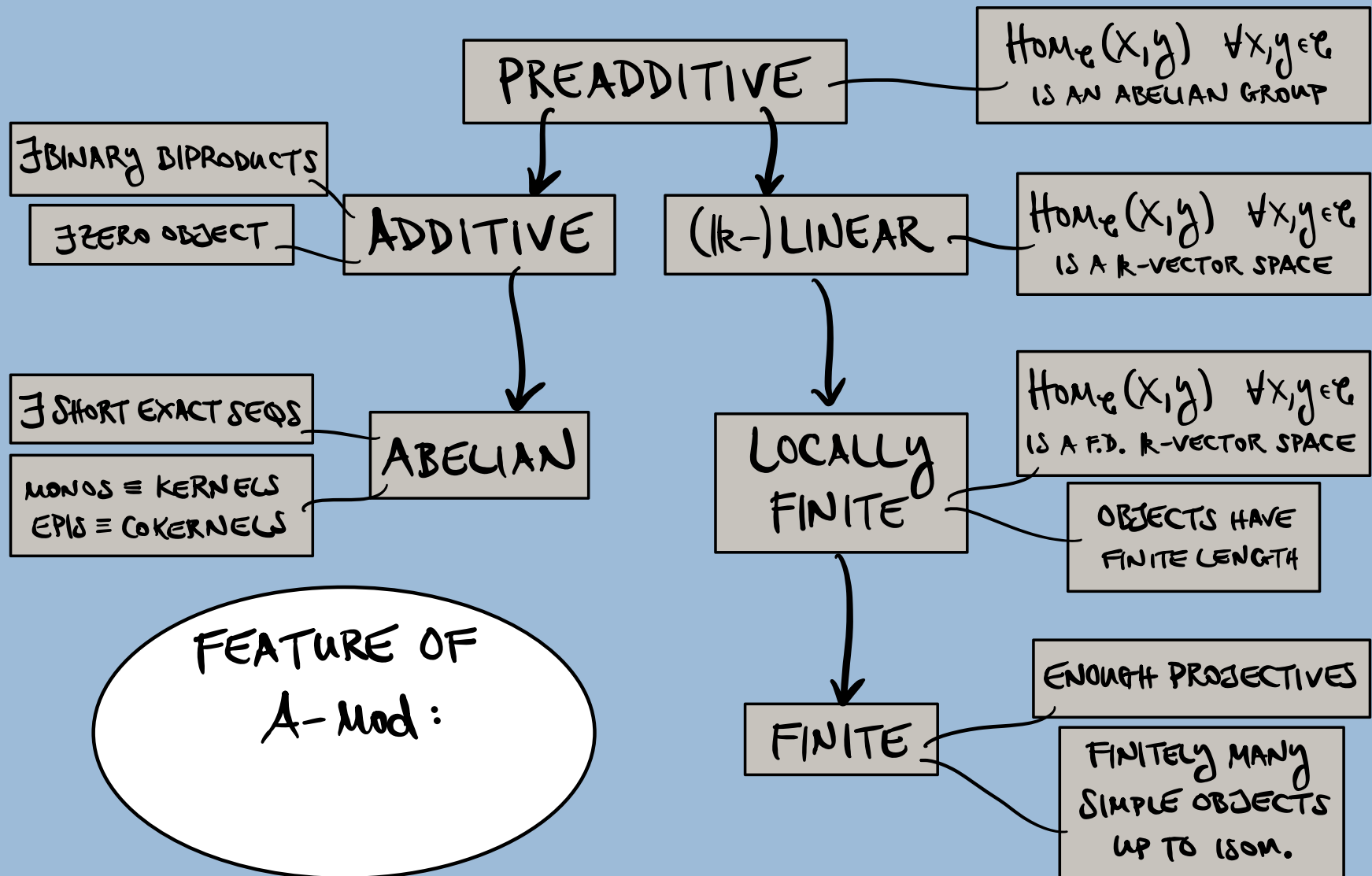
FEATURE OF

$\text{FdRep}(G)$

CATEGORY OF F.D. REPRS
OF A FINITE GROUP G

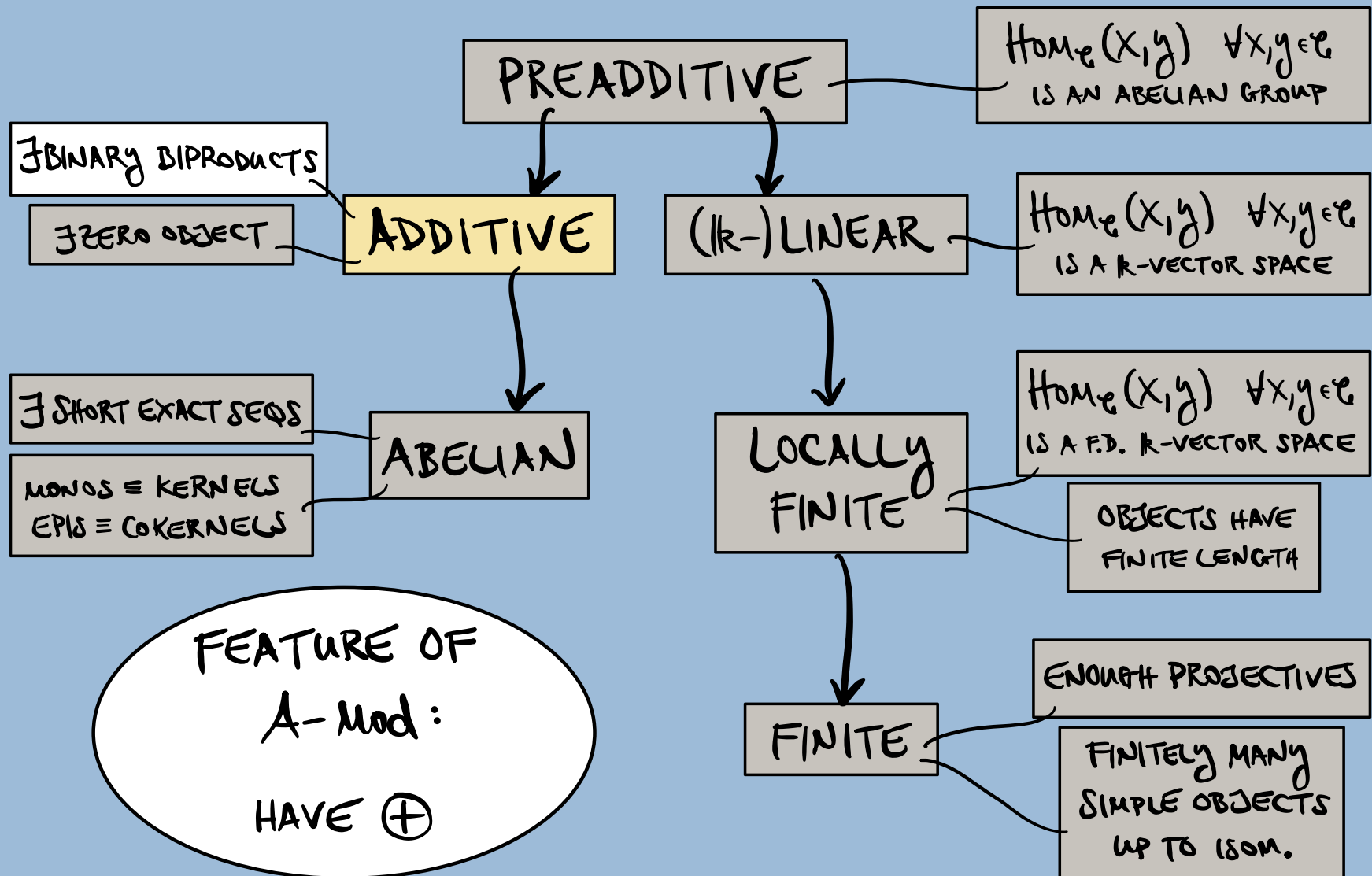
OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...



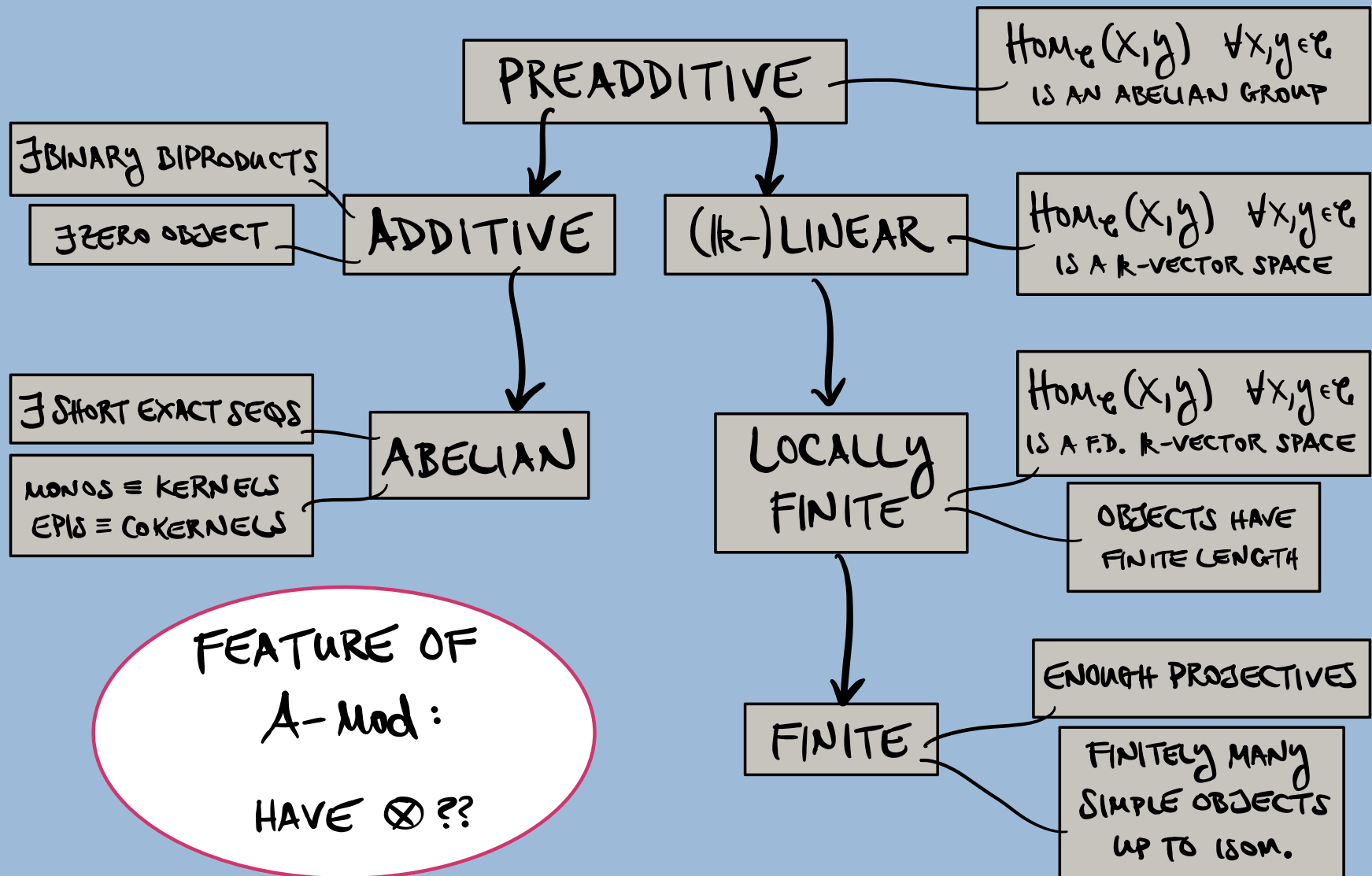
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$\exists$ BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

OPERATIONS ON CATEGORIES

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$\exists$ DISTINGUISHED OBJECT $1 \in \mathcal{C}$

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS...

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$\exists. (\mathcal{C}, \otimes, 1)$ BEHAVES LIKE A MONOID

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS...

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$\exists$ DISTINGUISHED OBJECT $\mathbb{1} \in \mathcal{C}$

$\exists. (\mathcal{C}, \otimes, \mathbb{1})$ BEHAVES LIKE A MONOID

$\exists$ NATURAL ISOMORPHISMS —

$$a := \{a_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)\}$$

$$l := \{l_x : \mathbb{1} \otimes x \xrightarrow{\sim} x\}$$

$$r := \{r_x : x \otimes \mathbb{1} \xrightarrow{\sim} x\}$$

SATISFYING "PENTAGON" & "TRIANGLE"
COHERENCE CONDITIONS

OPERATIONS ON CATEGORIES

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COHERENCE CONDITIONS

(k -)LINEAR

$\text{Hom}_{\mathcal{C}}(x, y) \forall x, y \in \mathcal{C}$
IS A k -VECTOR SPACE

OPERATIONS ON CATEGORIES

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IS A $\mathbb{K}$ -VECTOR SPACE

ASSUME

LINEAR MONOIDAL

$(X \otimes -), (- \otimes X) : \mathcal{C} \rightarrow \mathcal{C}$
ARE $\mathbb{K}$ -LINEAR

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COHERENCE CONDITIONS

(k -)LINEAR

$\text{Hom}_{\mathcal{C}}(x,y) \forall x,y \in \mathcal{C}$
IS A k -VECTOR SPACE

ASSUME

LINEAR MONOIDAL

$(x \otimes -), (- \otimes x) : \mathcal{C} \rightarrow \mathcal{C}$
ARE k -LINEAR



$$\begin{aligned} \text{id}_x \otimes (\lambda f + \lambda' f') \\ = \lambda(\text{id}_x \otimes f) + \lambda'(\text{id}_x \otimes f') \end{aligned}$$

$$\begin{aligned} (\lambda f + \lambda' f') \otimes \text{id}_x \\ = \lambda(f \otimes \text{id}_x) + \lambda'(f' \otimes \text{id}_x) \end{aligned}$$

$$\forall x \in \mathcal{C}; \quad f, f' \in \text{Hom}_{\mathcal{C}}(y,z); \quad \lambda, \lambda' \in k$$

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS...

MONOIDAL

$\exists$ BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

$\exists$ DISTINGUISHED OBJECT $\mathbb{1} \in \mathcal{C}$

$\exists. (\mathcal{C}, \otimes, \mathbb{1})$ BEHAVES LIKE A MONOID

($\mathbb{K}$ -)LINEAR

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
IS A $\mathbb{K}$ -VECTOR SPACE

ASSUME

LINEAR MONOIDAL

$(X \otimes -), (- \otimes X) : \mathcal{C} \rightarrow \mathcal{C}$
ARE $\mathbb{K}$ -LINEAR

FEATURE OF

$A\text{-Mod}$:

HAVE $\otimes$??

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

MONOIDAL

$\exists$ BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

$\exists$ DISTINGUISHED OBJECT $1 \in \mathcal{C}$

$\exists. (\mathcal{C}, \otimes, 1)$ BEHAVES LIKE A MONOID

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(k -)LINEAR

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
IS A k -VECTOR SPACE

ASSUME

LINEAR MONOIDAL

TAKE $\otimes := \otimes_k, \mathbb{1} = k :$

$(V, \triangleright) \otimes (V', \triangleright') := (V \otimes V', \triangleright)$

FEATURE OF

$A\text{-mod}$:

HAVE $\otimes$??

OPERATIONS ON CATEGORIES

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IS A k -VECTOR SPACE

ASSUME

LINEAR MONOIDAL

TAKE $\otimes := \otimes_k, \mathbb{1} = k$:

$(V, \triangleright) \otimes (V', \triangleright') := (V \otimes V', \triangleright)$

$\vdots$

Need $\Delta : A \rightarrow A \otimes A, \varepsilon : A \rightarrow k$

$$\begin{array}{ccc} A \otimes V \otimes V' & \xrightarrow{\triangleright} & V \otimes V' \\ \Delta \otimes \text{id} \otimes \text{id} \downarrow & \wr & \uparrow \triangleright \otimes \triangleright' \\ A \otimes A \otimes V \otimes V' & \xrightarrow{\text{id} \otimes \text{flip} \otimes \text{id}} & A \otimes V \otimes A \otimes V' \end{array}$$

$$A \otimes k \xrightarrow{\varepsilon \otimes \text{id}} k$$

FEATURE OF

A -mod:

HAVE $\otimes$??

OPERATIONS ON CATEGORIES

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$(k-)$ LINEAR

$\text{Hom}_{\mathcal{C}}(X, Y) \forall X, Y \in \mathcal{C}$
IS A k -VECTOR SPACE

ASSUME

LINEAR MONOIDAL

TAKE $\otimes := \otimes_k, 1 = k$:

$(V, \triangleright) \otimes (V', \triangleright') := (V \otimes V', \triangleright)$

$\vdots$

Need $\Delta : B \rightarrow B \otimes B, \varepsilon : B \rightarrow k$

$$\begin{array}{ccc} B \otimes V \otimes V' & \xrightarrow{\triangleright} & V \otimes V' \\ \Delta \otimes \text{id} \otimes \text{id} \downarrow & & \uparrow \triangleright \otimes \triangleright' \\ B \otimes B \otimes V \otimes V' & \xrightarrow{\text{id} \otimes \text{flip} \otimes \text{id}} & B \otimes V \otimes B \otimes V' \end{array}$$

$$B \otimes k \xrightarrow{\varepsilon \otimes \text{id}} k$$

FEATURE OF

B -Mod

FOR A k -BIALGEBRA B

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

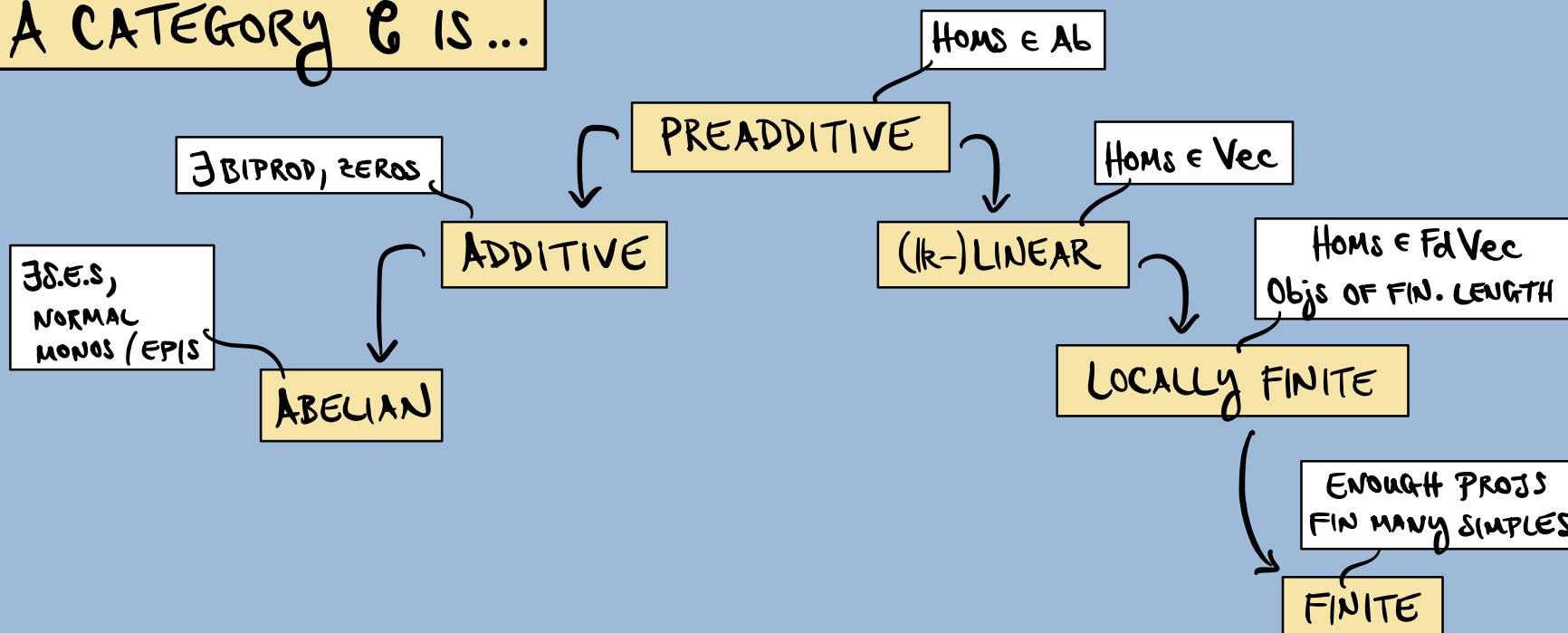
$\equiv$ ON THE ROAD
TO FUSION $\equiv$

FEATURE OF
 $\mathcal{B}$ -Mod
FOR A $\mathbb{K}$ -BIALGEBRA $\mathcal{B}$

HOMOLOGICAL CONDITIONS ON CATEGORIES

≡ ON THE ROAD
TO FUSION ≡

A CATEGORY $\mathcal{C}$ IS ...

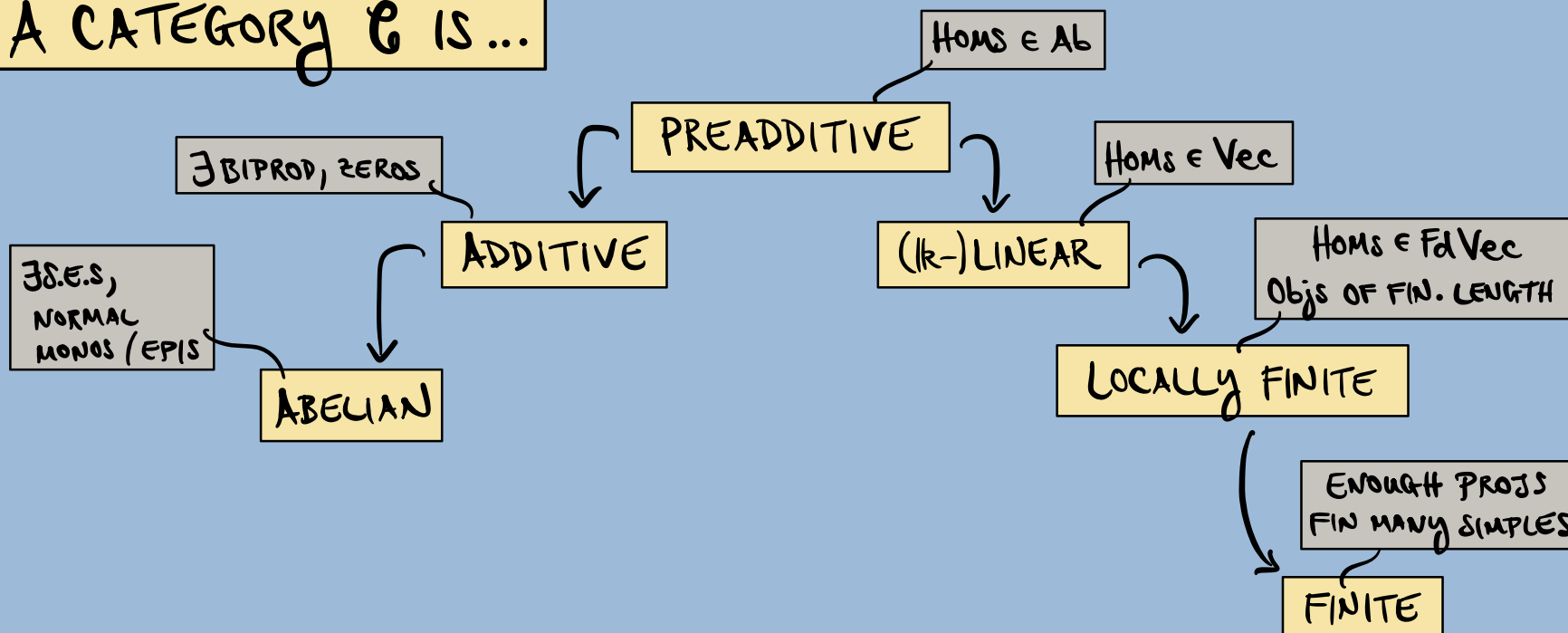


FEATURE OF
 $\mathcal{B}\text{-FdMod}$
FOR A F.D. $\mathbb{K}$ -BIALGEBRA $\mathcal{B}$

HOMOLOGICAL CONDITIONS ON CATEGORIES

≡ ON THE ROAD
TO FUSION ≡

A CATEGORY $\mathcal{C}$ IS ...



FEATURE OF

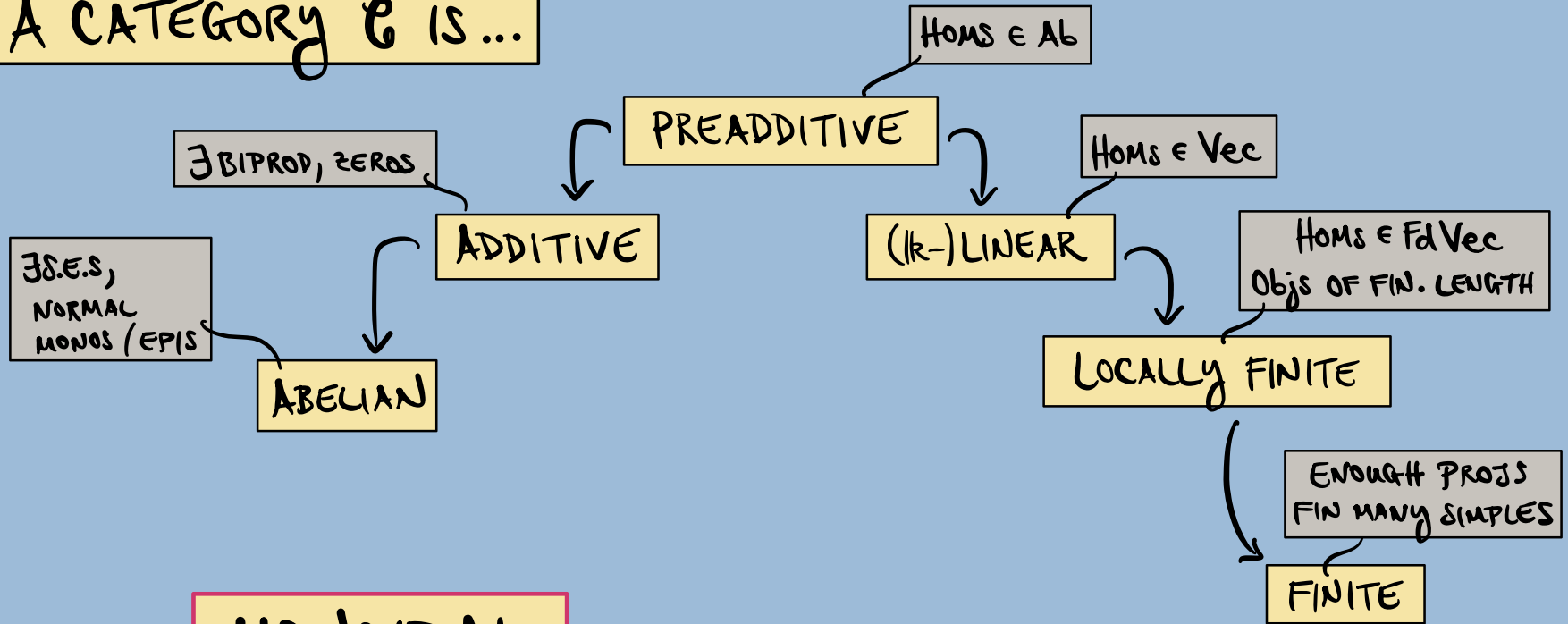
$\mathcal{B}\text{-FdMod}$

FOR A F.D. $\mathbb{K}$ -BIALGEBRA $\mathcal{B}$

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

≡ ON THE ROAD
TO FUSION ≡



MONOIDAL

FEATURE OF
 $\mathcal{B}\text{-FdMod}$
FOR A F.D. $\mathbb{K}$ -BIALGEBRA $\mathcal{B}$

$$\exists \otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad \mathbb{1} \in \mathcal{C}$$

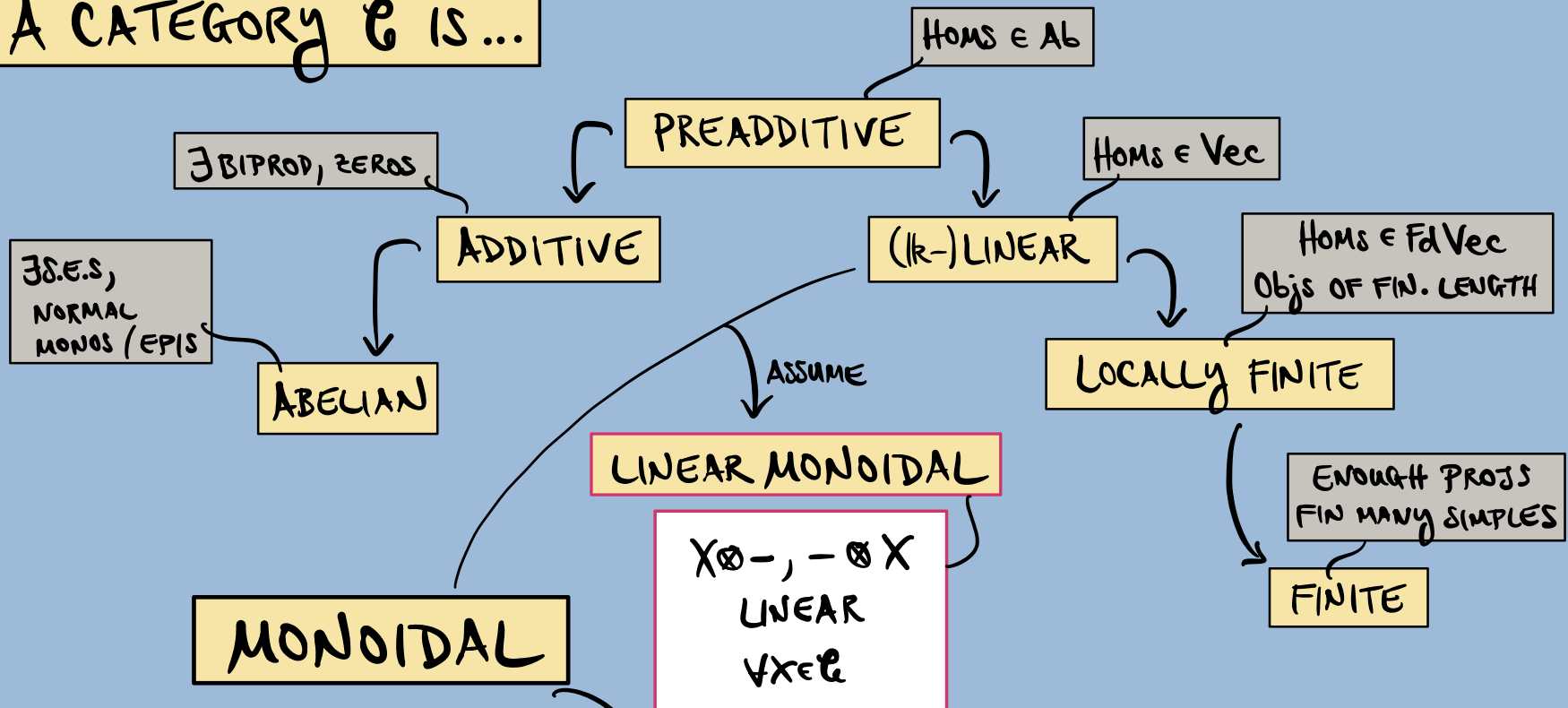
$$\Rightarrow$$

$$(\mathcal{C}, \otimes, \mathbb{1}) \simeq \text{"MONOID"}$$

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

≡ ON THE ROAD TO FUSION ≡



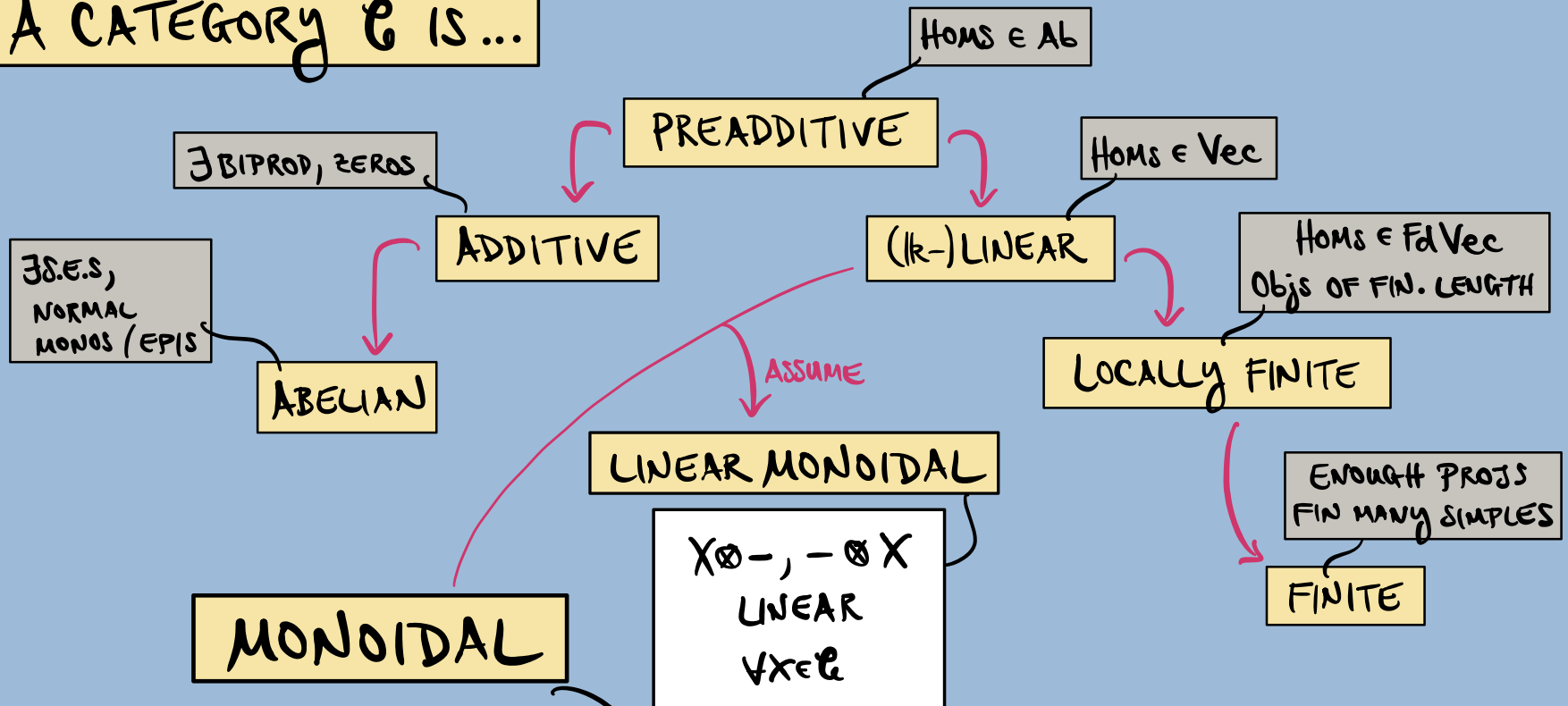
FEATURE OF
B-FdMod
FOR A F.D. K-BIALGEBRA B

$\exists \otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad \mathbb{1} \in \mathcal{C}$
 $\exists.$
 $(\mathcal{C}, \otimes, \mathbb{1}) \simeq \text{"MONOID"}$

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

≡ ON THE ROAD TO FUSION ≡



FEATURE OF
B-FdMod
FOR A F.D. k-BIALGEBRA B

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OPERATIONS ON CATEGORIES

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FEATURE OF
 $\mathcal{B}\text{-Mod}$
FOR A $\mathbb{k}$ -BIALGEBRA $\mathcal{B}$

$$\exists \otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad 1 \in \mathcal{C}$$

$\exists.$

$(\mathcal{C}, \otimes, 1) \approx \text{"MONOID"}$

OPERATIONS ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

MONOIDAL

FEATURE OF
?? H -Mod ??
FOR A k -HOPF ALG. H

$$\exists \otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, \quad 1 \in \mathcal{C} \\ \exists . \\ (\mathcal{C}, \otimes, 1) \approx \text{"MONOID"}$$

DUALITY ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

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RIGID

TO WORK WITH
DUAL VECTOR SPACES
IN A NICE WAY

FEATURE OF
 $H\text{-FdMod}$

FOR A k -HOPF ALG. H

DUALITY ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

MONOIDAL

$$\begin{aligned} \exists \otimes: \mathcal{C} \times \mathcal{C} &\rightarrow \mathcal{C}, \quad 1 \in \mathcal{C} \\ &\exists. \\ (\mathcal{C}, \otimes, 1) &\simeq \text{"MONOID"} \end{aligned}$$

RIGID

$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C}$ WITH

FEATURE OF
 $H\text{-FdMod}$

FOR A k -HOPF ALG. H

DUALITY ON CATEGORIES

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$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C}$ WITH

$\text{ev}_X^L: X^* \otimes X \rightarrow \mathbb{1}$



$\text{coev}_X^L: \mathbb{1} \rightarrow X \otimes X^*$



FEATURE OF
H-FdMod

FOR A $\mathbb{K}$ -HOPF ALG. H

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$\exists.$

$$\text{wavy line } X = \text{vertical line } X \quad \neq \quad \text{wavy line } X^* = \text{vertical line } X^*$$

FEATURE OF

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$\exists.$

$${}_X \cup^X = |^X \quad \neq \quad {}^{X^*} \cup_{X^*} = |^{X^*}$$

ALSO HAVE $({}^*X, ev_X^R, coev_X^R)$

FEATURE OF

H-FdMod

FOR A $\mathbb{K}$ -HOPF ALG. H

DUALITY ON CATEGORIES

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$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C}$ WITH

$$ev_X^L: X^* \otimes X \rightarrow 1 \quad \begin{array}{c} X^* \\ \cup \\ X \end{array}$$

$$coev_X^L: 1 \rightarrow X \otimes X^* \quad \begin{array}{c} X \\ \cap \\ X^* \end{array}$$

$\exists.$

$$\begin{array}{c} X \\ \cup \\ X \end{array} = \begin{array}{c} | \\ X \end{array} \quad \neq \quad \begin{array}{c} X^* \\ \cap \\ X^* \end{array} = \begin{array}{c} | \\ X^* \end{array}$$

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FOR ANTIPODE S OF H :
 $(h \triangleright f)(v) := f(S(h) \triangleright v)$
 $h \in H, f \in V^*, v \in V$

FEATURE OF
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DUALITY ON CATEGORIES

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RIGID

PIVOTAL

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NAT'L ISOM.

$$j: Id_{\mathcal{C}} \xrightarrow[\sim]{\otimes} (-)^{**}$$

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DUALITY ON CATEGORIES

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FEATURE OF
H-FdMod

FOR A $\mathbb{K}$ -HOPF ALG. H

FEATURE OF
H-FdMod

FOR A "PIVOTAL" $\mathbb{K}$ -HOPF ALG H
E.G., WITH $S^2 = id_H$

DUALITY ON CATEGORIES

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 $\exists. (\mathcal{C}, \otimes, 1) \approx \text{"MONOID"}$

RIGID

PIVOTAL

CAN MEASURE SIZE
OF OBJECTS VIA

$$\dim_j X = \text{tr}_{X^*}^{X^*} j^{-1} \text{tr}_X^{X^*}$$

"PIV./QUANTUM DIM'N"

$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C}$ WITH

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$$coev_X^L: 1 \rightarrow X \otimes X^* \quad \begin{array}{c} X \\ \cap \\ X^* \end{array}$$

$\exists.$

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FEATURE OF
H-FdMod

FOR A $\mathbb{K}$ -HOPF ALG. H

FEATURE OF
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FOR A "PIVOTAL" $\mathbb{K}$ -HOPF ALG H
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DUALITY ON CATEGORIES

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RIGID

PIVOTAL

CAN MEASURE SIZE
OF OBJECTS VIA

$$\dim_j X = \text{tr}_{X^*}^{X^*} \left(\begin{array}{c} X^* \\ \downarrow j^{-1} \\ X \end{array} \right)$$

"PIV./QUANTUM DIM'N"

$\exists$ MONOIDAL
NAT'L ISOM.

$$j: \text{Id}_{\mathcal{C}} \xrightarrow{\otimes} (-)^{**}$$

FOR ANTIPODE S OF H :

$$(h \triangleright f)(v) := f(S(h) \triangleright v)$$

$h \in H, f \in V^*, v \in V$

SPHERICAL

$$\dim_j X = \dim_j X^* \quad \forall X$$

& MORE

$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C}$ WITH
 $\text{ev}_X^L: X^* \otimes X \rightarrow 1$ 
 $\text{coev}_X^L: 1 \rightarrow X \otimes X^*$ 

$\exists.$

$$X \text{ (cup) }^X = |^X \quad \& \quad X^* \text{ (cap) }_{X^*} = |_{X^*}$$

ALSO HAVE $({}^*X, \text{ev}_X^R, \text{coev}_X^R)$

FEATURE OF
H-FdMod

FOR A $\mathbb{K}$ -HOPF ALG. H

FEATURE OF
H-FdMod

FOR A "PIVOTAL" $\mathbb{K}$ -HOPF ALG H
 E.G., WITH $S^2 = \text{id}_H$

DUALITY ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

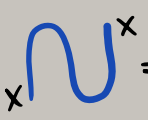
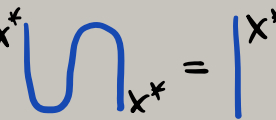
MONOIDAL

$\exists \otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, 1 \in \mathcal{C}$
 $\exists. (\mathcal{C}, \otimes, 1) \approx \text{"MONOID"}$

RIGID

PIVOTAL

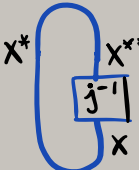
$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C}$ WITH
 $ev_X^L: X^* \otimes X \rightarrow 1$ 
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$\exists.$
 $=$ 
 ALSO HAVE $({}^*X, ev_X^R, coev_X^R)$

$\exists$ MONOIDAL
 NAT'L ISOM.

$j: Id_{\mathcal{C}} \xrightarrow{\otimes} (-)^{**}$

CAN MEASURE SIZE
 OF OBJECTS VIA

$dim_j X = \text{tr}(j^{-1})$ 

"PIV./QUANTUM DIM'N"

SPHERICAL

$dim_j X = dim_j X^* \forall X$
 $\& \text{ MORE}$

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FEATURE OF
 $H\text{-FdMod}$
 FOR A $\mathbb{K}$ -HOPF ALG. H

FEATURE OF
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 E.G., WITH $S^2 = id_H$

FEATURE OF
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 FOR A "SPHERICAL"
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DUALITY ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS...

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$$(\mathcal{C}, \otimes, \mathbb{1}) \approx \text{"MONOID"}$$

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$$\forall X \in \mathcal{C}: \exists X^* \in \mathcal{C} \text{ WITH}$$

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$\exists.$

$$x \text{ wavy }^X = |^X \quad \neq \quad x^* \text{ wavy }_{X^*} = |_{X^*}$$

ALSO HAVE $({}^*X, \text{ev}_X^R, \text{coev}_X^R)$

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CAN MEASURE SIZE
OF OBJECTS VIA

$$\dim_j X = \begin{array}{c} X^* \\ \cup \\ j^{-1} \\ \cup \\ X \end{array} X^{**}$$

"PIV./QUANTUM DIM'N"

SPHERICAL

$$\dim_j X = \dim_j X^* \quad \forall X$$

$\neq$ MORE

FOR ANTIPODE S OF H :

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FEATURE OF
H-FdMod
FOR A $\mathbb{K}$ -HOPF ALG. H

FEATURE OF
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FOR A "PIVOTAL" $\mathbb{K}$ -HOPF ALG H
E.G., WITH $S^2 = \text{id}_H$

FEATURE OF
H-FdMod
FOR A "SPHERICAL"
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DUALITY ON CATEGORIES

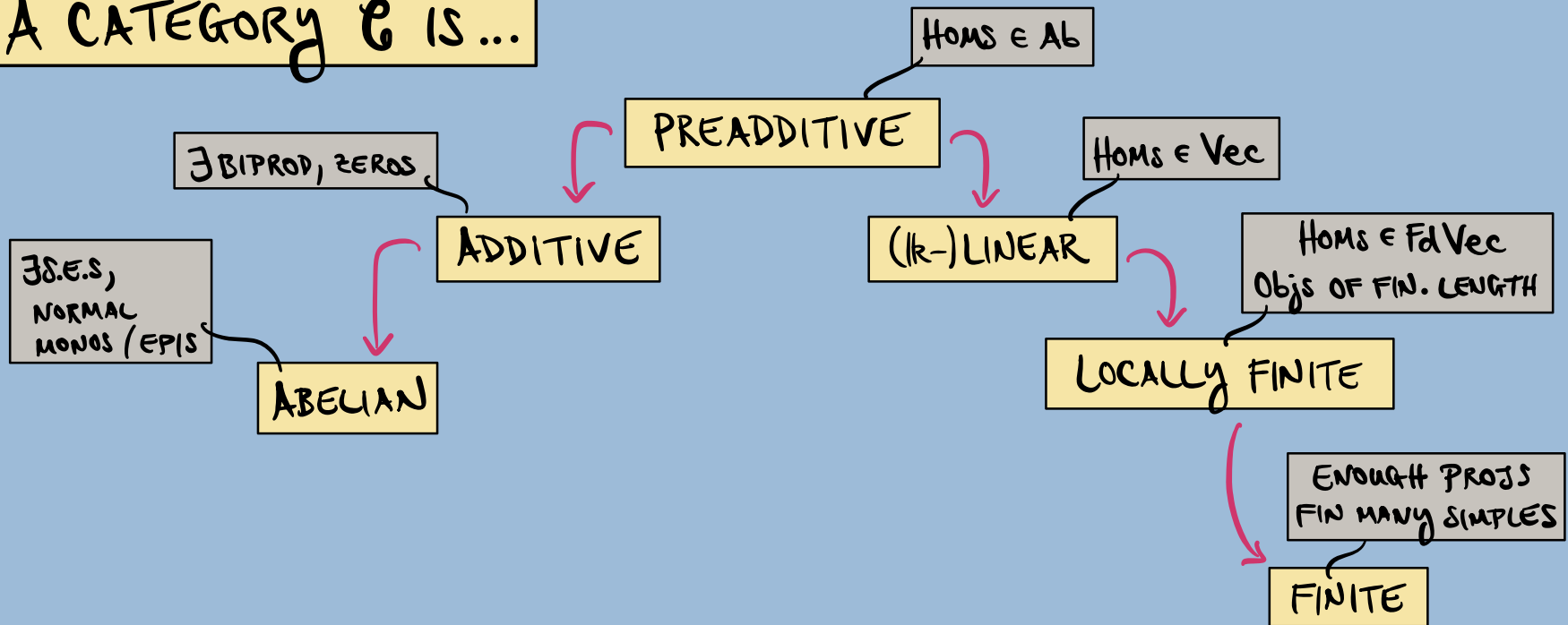
A CATEGORY $\mathcal{C}$ IS ...

$\equiv$ ON THE ROAD $\equiv$
TO FUSION

DUALITY ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

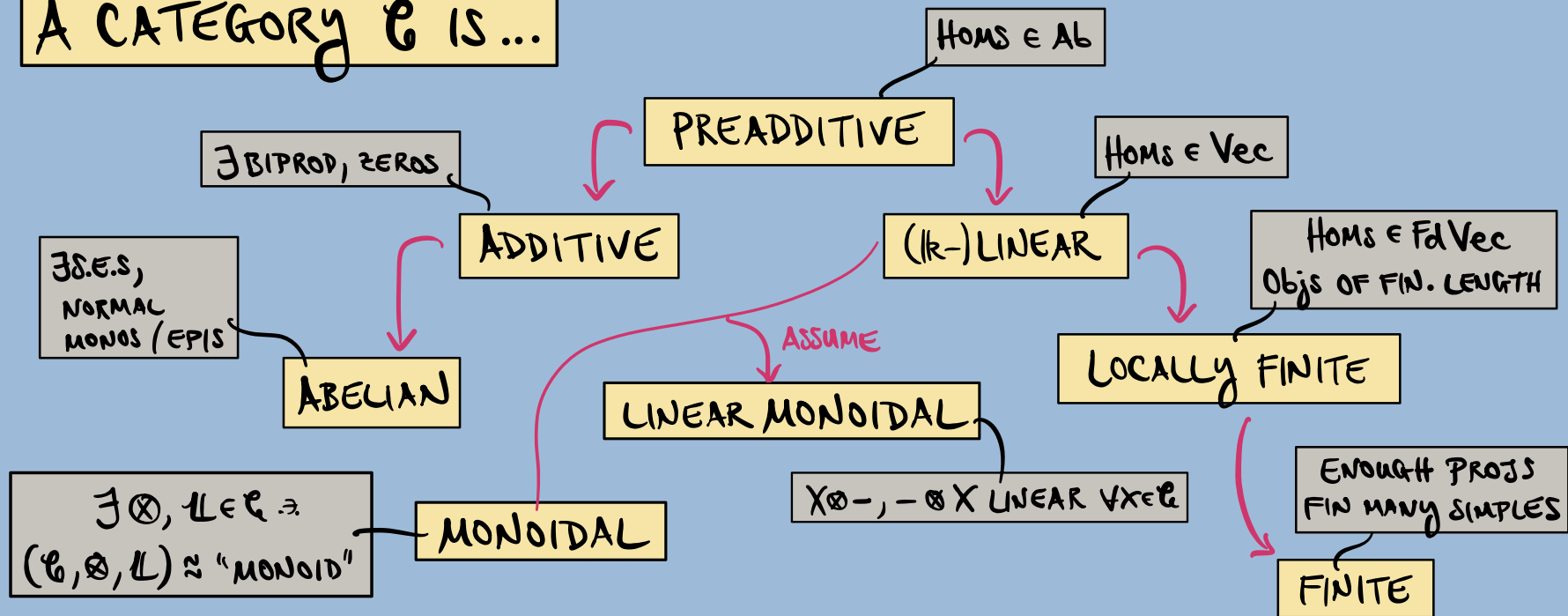
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DUALITY ON CATEGORIES

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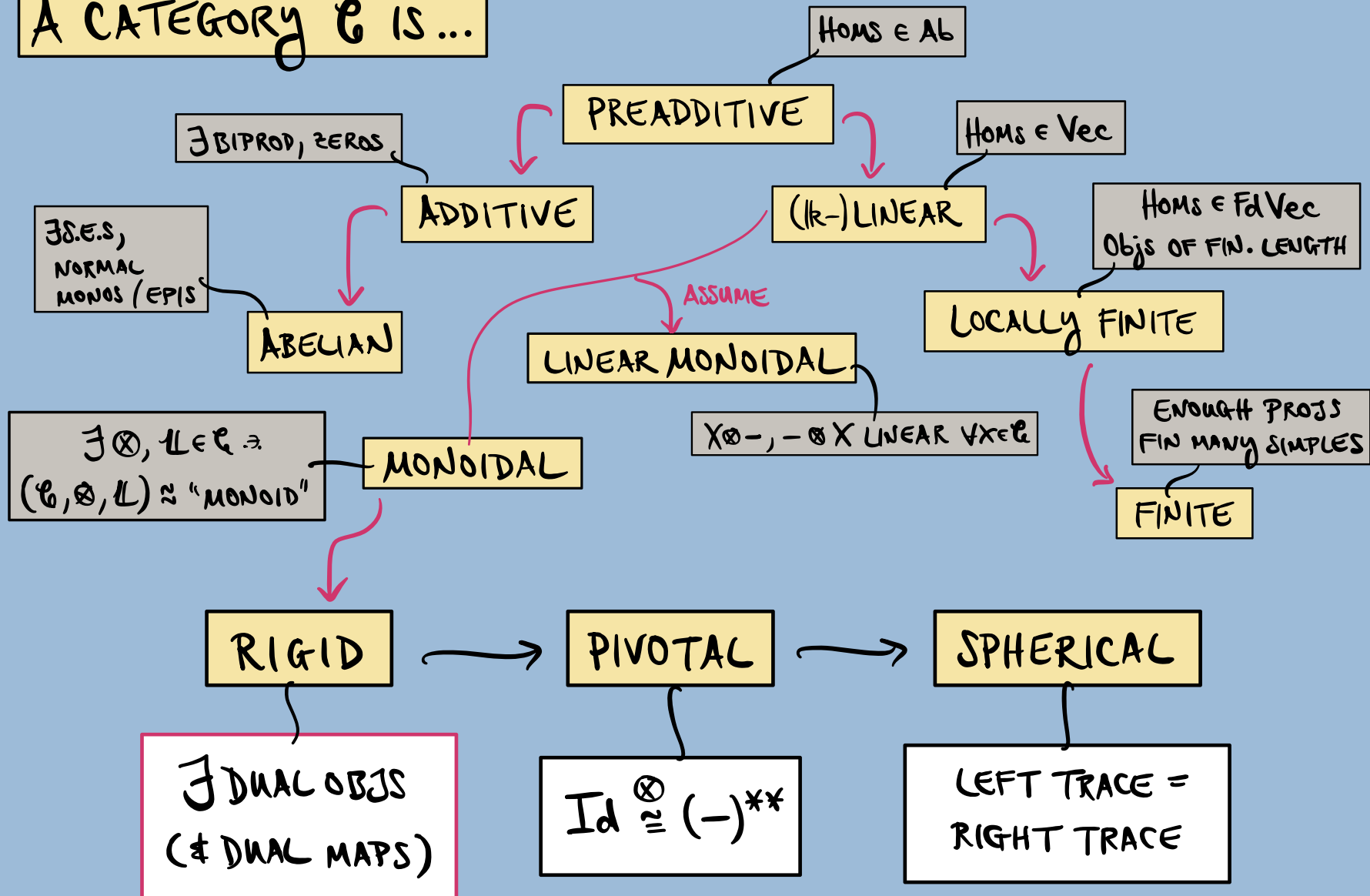
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DUALITY ON CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

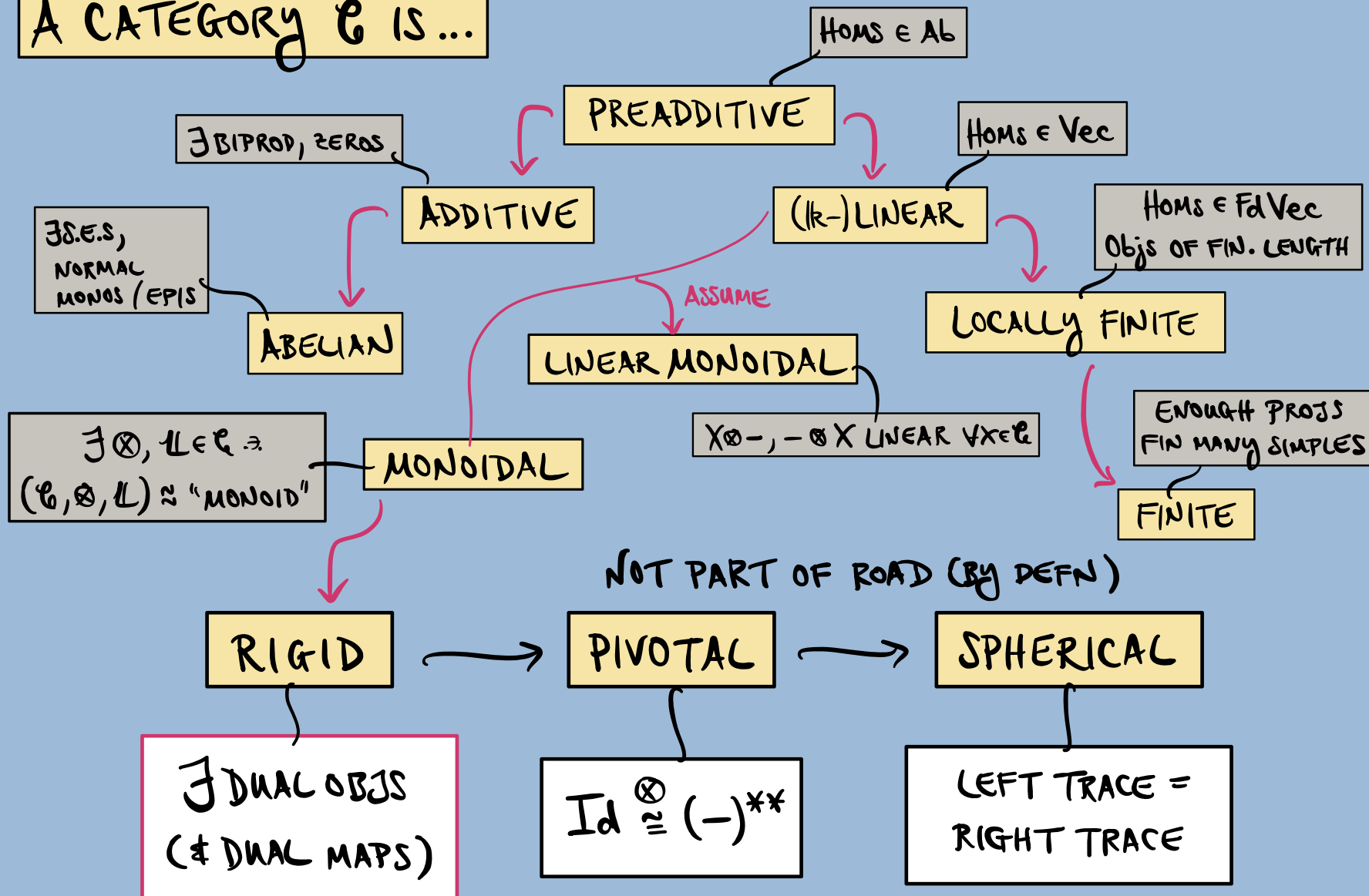
≡ ON THE ROAD
TO FUSION ≡



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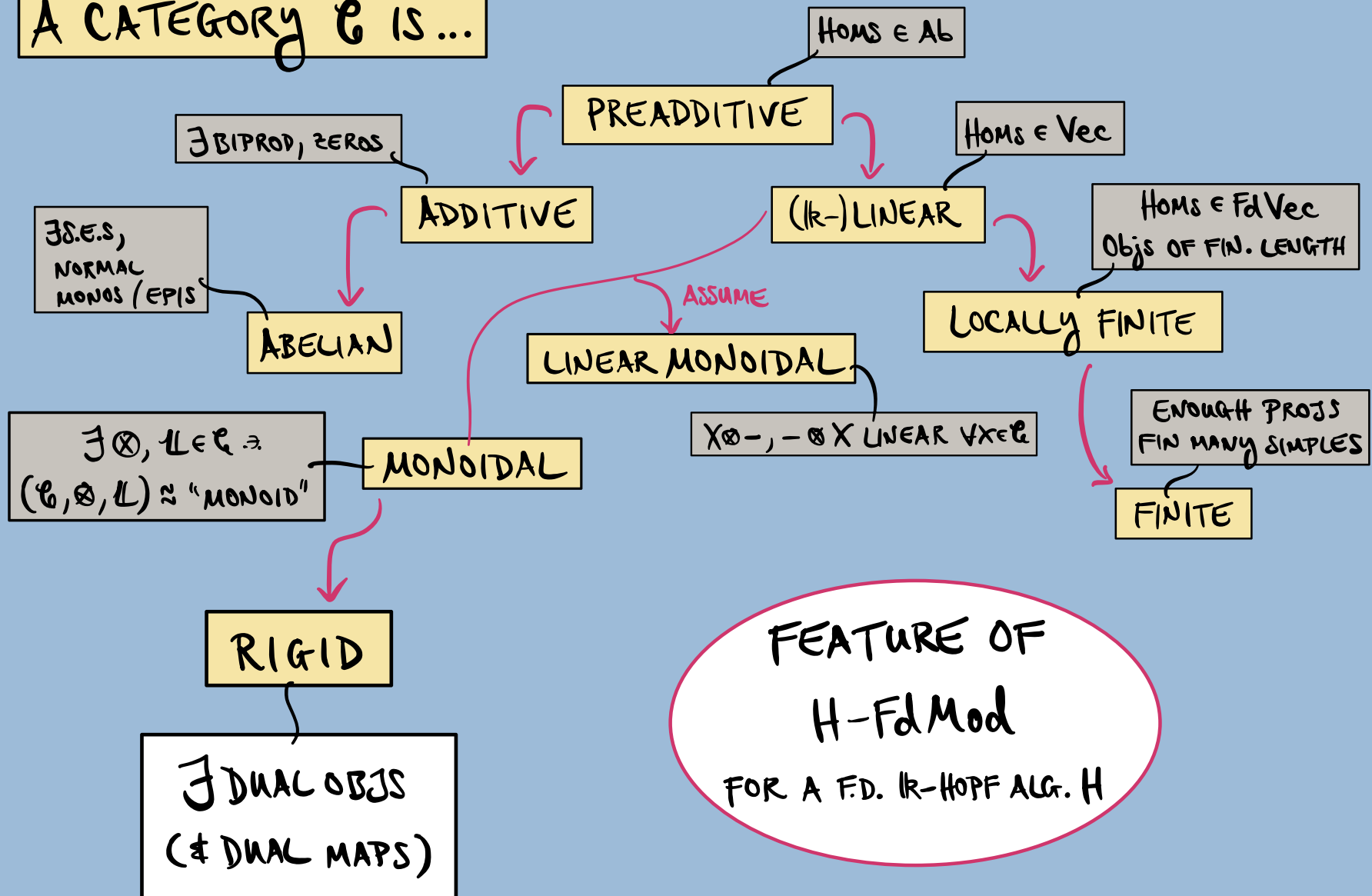
$\equiv$ ON THE ROAD $\equiv$
TO FUSION



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≡ ON THE ROAD
TO FUSION ≡

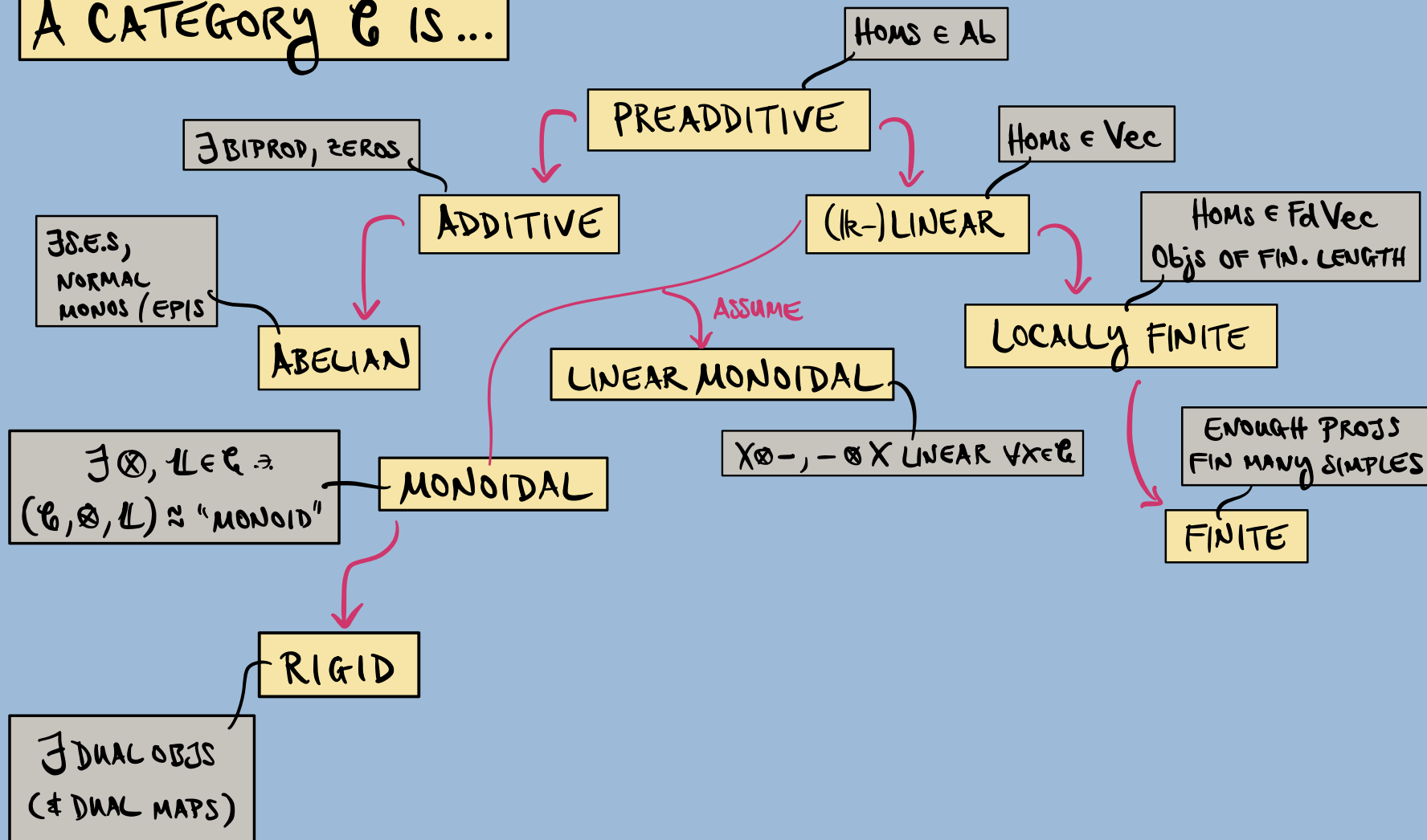


FEATURE OF
 $H\text{-FdMod}$
FOR A F.D. K -HOPF ALG. H

TENSOR CATEGORIES

A CATEGORY $\mathcal{C}$ IS ...

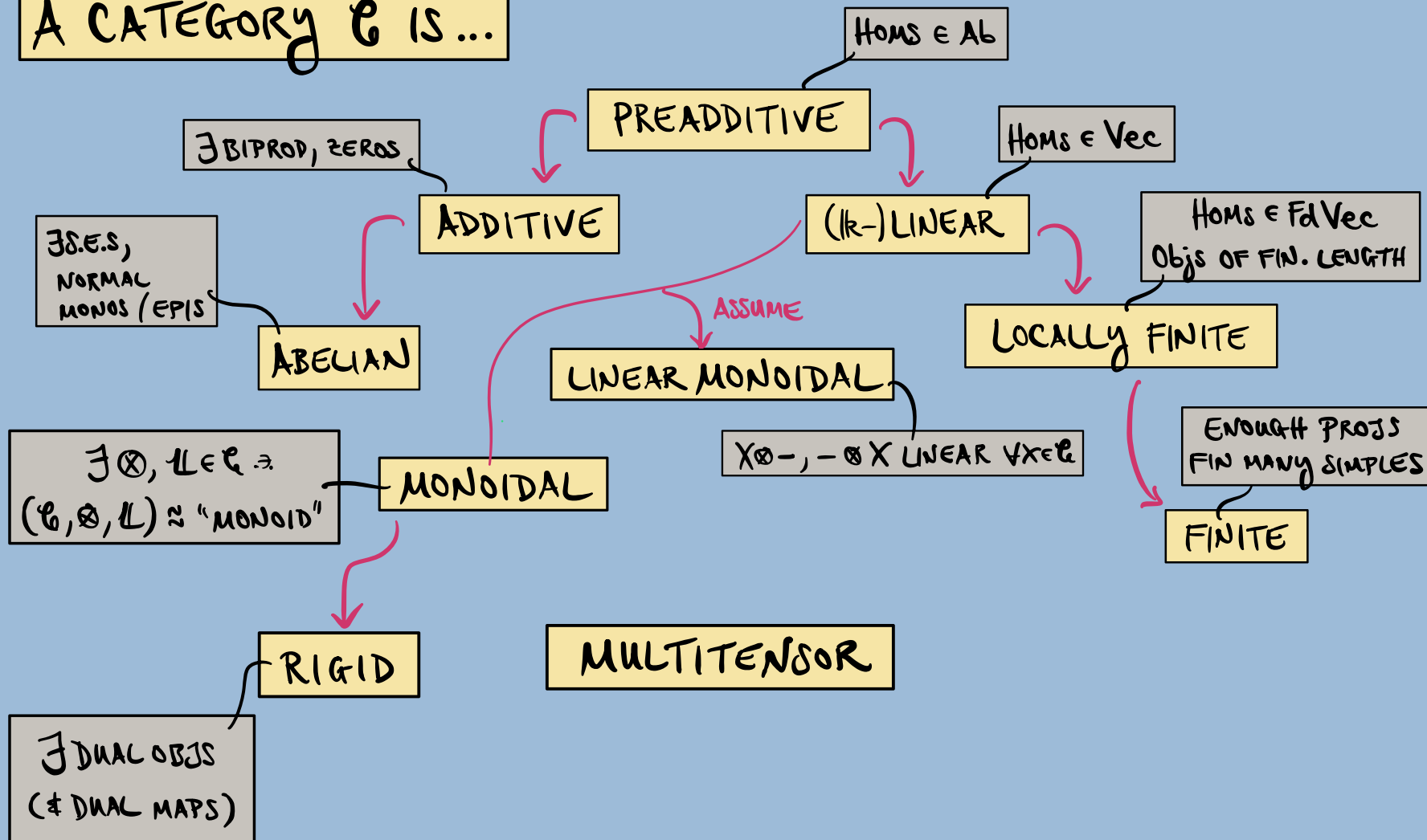
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TENSOR CATEGORIES

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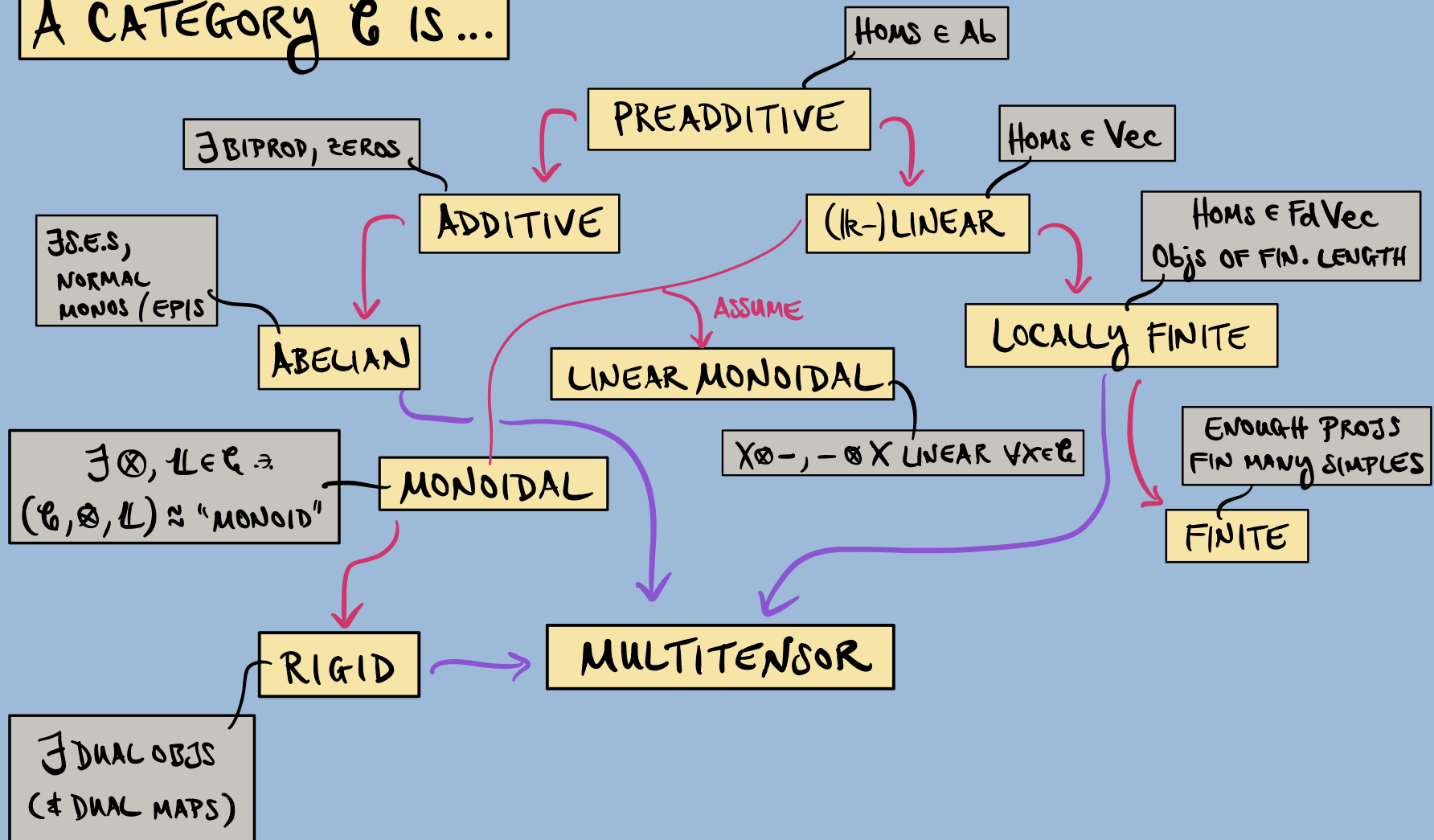
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TO FUSION ≡



TENSOR CATEGORIES

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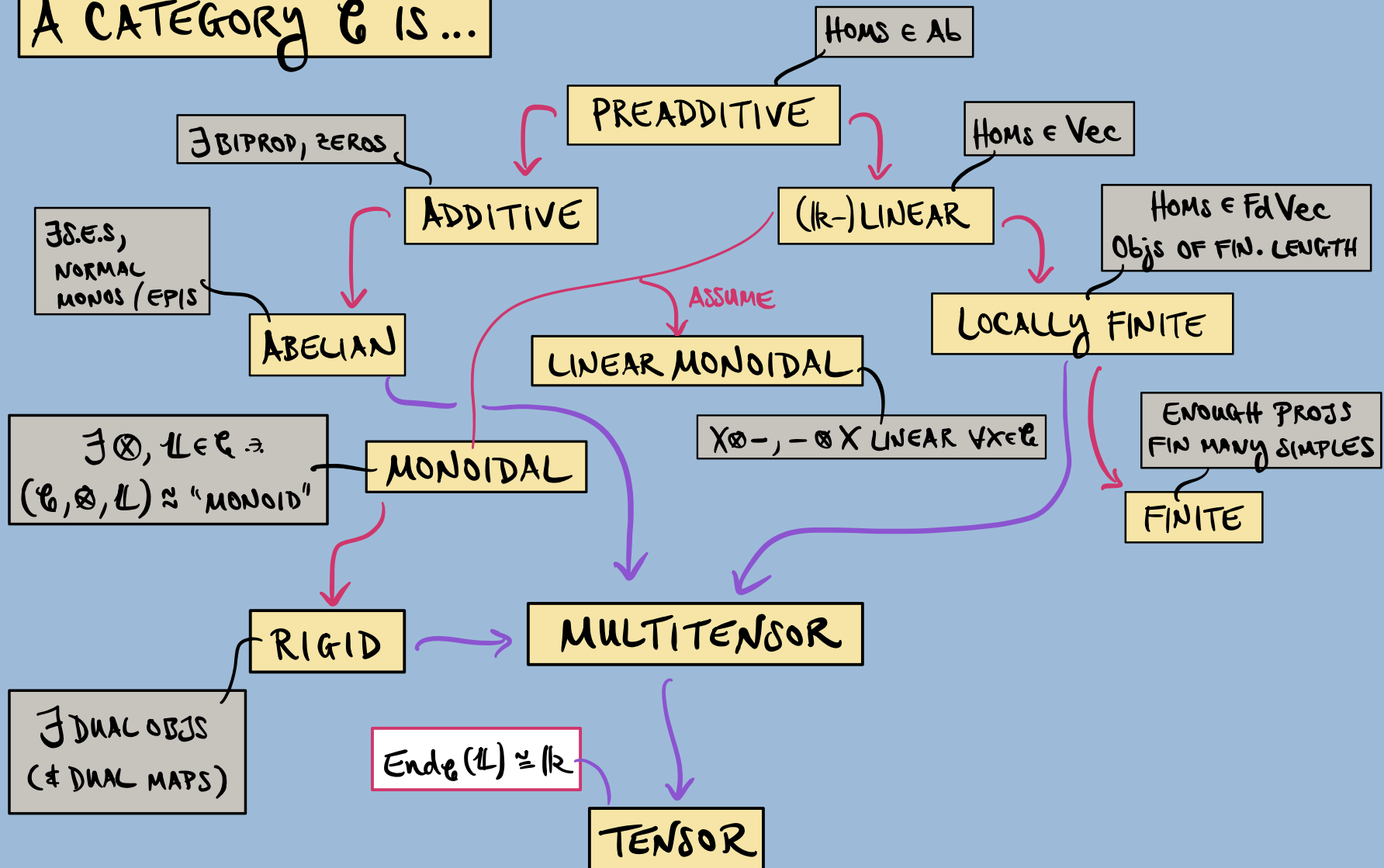
≡ ON THE ROAD
TO FUSION ≡



TENSOR CATEGORIES

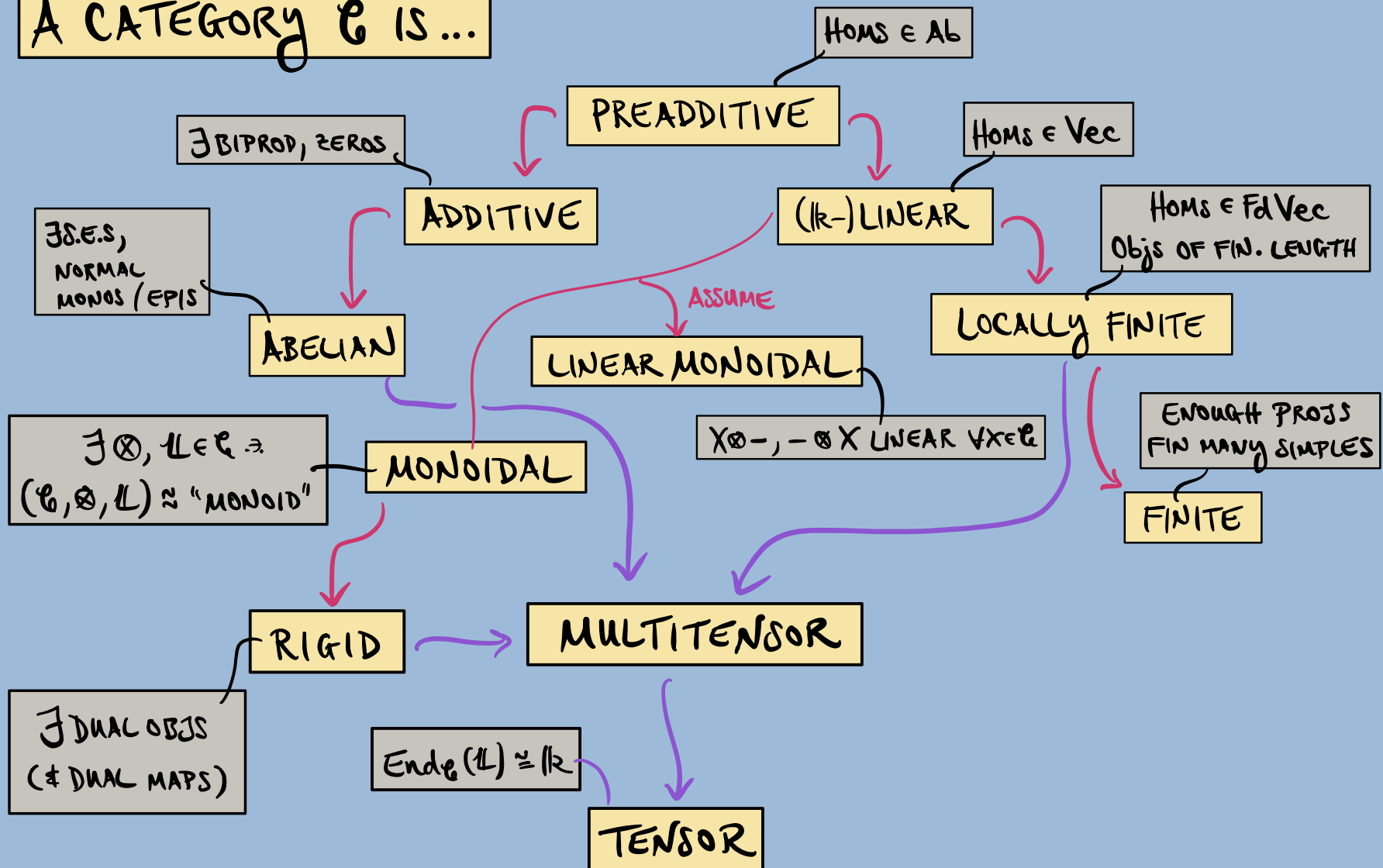
A CATEGORY $\mathcal{C}$ IS ...

≡ ON THE ROAD TO FUSION ≡



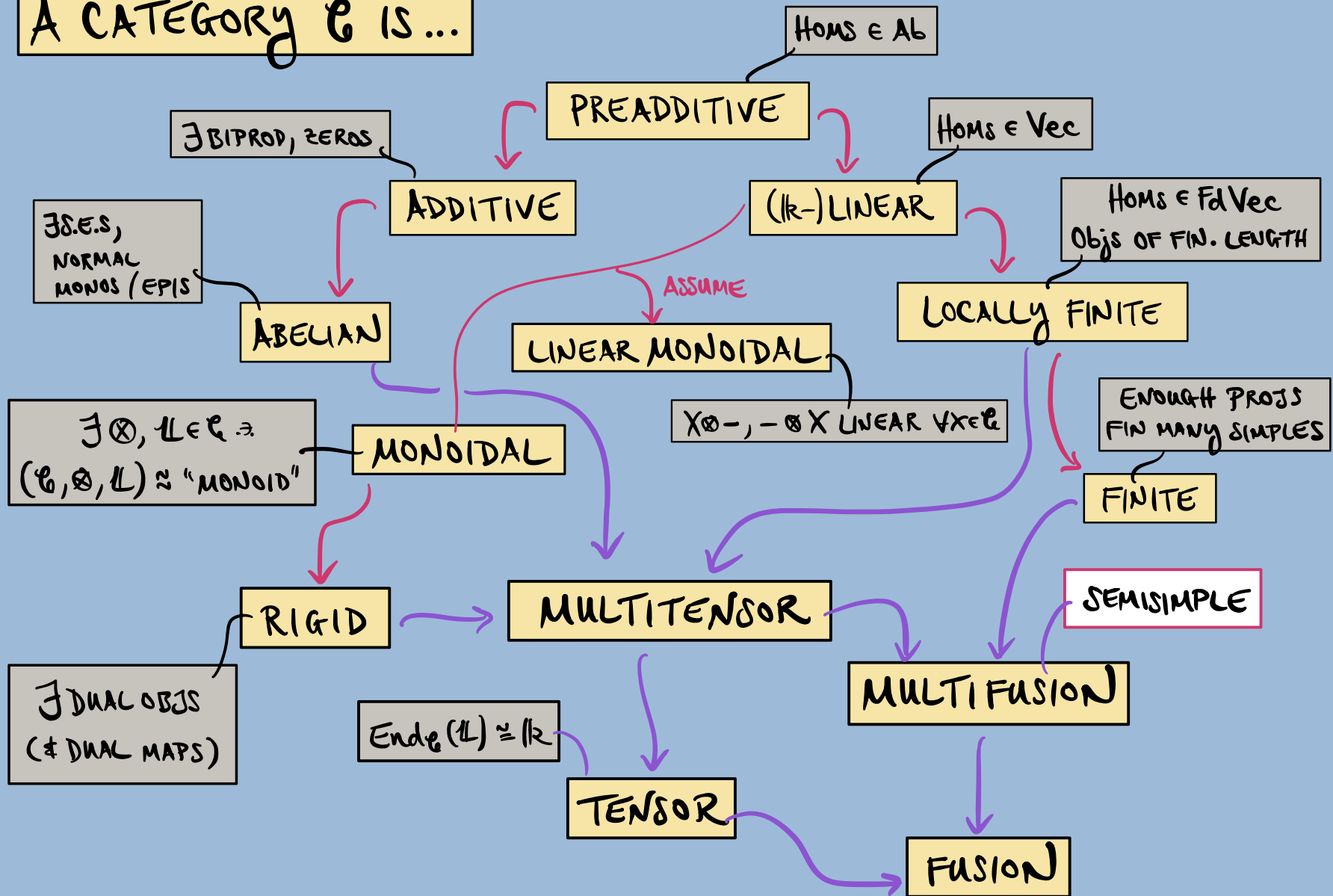
FUSION CATEGORIES!

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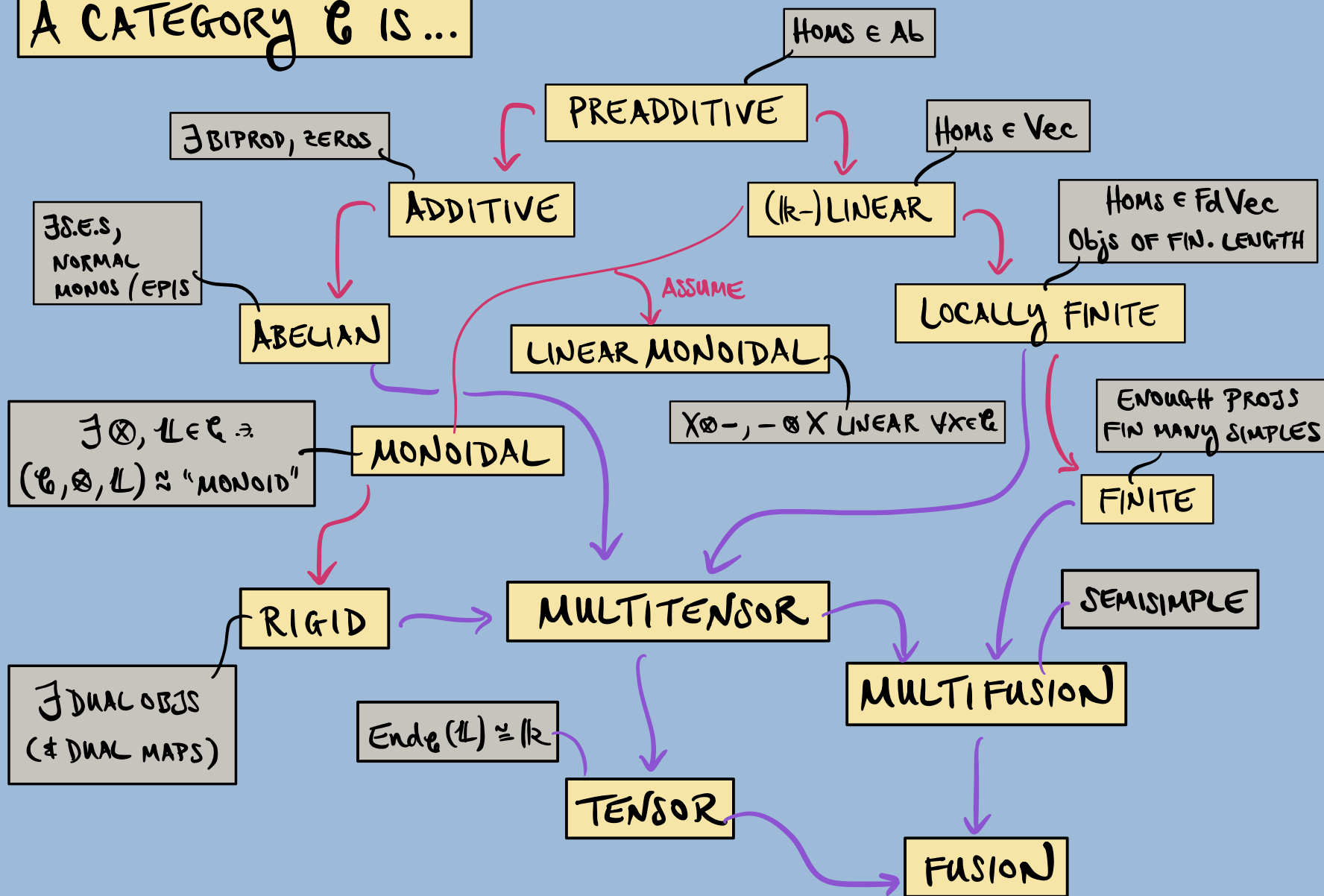
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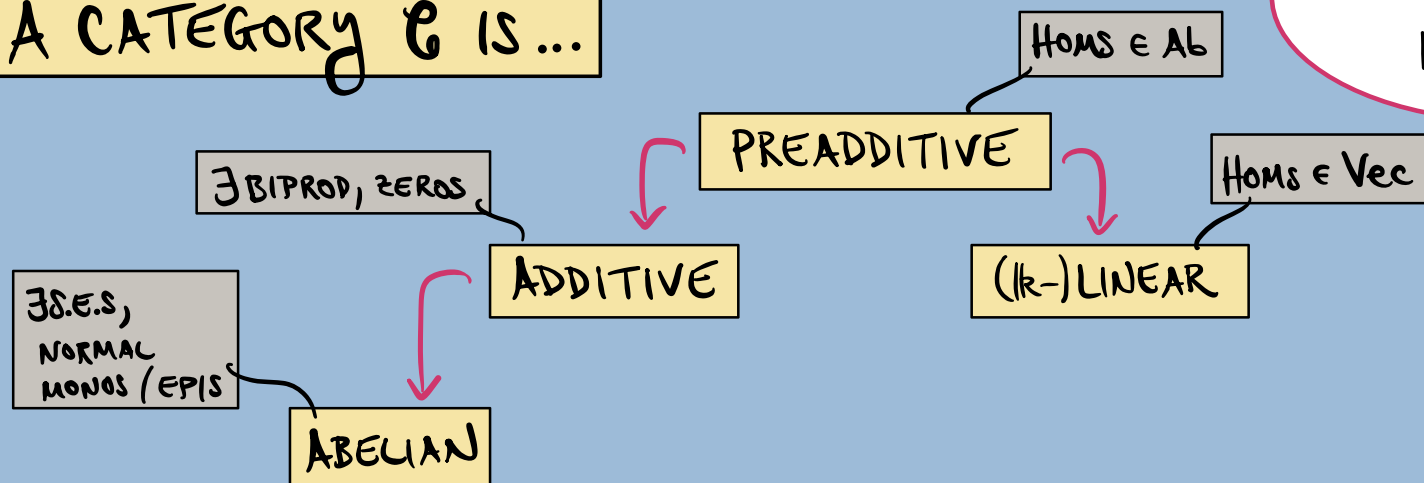
FUSION CATEGORIES!

A CATEGORY $\mathcal{C}$ IS ...



ON CATEGORIES OF REPS

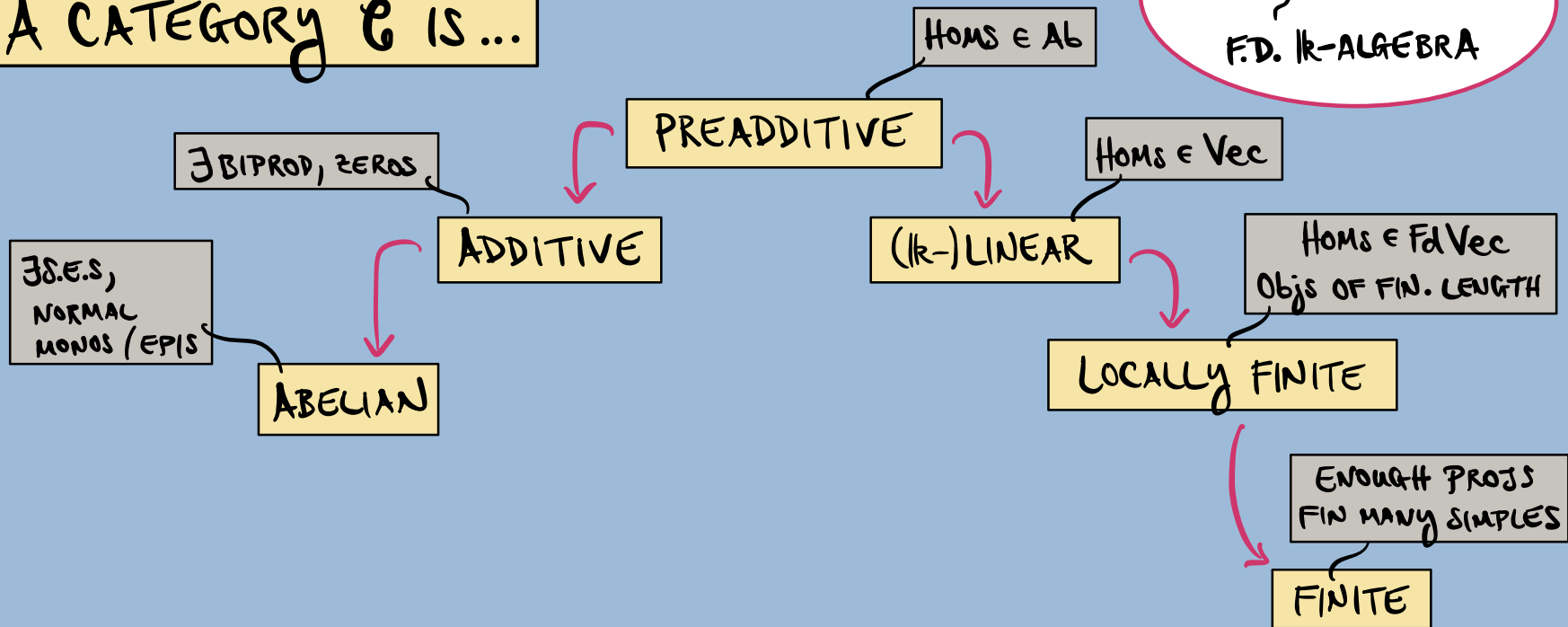
A CATEGORY $\mathcal{C}$ IS ...



Ex. $A\text{-Mod}$
 $\uparrow$
 $R\text{-ALGEBRA}$

ON CATEGORIES OF REPS

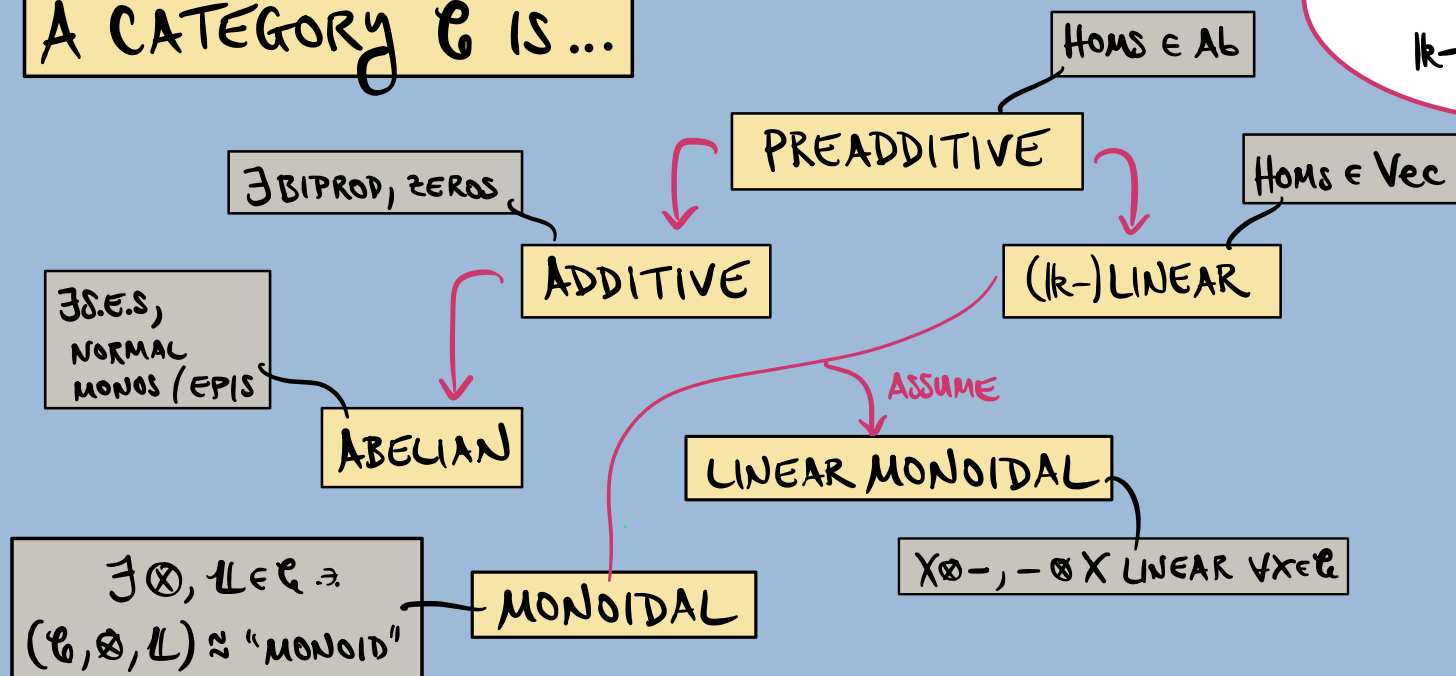
A CATEGORY $\mathcal{C}$ IS ...



Ex. $A\text{-FdMod}$
 $\uparrow$
F.D. K -ALGEBRA

ON CATEGORIES OF REPS

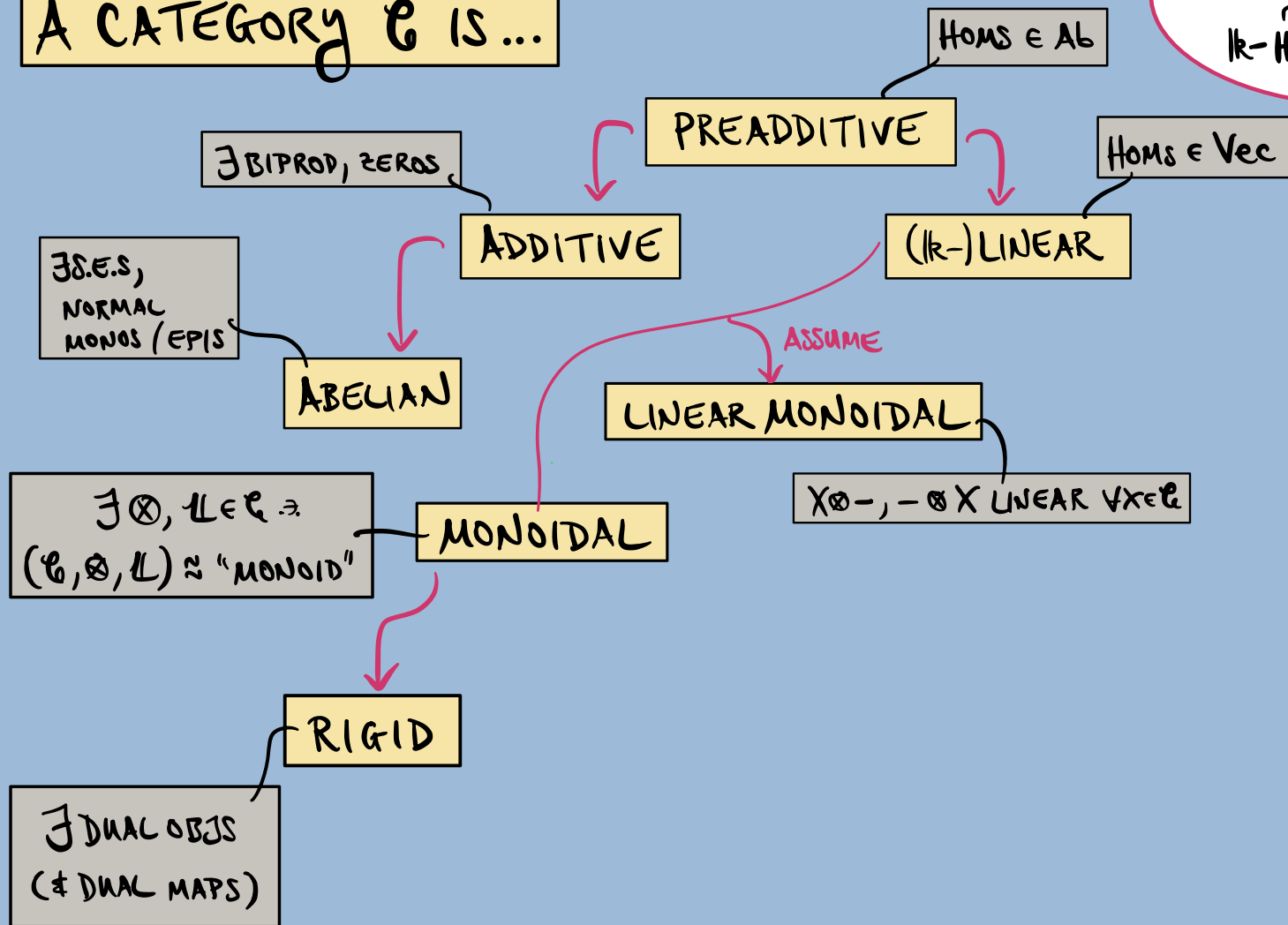
A CATEGORY $\mathcal{C}$ IS ...



Ex. $\mathcal{B}\text{-Mod}$
 $\uparrow$
 $\mathbb{K}\text{-BIALGEBRA}$

ON CATEGORIES OF REPS

A CATEGORY $\mathcal{C}$ IS ...

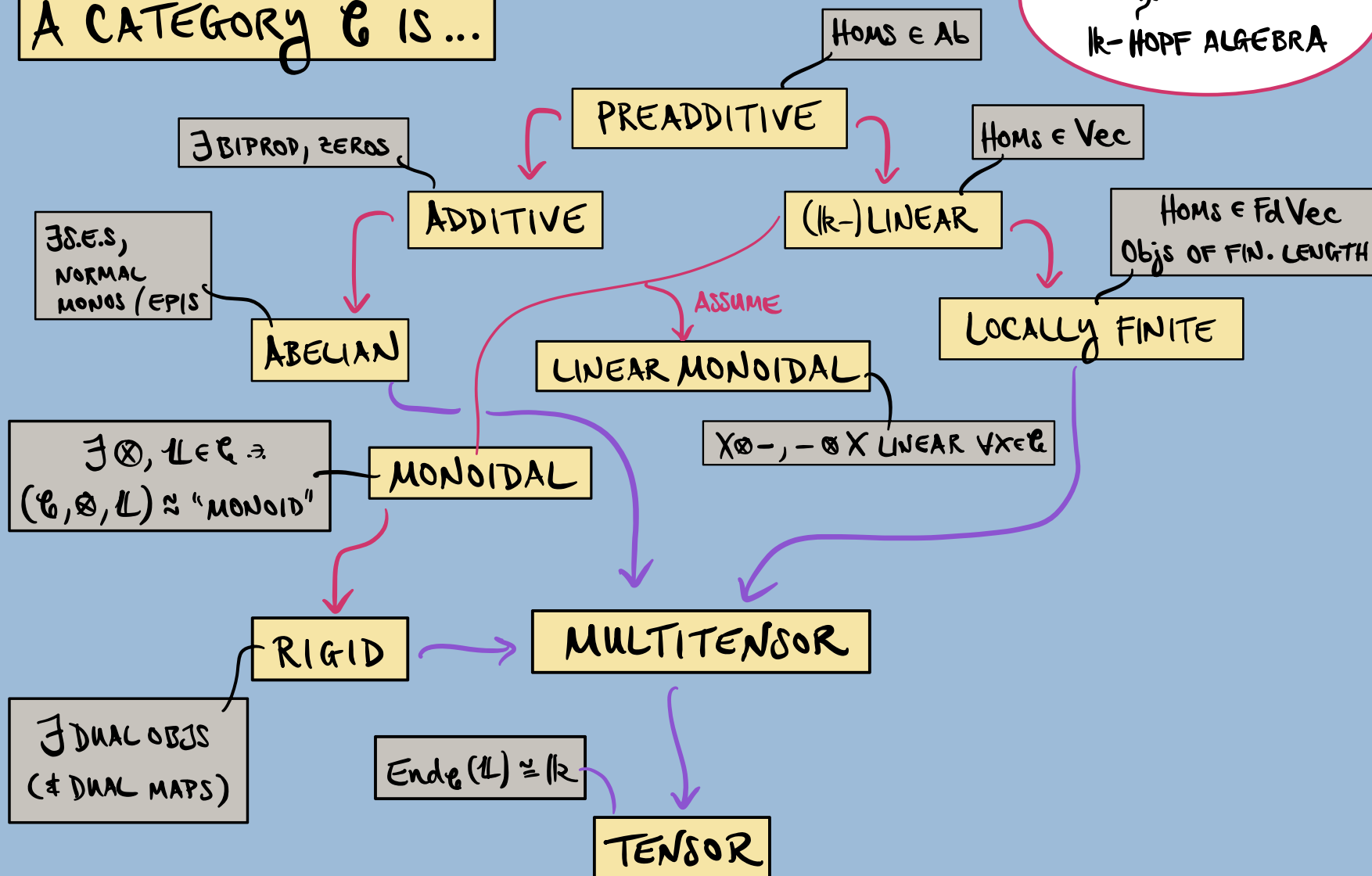


Ex. $H\text{-FdMod}$
 $\uparrow$
 $k\text{-HOPF ALGEBRA}$

ON CATEGORIES OF REPS

A CATEGORY $\mathcal{C}$ IS ...

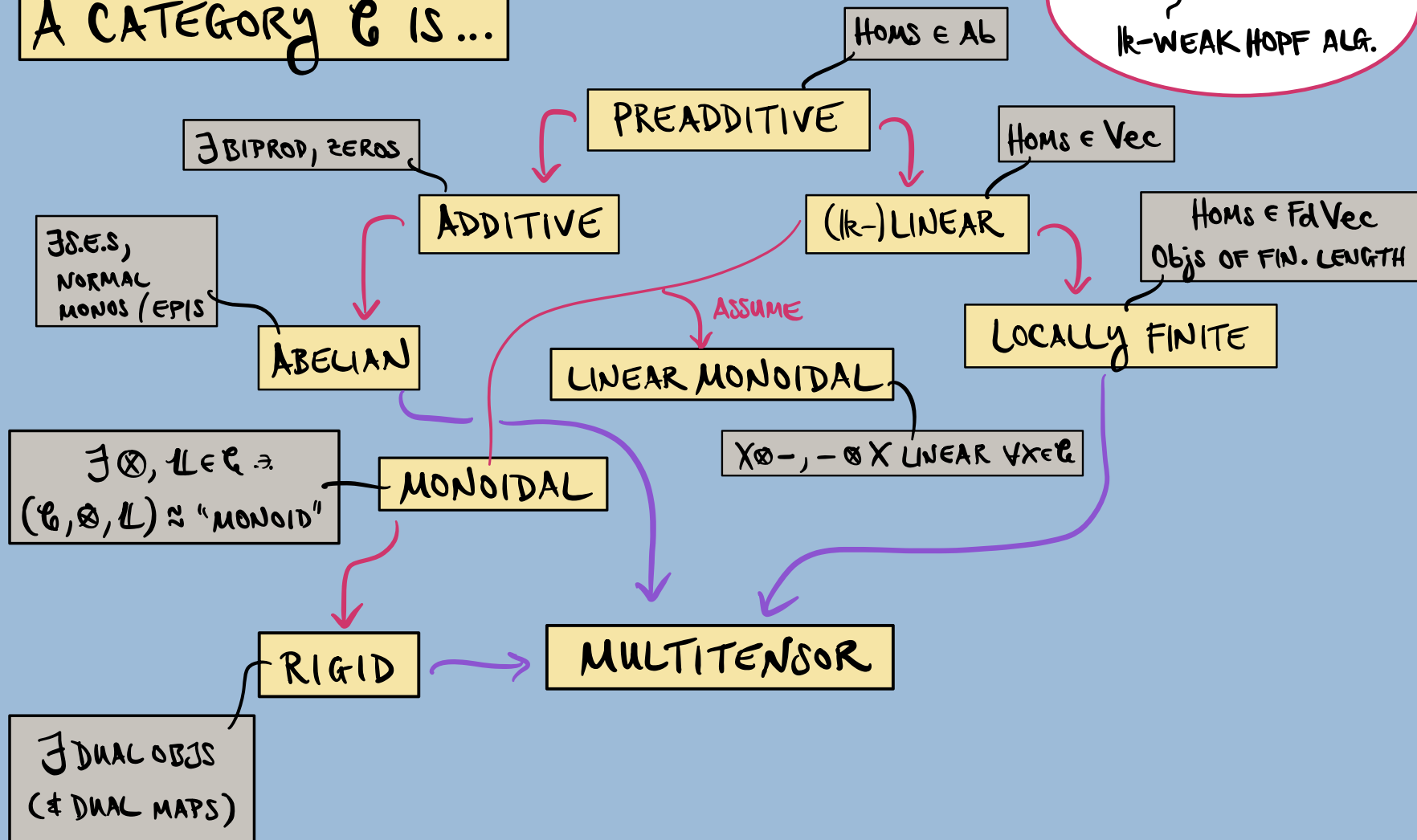
Ex. $H\text{-FdMod}$
 $\uparrow$
 $\mathbb{K}\text{-HOPF ALGEBRA}$



ON CATEGORIES OF REPS

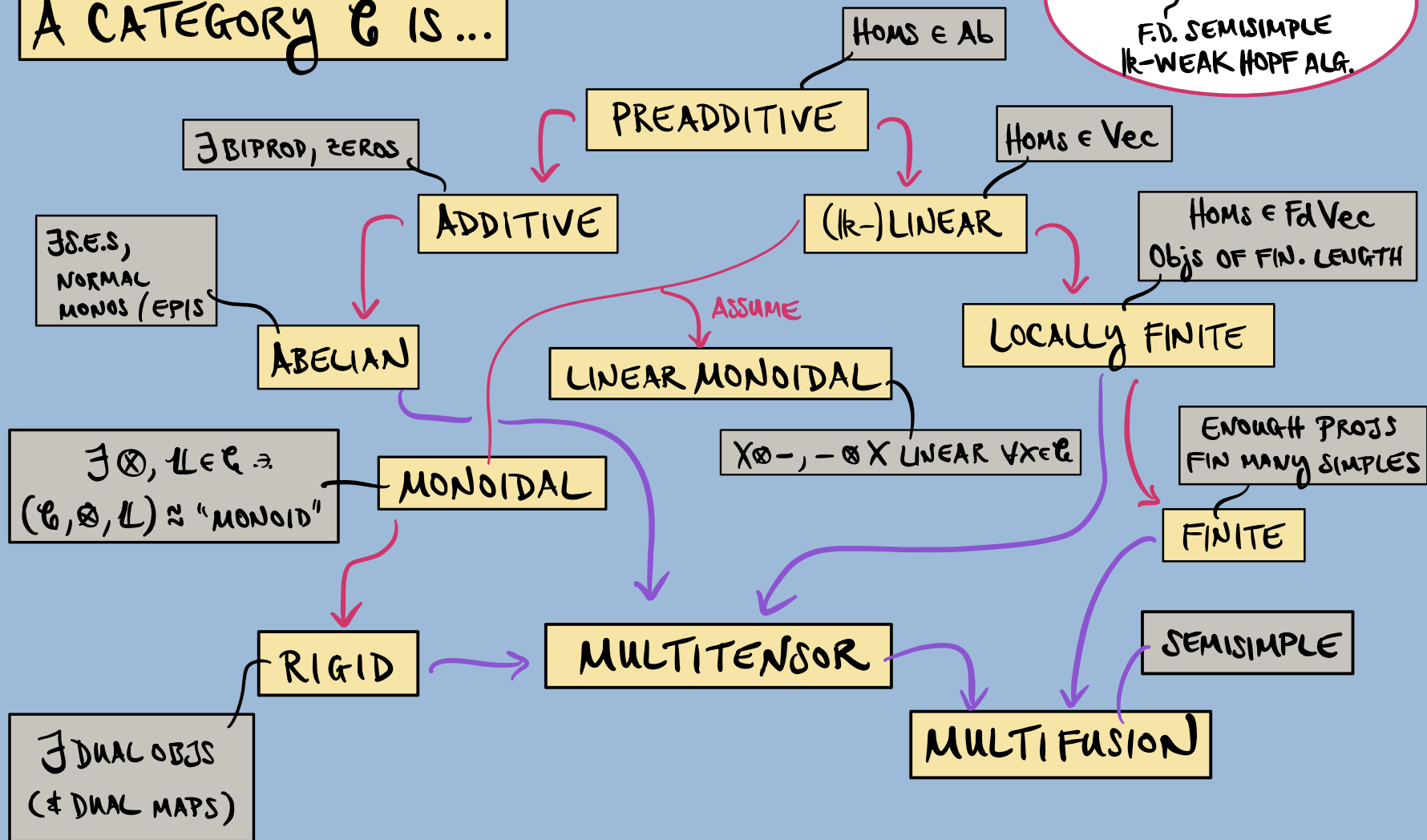
A CATEGORY $\mathcal{C}$ IS ...

Ex. $H\text{-FdMod}$
 $\uparrow$
 $k\text{-WEAK HOPF ALG.}$



ON CATEGORIES OF REPS

A CATEGORY $\mathcal{C}$ IS ...

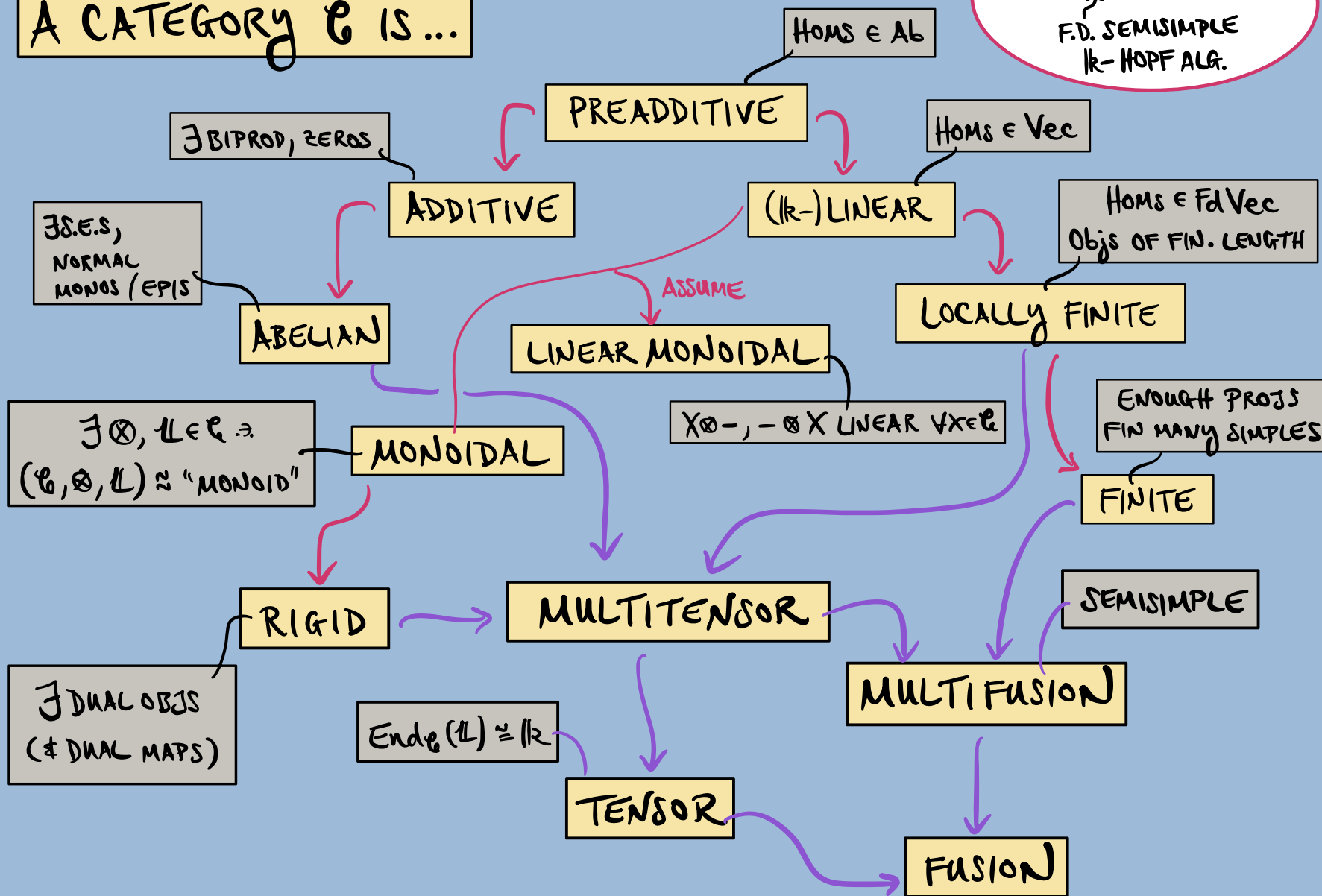


ON CATEGORIES OF REPS

A CATEGORY $\mathcal{C}$ IS ...

Ex. $H\text{-FdMod}$

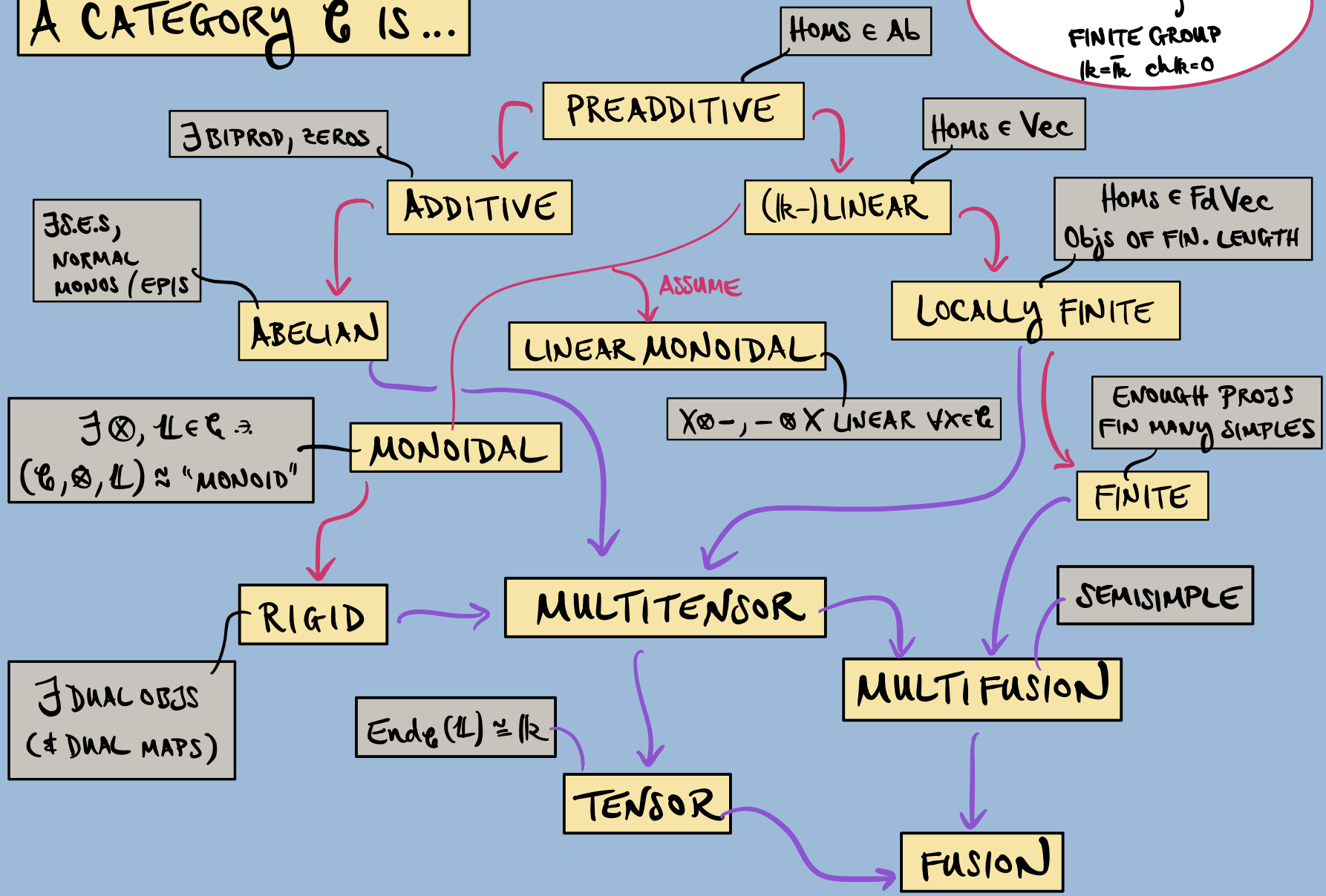
F.D. SEMISIMPLE
 $\mathbb{K}$ -HOPF ALG.



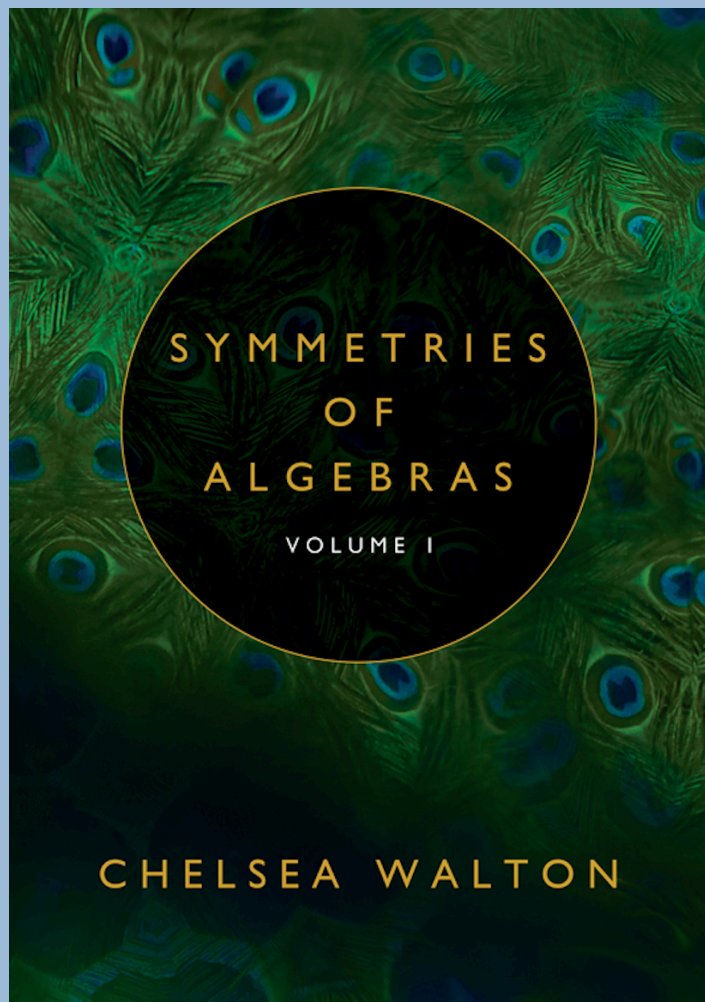
ON CATEGORIES OF REPS

A CATEGORY $\mathcal{C}$ IS ...

Ex. $\text{Rep}_k(G)$
 $\uparrow$
 FINITE GROUP
 $k = \bar{k}$ $\text{char } k = 0$



COMMERCIAL BREAK



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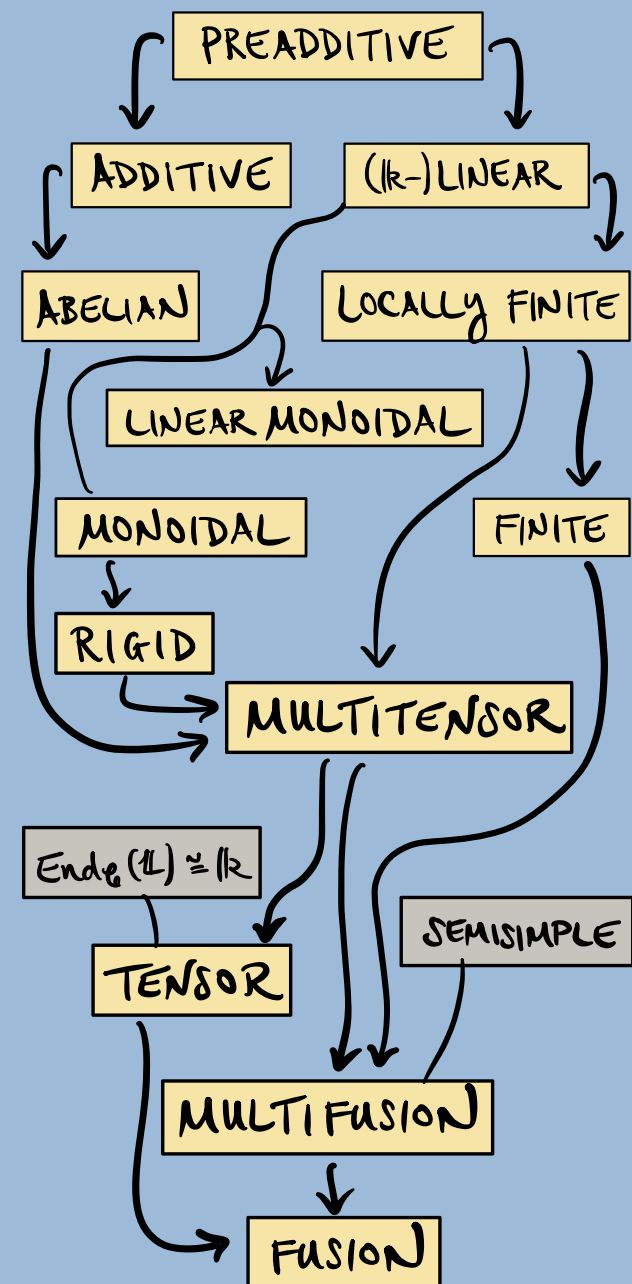
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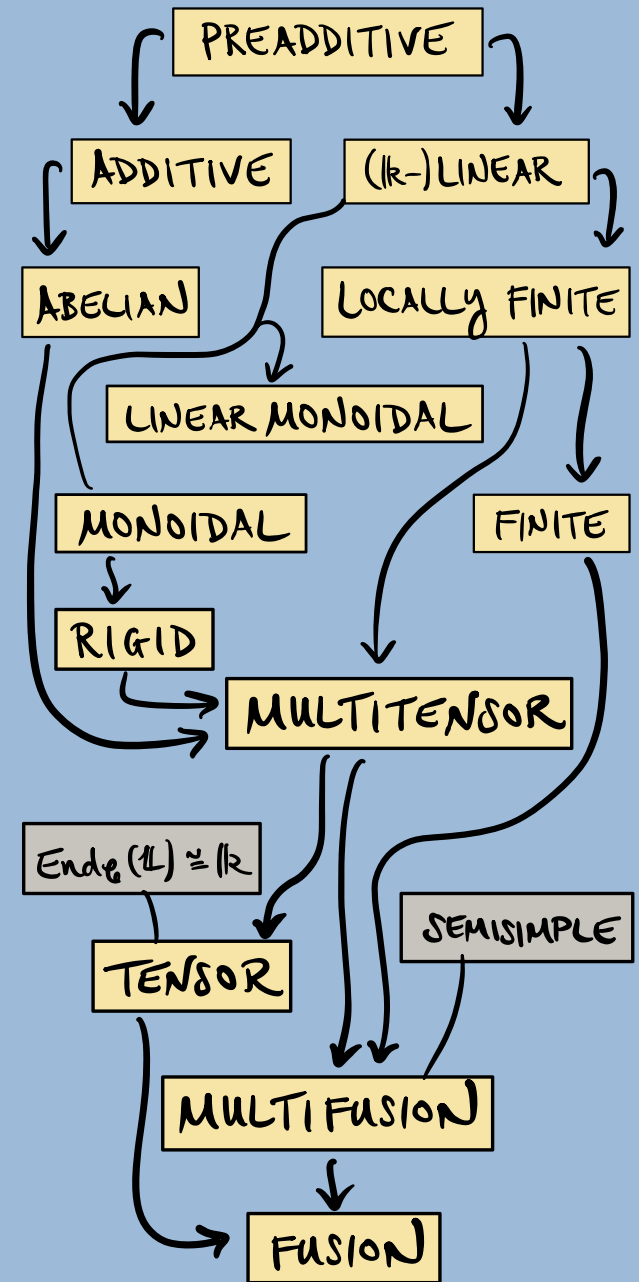
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GREATLY APPRECIATED ☺

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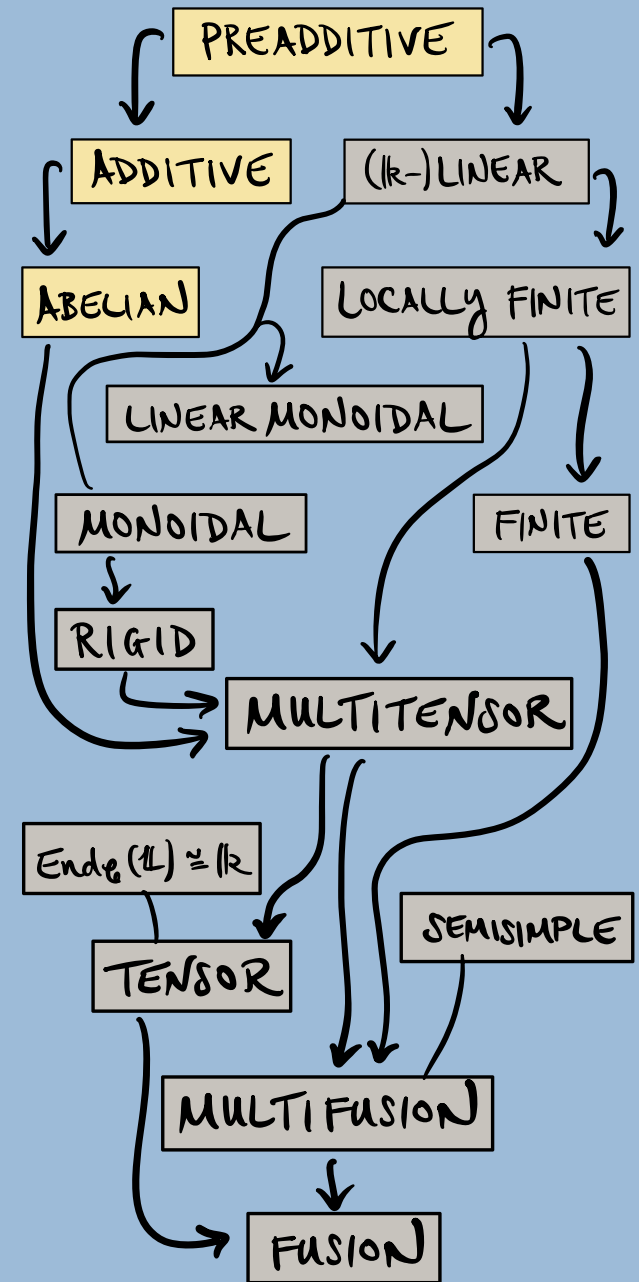
DETOURS OFF THE ROAD TO FUSION

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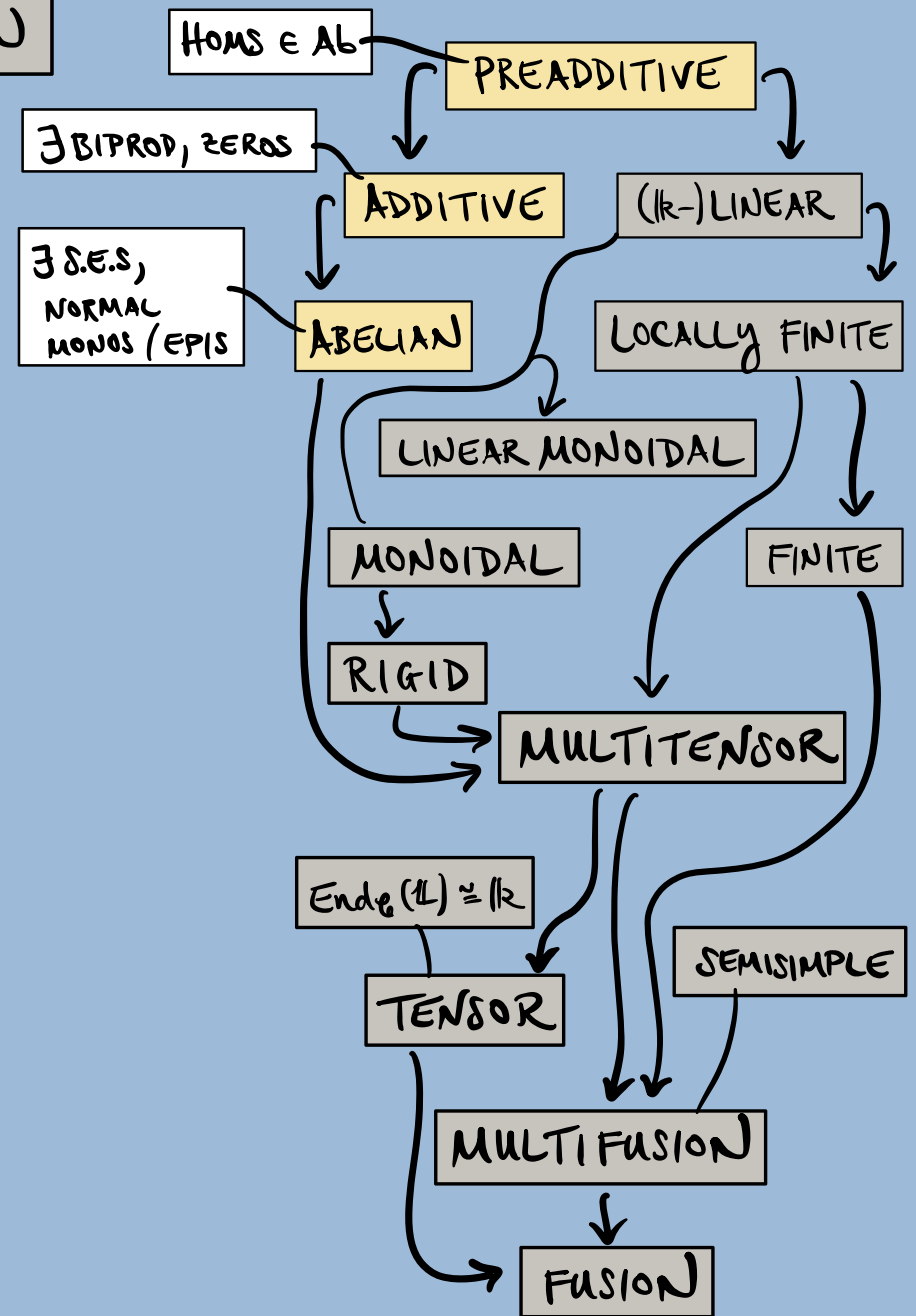
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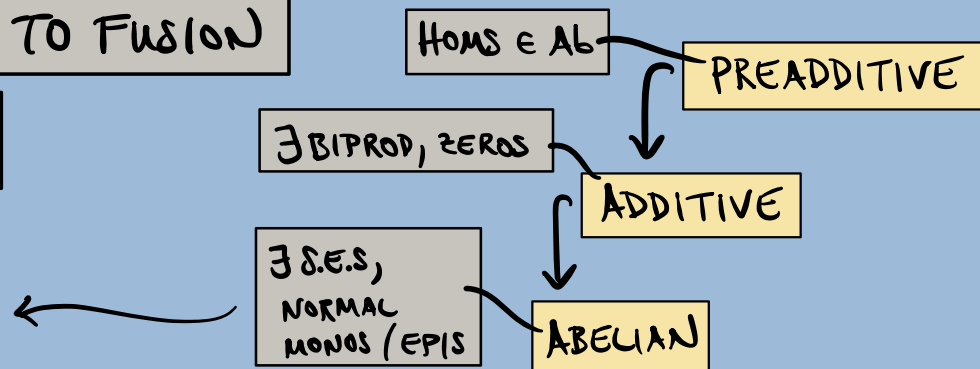
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DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$\exists$ PUSHOUTS & PULLBACKS,
 $\exists$ (CO)EQUALIZERS



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$\exists$ PUSHOUTS & PULLBACKS,
 $\exists$ (CO)EQUALIZERS

(CO) COMPLETE

$\exists$ (CO)PRODUCTS,
 $\exists$ (CO)EQUALIZERS

$\text{Homs} \in \text{Ab}$

PREADDITIVE

$\exists$ BIPROD, ZEROS

ADDITIVE

$\exists$ S.E.S,
NORMAL
MONOS / EPIS

ABELIAN



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EXAMPLES:

$A\text{-Mod}$

Set Group Ring Top
(NON-ABELIAN)

$\text{Homs} \in \text{Ab}$

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EXACT

(IN THE SENSE OF QUILLEN)

ADDITIVE &
 $\exists$ "EXACT TRIPLES"
 $X' \rightarrow X \rightarrow X''$

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$\left\{ \begin{array}{l} \text{KERNEL} \\ \text{COKERNEL} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \text{MONIC} \\ \text{EPIC} \end{array} \right\}$

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"ADMISSIBLE"

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Ex. $\text{Ab}^{\text{Tors(Free)}}$

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TRIANGULATED

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 $X' \rightarrow X \rightarrow X'' \rightarrow X'[1]$

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USED IN
DERIVED
CATEGORY THEORY

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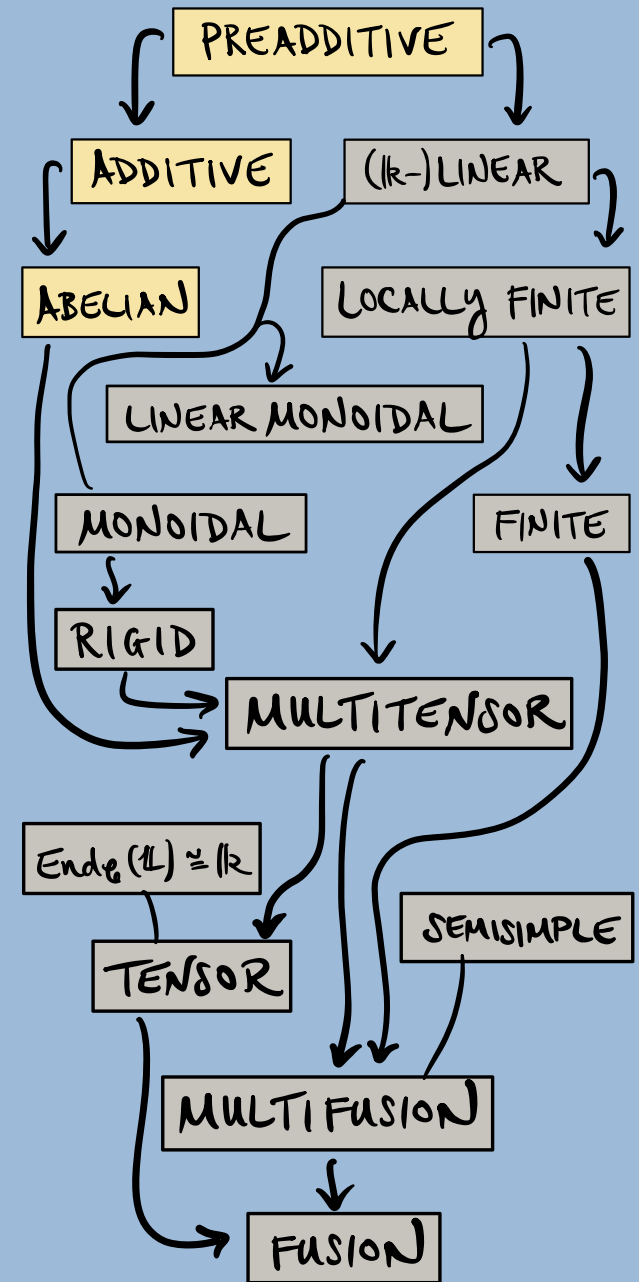
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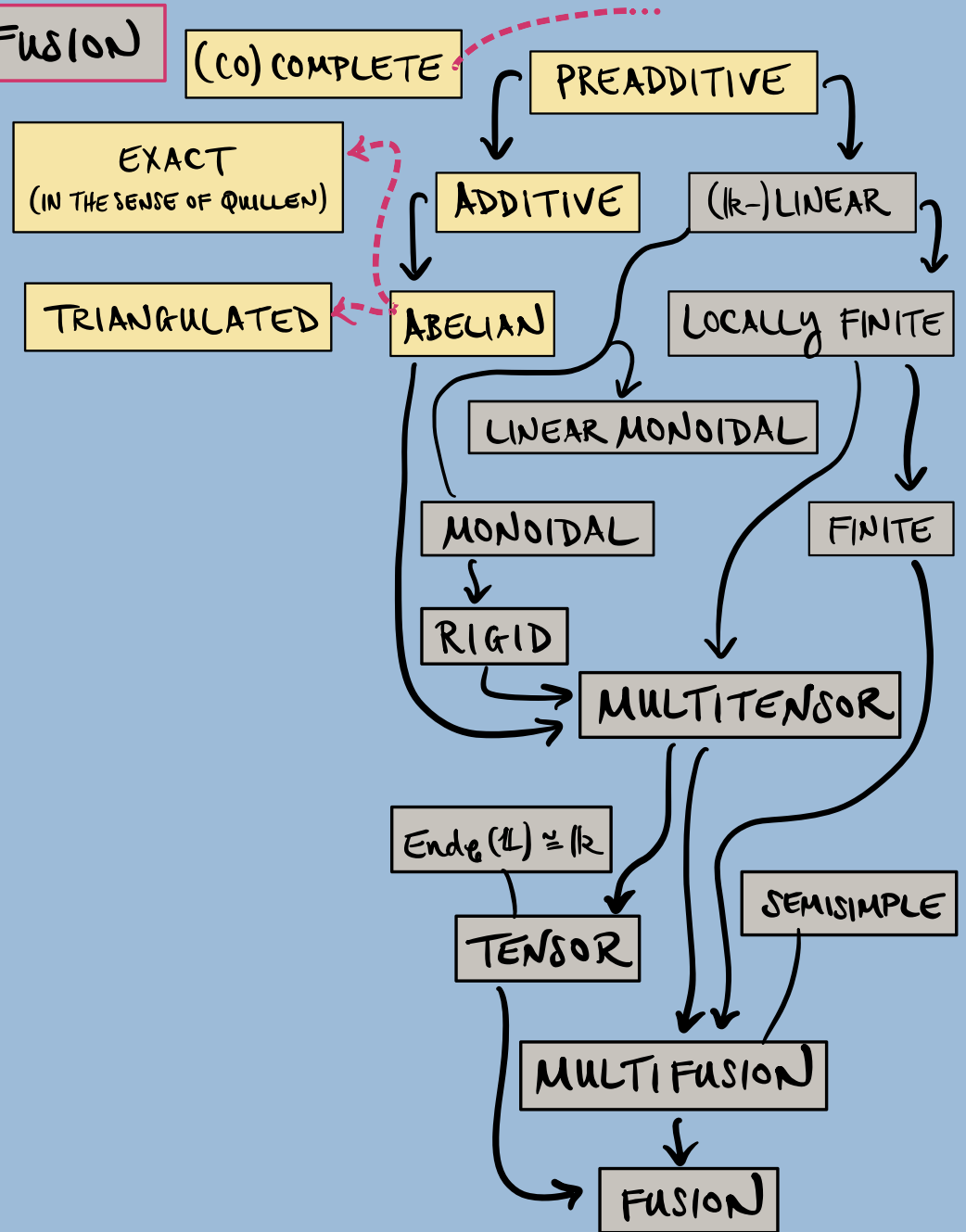
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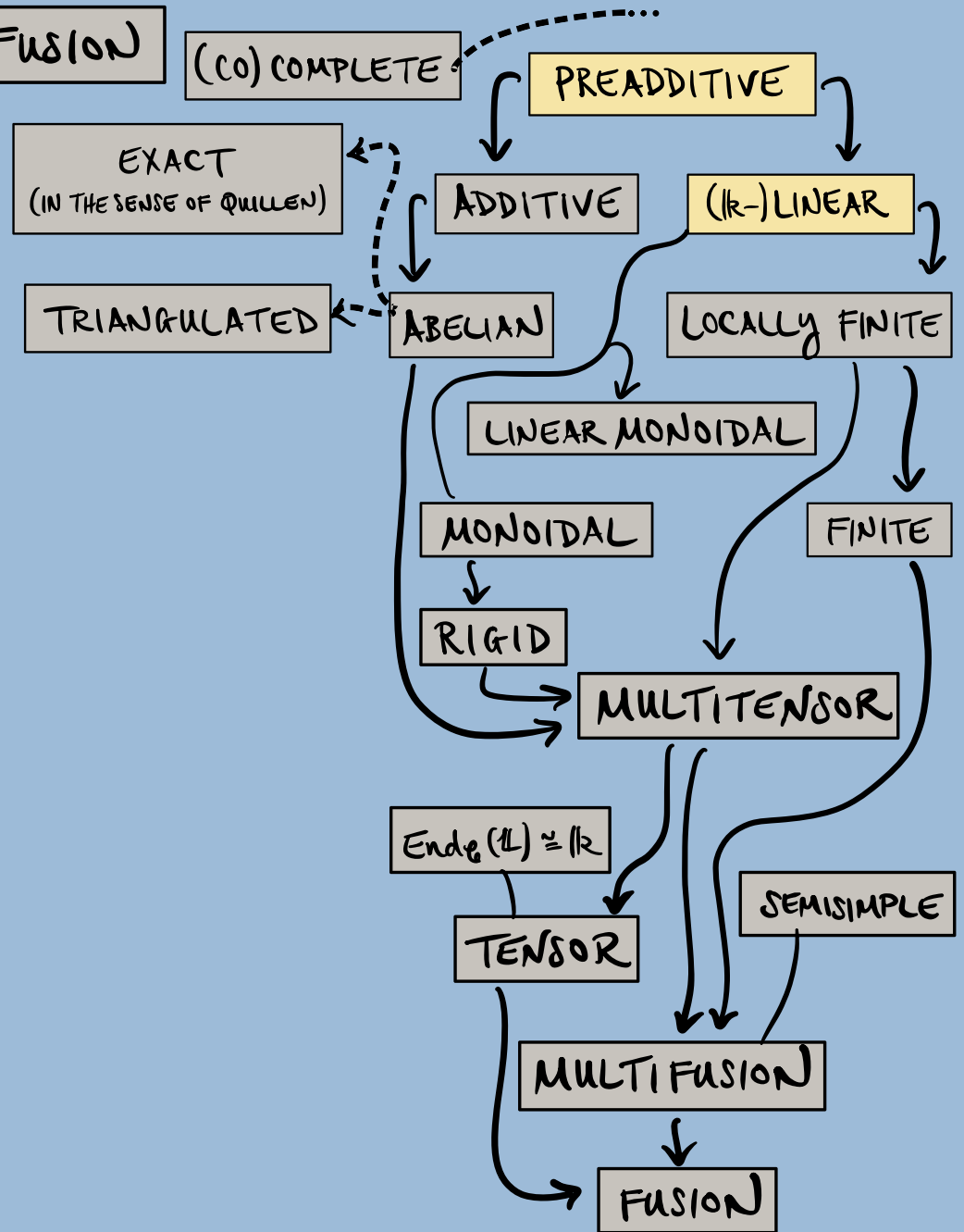
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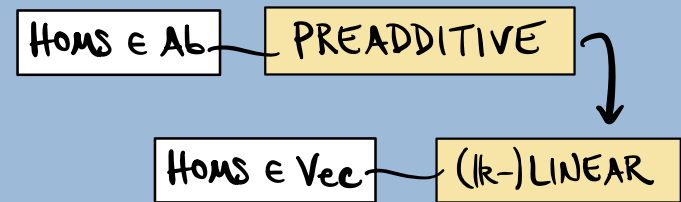
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DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$(\text{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$
 $\nmid (\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{K})$
ARE MONOIDAL

$\text{HomS} \in \text{Ab}$

PREADDITIVE

$\text{HomS} \in \text{Vec}$

$(\mathbb{K}-)$ LINEAR

... HomS AS OBJECTS OF MONOIDAL CATEGORIES $(\mathcal{V}, \otimes^{\mathcal{V}}, \mathbb{1}^{\mathcal{V}})$...

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$(\mathcal{A}b, \otimes_{\mathbb{Z}}, \mathbb{Z})$
 $\neq (\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{K})$
ARE MONOIDAL

$\text{HOMs} \in \mathcal{A}b$

PREADDITIVE



$\text{HOMs} \in \text{Vec}$

($\mathbb{K}$ -)LINEAR

...HOMs AS OBJECTS OF MONOIDAL CATEGORIES $(\mathcal{V}, \otimes^{\mathcal{V}}, \mathbb{1}^{\mathcal{V}})$...

$\mathcal{A}$ IS AN ENRICHED
CATEGORY OVER $\mathcal{V}$
($\equiv \mathcal{V}$ -CATEGORY):

- A COLLECTION OF OBJECTS,
- "HOM OBJECT" $\mathcal{A}(x, y) \in \mathcal{V} \quad \forall x, y \in \mathcal{A}$,

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$(\mathcal{A}b, \otimes_{\mathbb{Z}}, \mathbb{Z})$
 $\neq (\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{K})$
 ARE MONOIDAL

$\text{HOMs} \in \mathcal{A}b$

PREADDITIVE



$\text{HOMs} \in \text{Vec}$

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- "COMPOSITION MAP" $\forall x, y, z \in \mathcal{A}$

$$\gamma_{x, y, z} : \mathcal{A}(y, z) \otimes^{\mathcal{V}} \mathcal{A}(x, y) \longrightarrow \mathcal{A}(x, z),$$

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

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 ARE MONOIDAL

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PREADDITIVE



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SAT. ASSOC. & UNIT. AXIOMS

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$(\mathcal{A}b, \otimes_{\mathbb{Z}}, \mathbb{Z})$
 $\neq (\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{K})$
 ARE MONOIDAL

$\text{Homs} \in \mathcal{A}b$

PREADDITIVE

$\text{Homs} \in \text{Vec}$

$(\mathbb{K}-)\text{LINEAR}$

...HOMS AS OBJECTS OF MONOIDAL CATEGORIES $(\mathcal{V}, \otimes^{\mathcal{V}}, \mathbb{1}^{\mathcal{V}}) \dots$

$\mathcal{A}$ IS AN ENRICHED
 CATEGORY OVER $\mathcal{V}$
 ($\equiv \mathcal{V}$ -CATEGORY):

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- "HOM OBJECT" $\mathcal{A}(x, y) \in \mathcal{V} \quad \forall x, y \in \mathcal{A}$,
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SAT. ASSOC. & UNIT. AXIOMS

EXAMPLES:

PREADDITIVE $\equiv \text{ENRICHED}/_{\mathcal{A}b}$

LINEAR $\equiv \text{ENRICHED}/_{\text{Vec}}$

DETOURS OFF THE ROAD TO FUSION

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SAT. ASSOC. & UNIT. AXIOMS

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SAT. ASSOC. & UNIT. AXIOMS

EXAMPLES:

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 $\equiv (\text{Set}, \times, \{*\})$ -CATEG.

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SAT. ASSOC. & UNIT. AXIOMS

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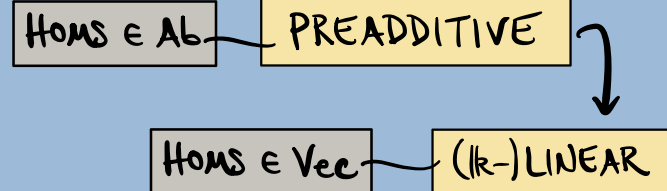
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DETOURS OFF THE ROAD TO FUSION



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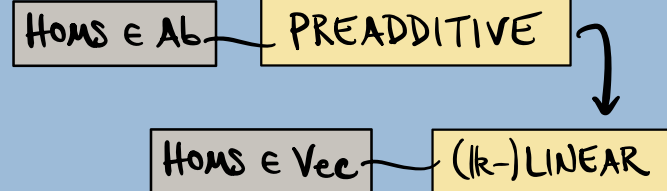
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SAT. ASSOC. & UNIT. AXIOMS

DETOURS OFF THE ROAD TO FUSION



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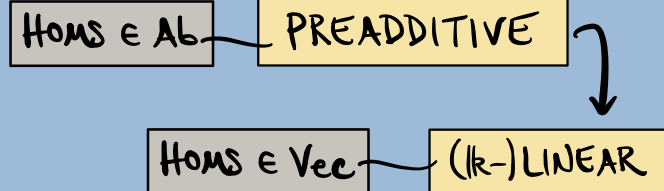
$$\nu_x : \mathbb{1}^{\mathcal{V}} \longrightarrow \mathcal{A}(x, x)$$

SAT. ASSOC. & UNIT. AXIOMS

UNDERLYING CATEGORY $\mathcal{A}_0$:

- $\text{Ob}(\mathcal{A}_0) = \text{Ob}(\mathcal{A})$
- $f : x \rightarrow y \equiv \mathbb{1}^{\mathcal{V}} \rightarrow \mathcal{A}(x, y)$

DETOURS OFF THE ROAD TO FUSION



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- $x \xrightarrow{f} y \xrightarrow{g} z \equiv$

$$\mathbb{1}^{\mathcal{V}} \cong \mathbb{1}^{\mathcal{V}} \otimes^{\mathcal{V}} \mathbb{1}^{\mathcal{V}}$$

$$\xrightarrow{g \otimes f} \mathcal{A}(y, z) \otimes^{\mathcal{V}} \mathcal{A}(x, y)$$

$$\xrightarrow{\gamma_{x, y, z}} \mathcal{A}(x, z)$$

DETOURS OFF THE ROAD TO FUSION

EXTENSIVE THEORY OF $\mathcal{V}$ -CATEGORIES ... $\mathcal{V}$ -FUNCTORS ... $\mathcal{V}$ -EQUIVALENCE ... $\mathcal{V}$ -ADJUNCTION...

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SAT. ASSOC. & UNIT. AXIOMS

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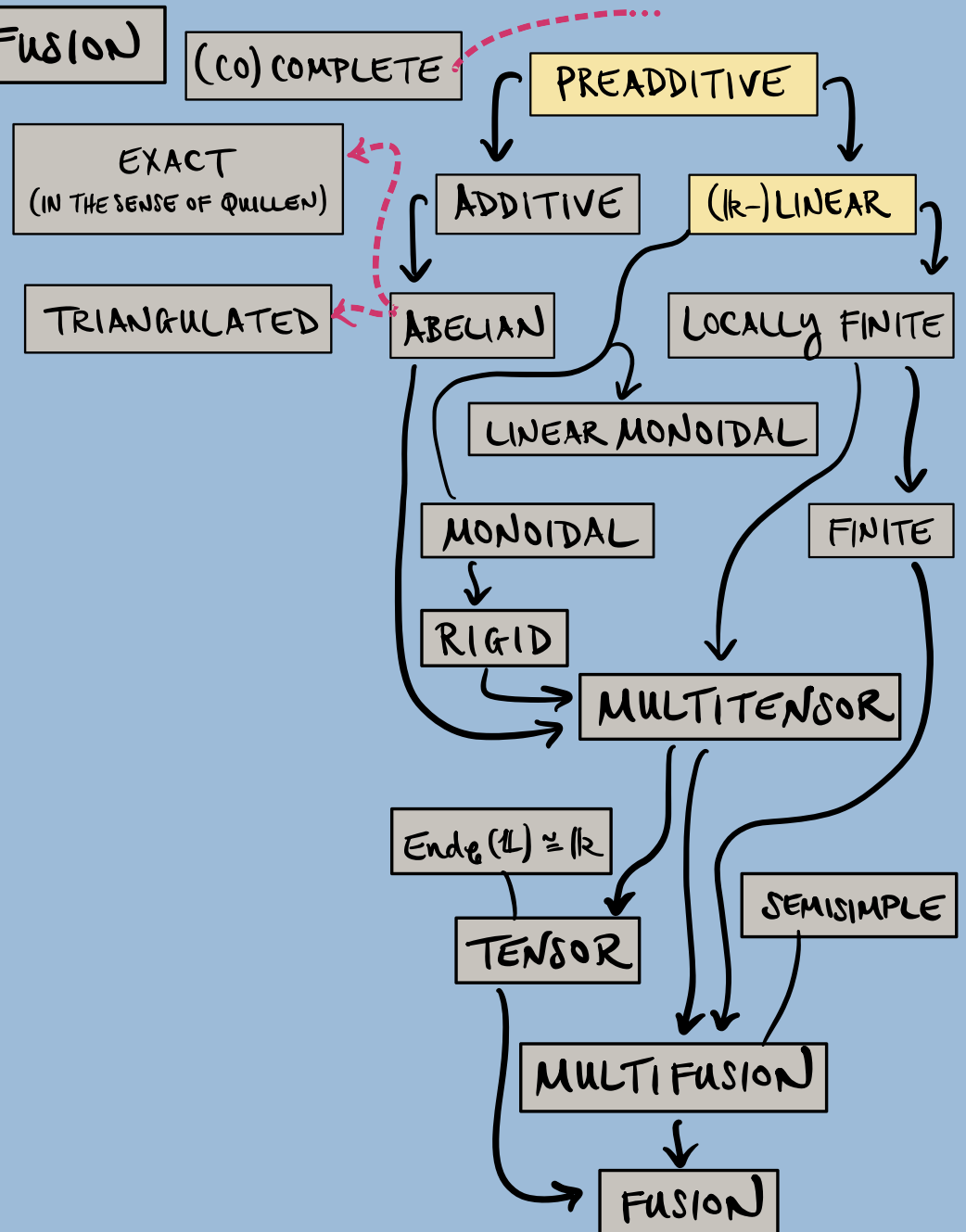
$$\mathbb{1}^{\mathcal{V}} \cong \mathbb{1}^{\mathcal{V}} \otimes^{\mathcal{V}} \mathbb{1}^{\mathcal{V}}$$

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DETOURS OFF THE ROAD TO FUSION

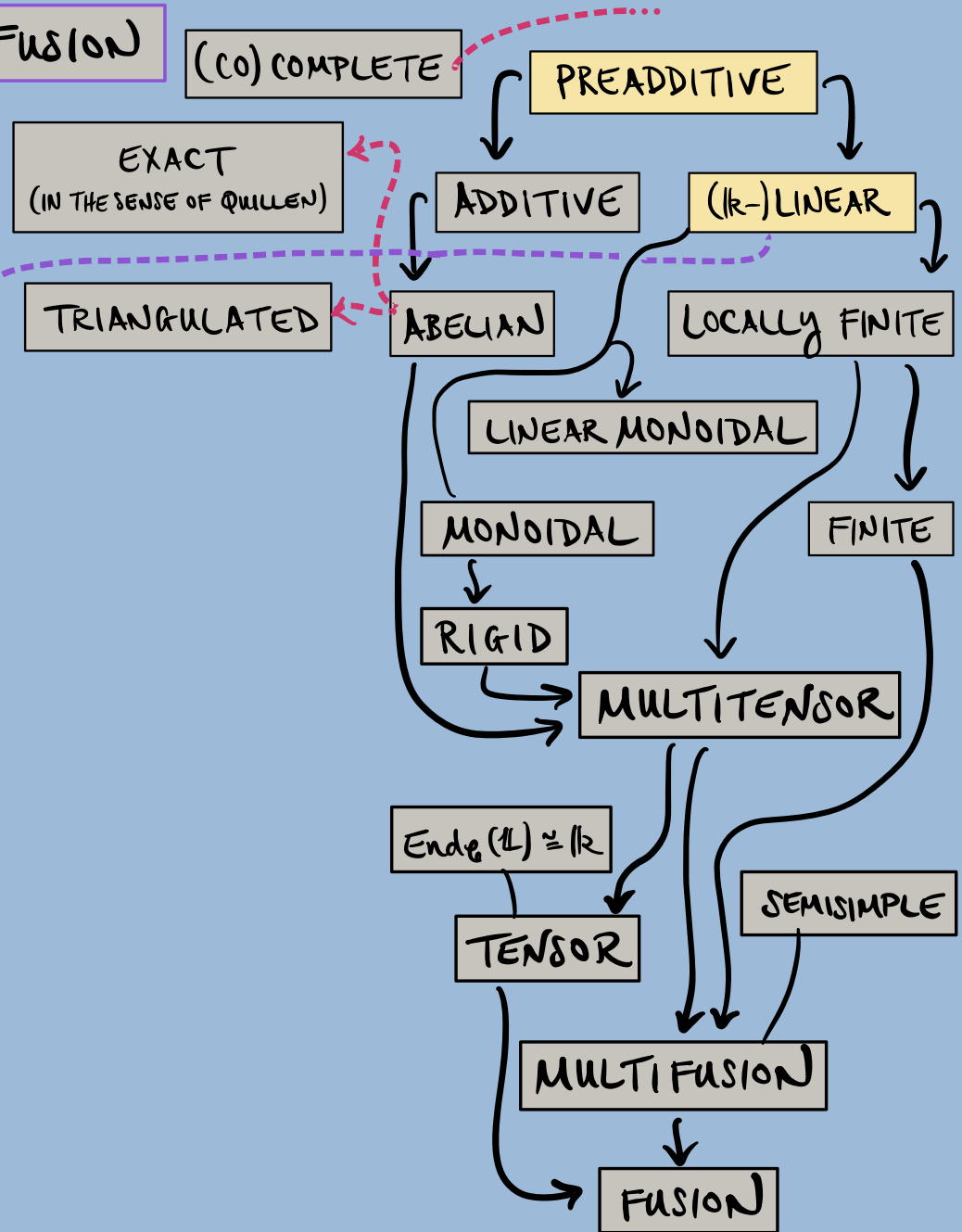
A CATEGORY $\mathcal{C}$ IS ...



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

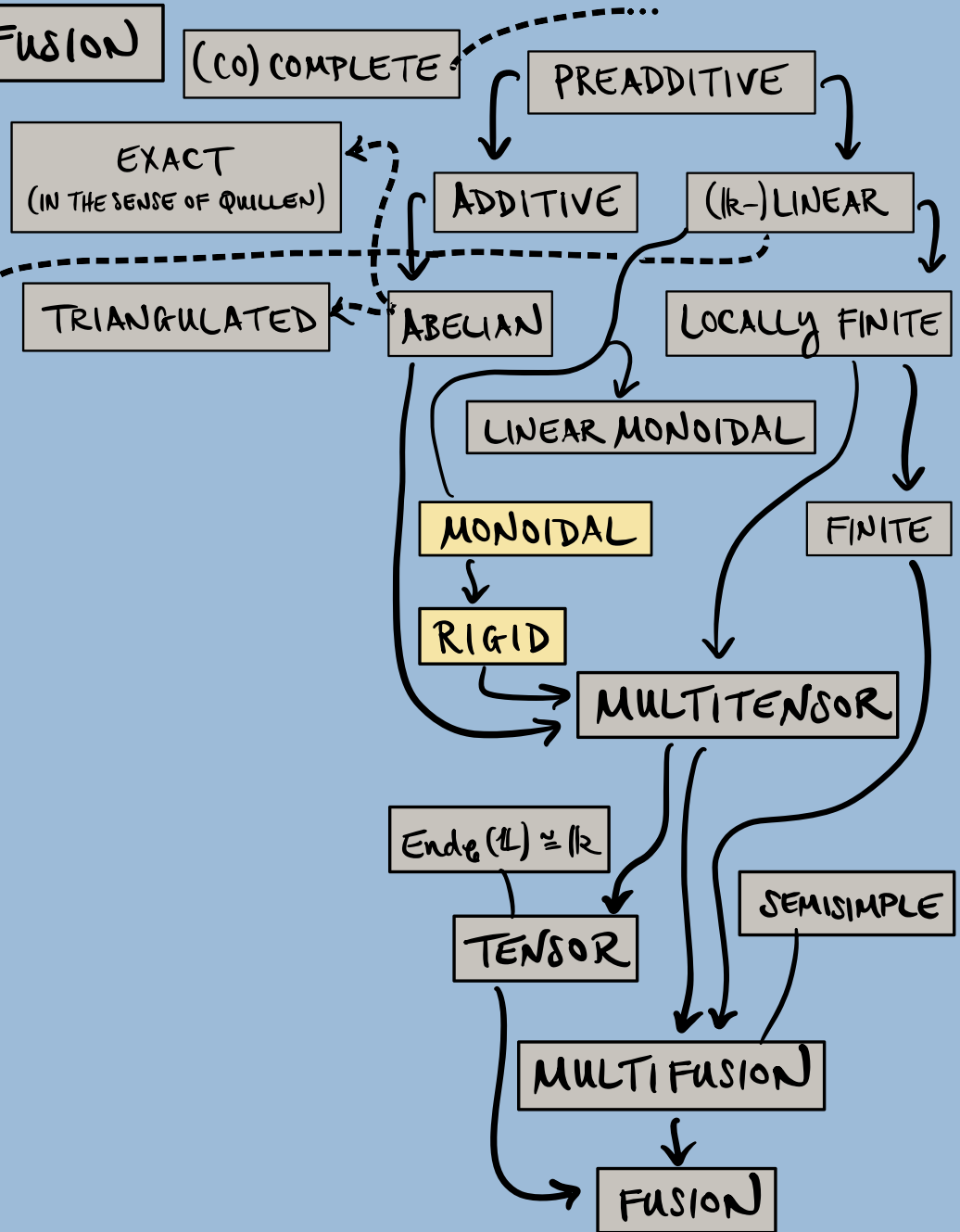
CATEGORY $\mathcal{A}$ ENRICHED OVER
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DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

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DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$\exists \otimes, 1_L \in \mathcal{C} \Rightarrow$
 $(\mathcal{C}, \otimes, 1_L) \approx \text{"MONOID"}$

MONOIDAL



RIGID

$\exists$ DUAL OBJS ($\neq$ DUAL MAPS)

DETOURS OFF THE ROAD TO FUSION

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$\forall X \in \mathcal{C} : \exists X^* \in \mathcal{C}$ WITH

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$coev_X^L : 1 \rightarrow X \otimes X^*$ 

$\exists$.

 $= |^X$ $\neq$  $= |^{X^*}$

ALSO HAVE $({}^*X, ev_X^R, coev_X^R)$

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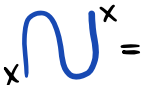
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$(- \otimes y) \dashv (- \otimes y^*)$
 $Hom_{\mathcal{C}}(X \otimes y, z) \cong Hom_{\mathcal{C}}(X, z \otimes y^*)$
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DETOURS OFF THE ROAD TO FUSION

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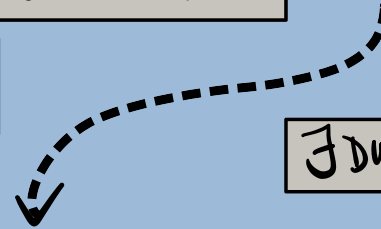
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A GROTHENDIECK-VERDIER (GV) CATEGORY

$\exists$ "DUALIZING" OBJECT $K \in \mathcal{C}$

$\neq D := D_K : \mathcal{C} \rightarrow \mathcal{C}$ ANTI-EQUIVALENCE :

$$\text{Hom}_{\mathcal{C}}(X \otimes y, K) \cong \text{Hom}_{\mathcal{C}}(X, D(y)) \quad \forall X, y \in \mathcal{C}$$

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EXAMPLES —

- RIGID CATEGORIES : $K = \mathbb{1}, D = (-)^*$

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- RIGID CATEGORIES : $K = \mathbb{1}, D = (-)^*$
- A-Fd Bimod, $A = \text{F.D. ALG.}$ [FSSW: 2306.17668]
 $K = \text{VS DUAL OF } A$

$\forall X \in \mathcal{C} : \exists X^* \in \mathcal{C}$ WITH

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$$\text{cup}_X = |^X \quad \neq \quad \text{cap}_{X^*} = |^{X^*}$$

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 $K = \text{VS DUAL OF } A$
- $H\text{-FdMod}$, $H = \text{HOPF ALGEBROID}$ [A: 2308.01029]
 $K = \text{VS DUAL OF BASE ALG OF } H$

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- CERTAIN REP'N CATEGORIES OF VOAs
[ALSW: 2107.05718]

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$(-)^*, {}^*(-) : \mathcal{C} \rightarrow \mathcal{C}$
ARE ANTI-EQUIVALENCES

$$(- \otimes y) \rightarrow (- \otimes y^*)$$

$$\text{Hom}_{\mathcal{C}}(X \otimes y, z) \cong \text{Hom}_{\mathcal{C}}(X, z \otimes y^*)$$

$$\forall X, z \in \mathcal{C}$$

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

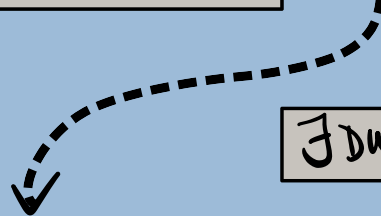
$\exists \otimes, 1 \in \mathcal{C} \Rightarrow$
 $(\mathcal{C}, \otimes, 1) \approx \text{"MONOID"}$

MONOIDAL



RIGID

$\exists$ DUAL OBJS ($\neq$ DUAL MAPS)



A GROTHENDIECK-VERDIER (GV) CATEGORY

$\exists$ "DUALIZING" OBJECT $K \in \mathcal{C}$

$\neq D := D_K : \mathcal{C} \rightarrow \mathcal{C}$ ANTI-EQUIVALENCE :

$$\text{Hom}_{\mathcal{C}}(X \otimes Y, K) \cong \text{Hom}_{\mathcal{C}}(X, D(Y)) \quad \forall X, Y \in \mathcal{C}$$

$\exists$ SECOND MONOIDAL
STRUCTURE :

$$X \odot Y := D^{-1}(D(Y) \otimes D(X))$$

EXAMPLES —

- RIGID CATEGORIES : $K = 1, D = (-)^*$
- $A\text{-FdBimod}$, $A = \text{F.D. ALG.}$ [FSSW: 2306.17668]
 $K = \text{VS DUAL OF } A$
- $H\text{-FdMod}$, $H = \text{HOPF ALGEBROID}$ [A: 2308.01029]
 $K = \text{VS DUAL OF BASE ALG OF } H$
- CERTAIN REPIN CATEGORIES OF VOAs
[ALSW: 2107.05718]

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

$$\exists \otimes, \mathbb{1} \in \mathcal{C} \Rightarrow (\mathcal{C}, \otimes, \mathbb{1}) \approx \text{"MONOID"}$$

MONOIDAL



RIGID

$$\exists \text{ DUAL OBJS (\& DUAL MAPS)}$$

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- CERTAIN REPIN CATEGORIES OF VOAs
[ALSW: 2107.05718]

$$\begin{aligned} X \odot Y &:= *(Y^* \otimes X^*) \\ &\cong *(X \otimes Y)^* \\ &\cong X \odot Y \end{aligned}$$

DETOURS OFF THE ROAD TO FUSION

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- CERTAIN REPIN CATEGORIES OF VOAs
[ALSW: 2107.05718]

$$\begin{aligned} X \odot Y &:= *(Y^* \otimes X^*) \\ &\cong *((X \otimes Y)^*) \\ &\cong X \otimes Y \end{aligned}$$

$$\begin{aligned} \otimes &:= \otimes_A \quad \left(\begin{array}{l} \text{COEQUALIZER OF} \\ A\text{-ACTION MAPS} \end{array} \right) \\ \odot &:= \otimes^A \quad \left(\begin{array}{l} \text{EQUALIZER OF} \\ A^*\text{-COACTION MAPS} \end{array} \right) \end{aligned}$$

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

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MONOIDAL



RIGID

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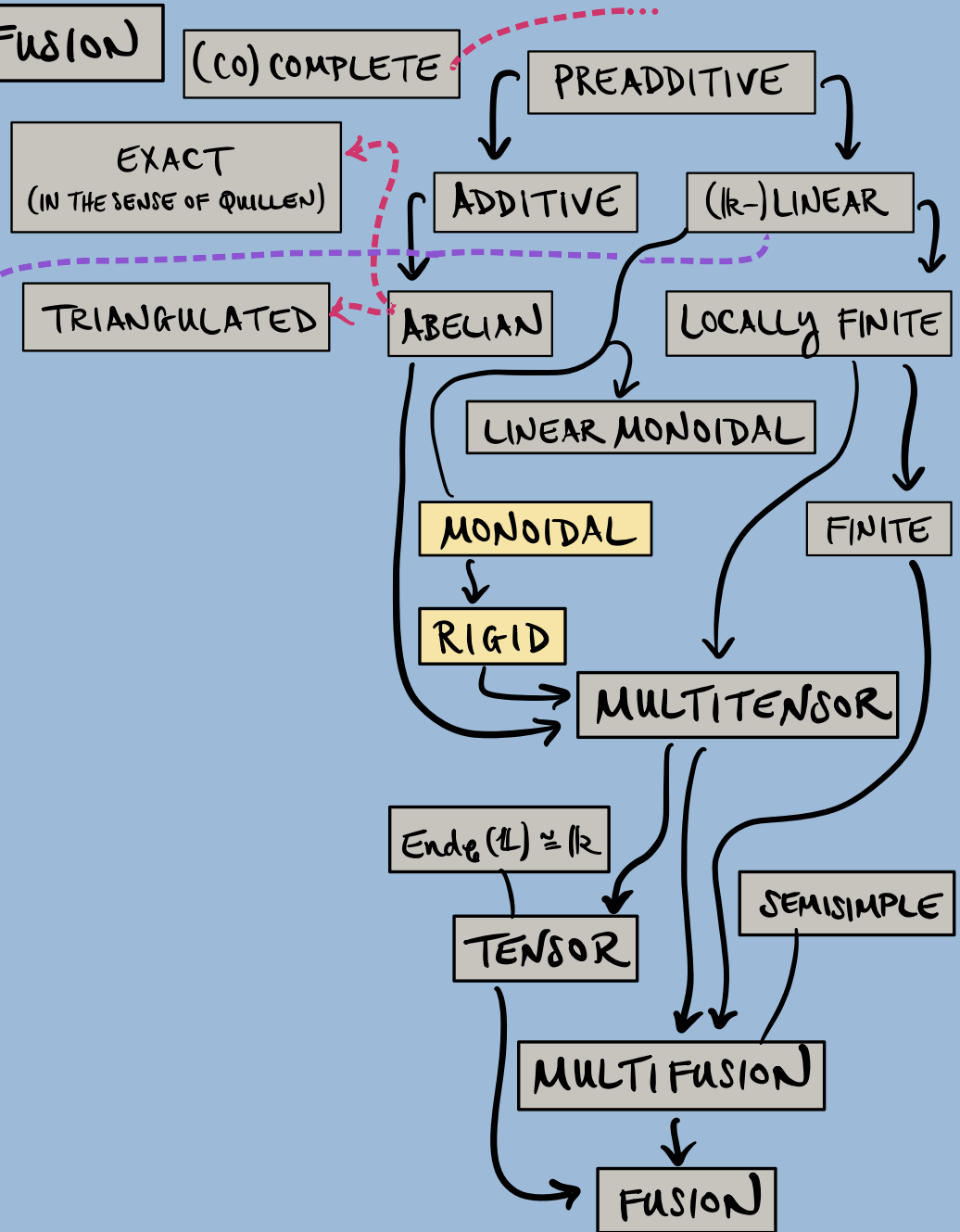
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DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

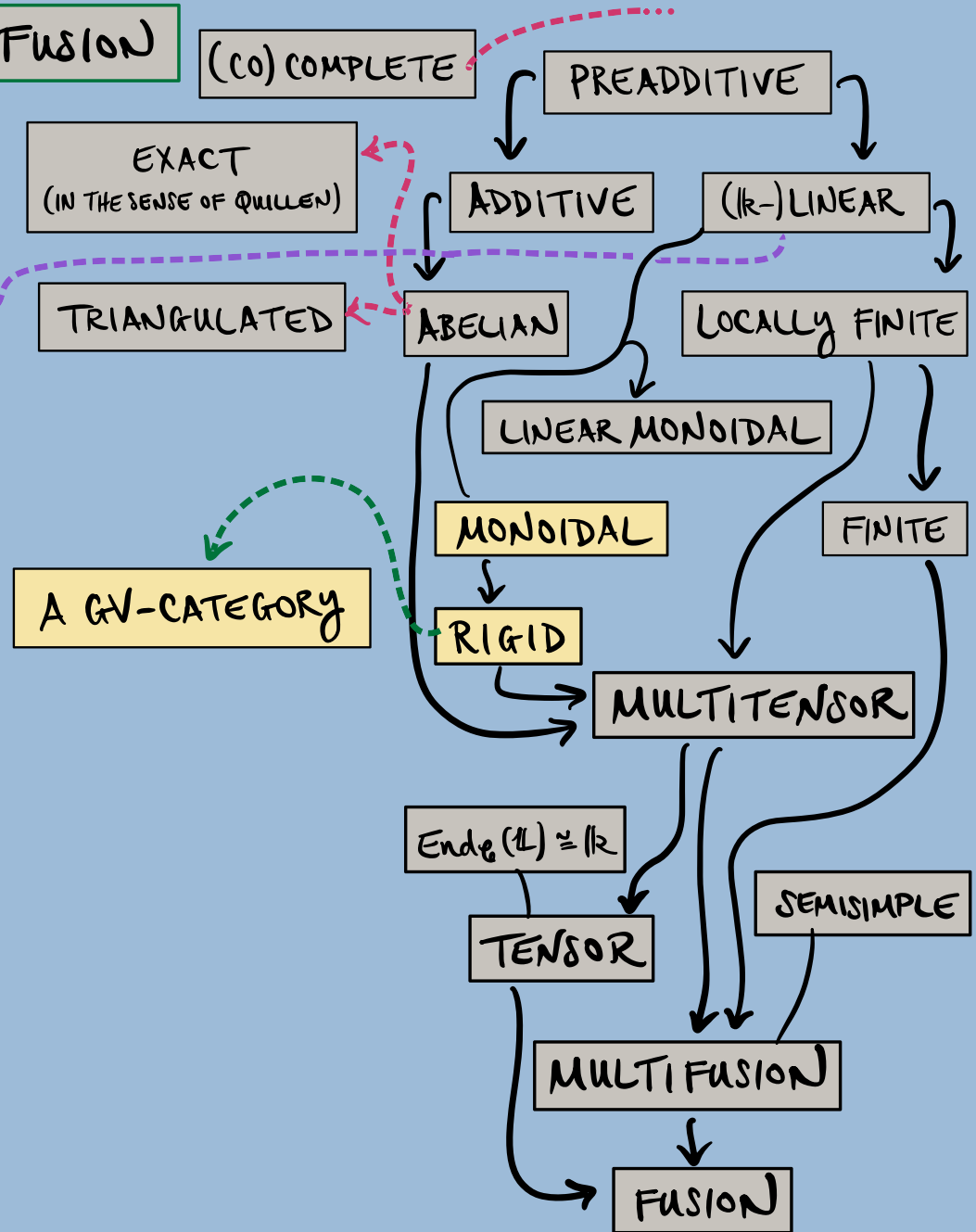
CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

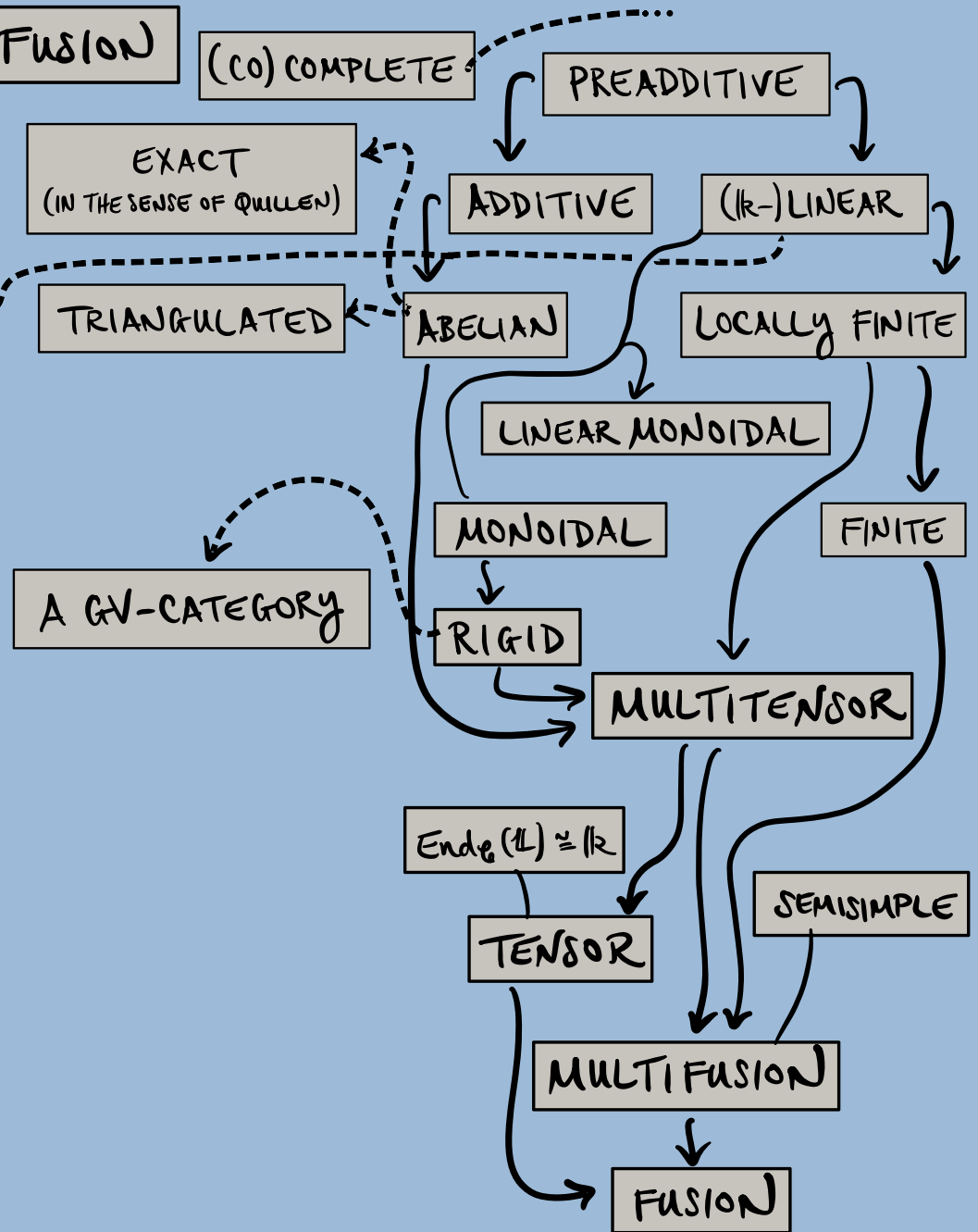
CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

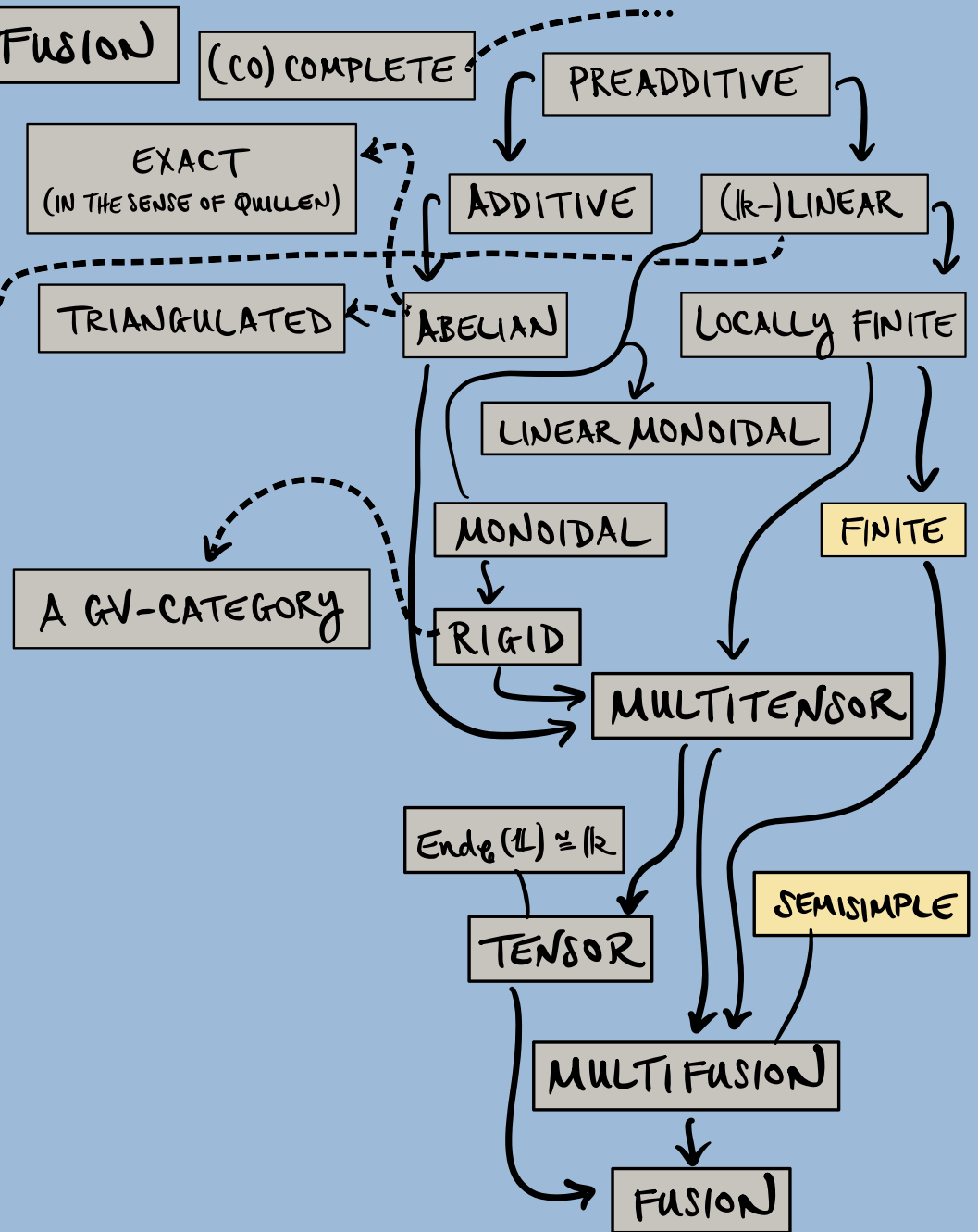
CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

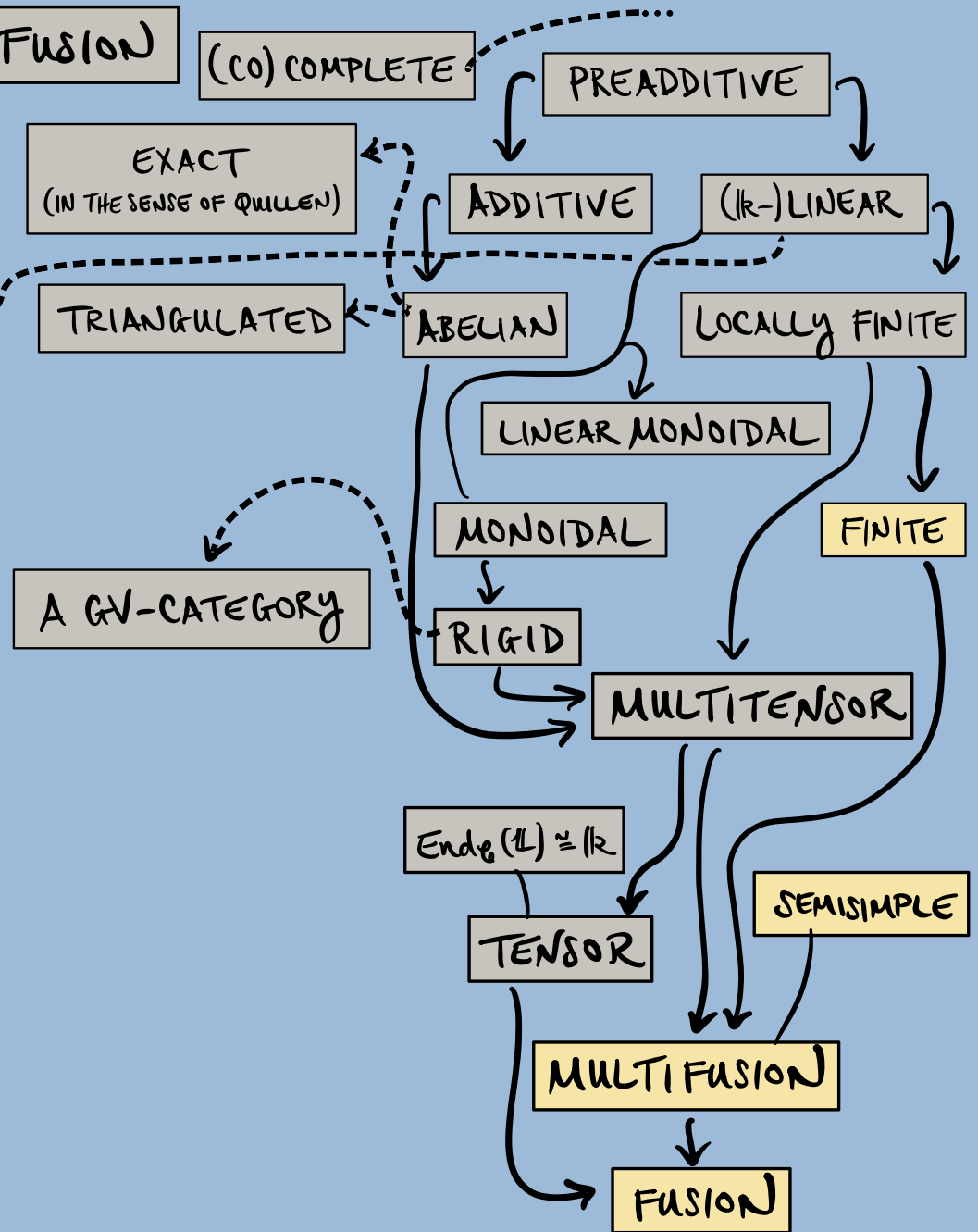
CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

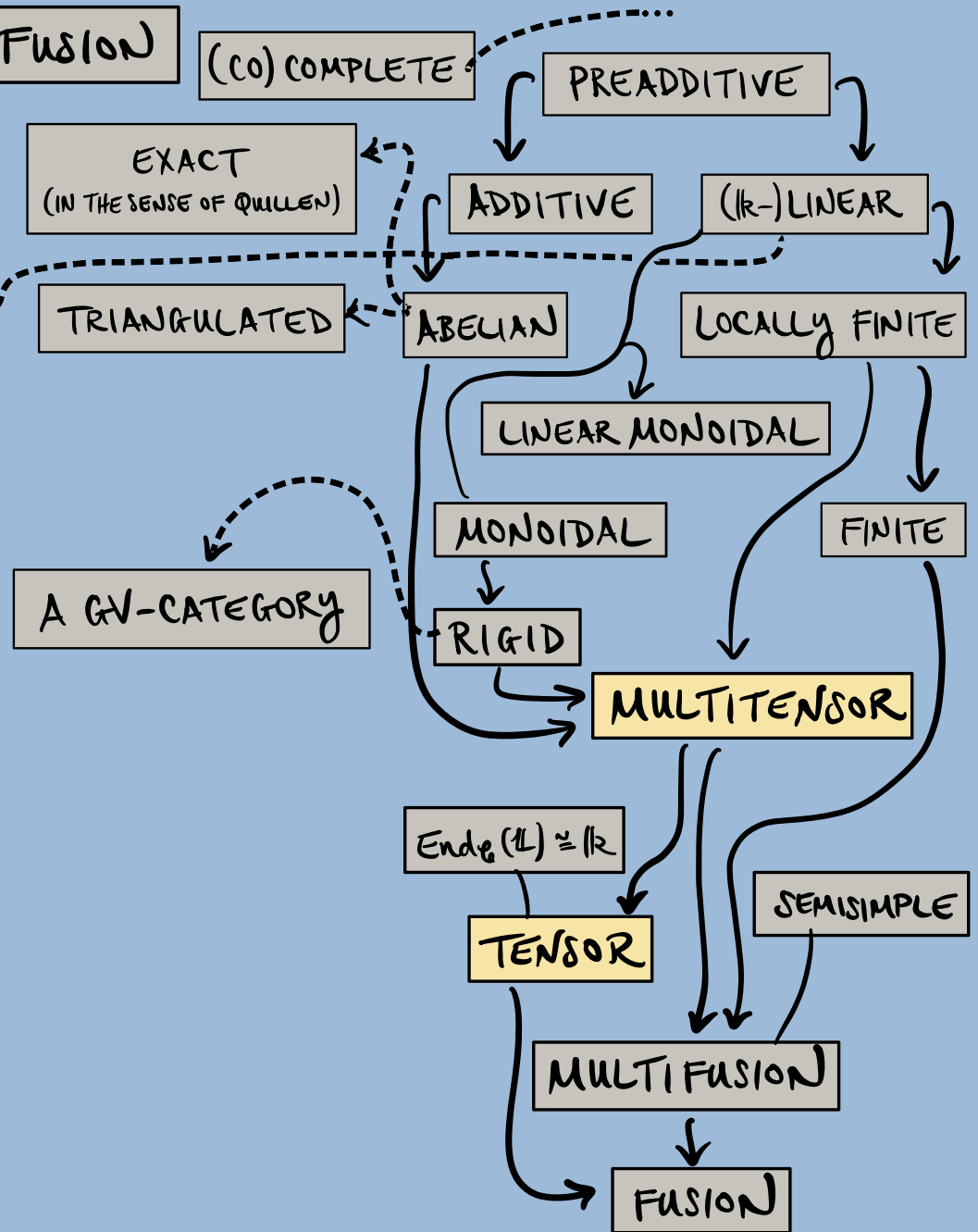
CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

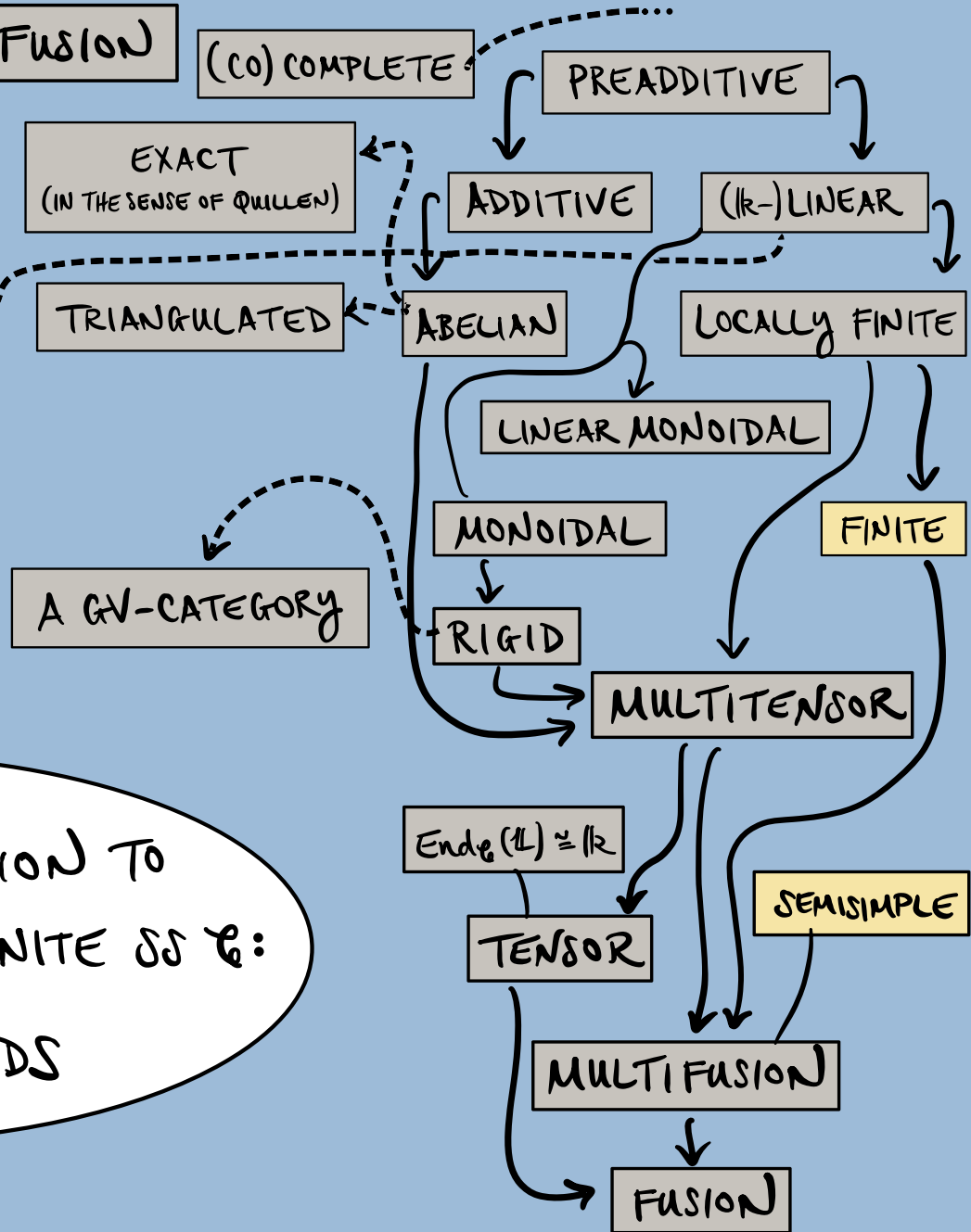
CATEGORY $\mathcal{A}$ ENRICHED OVER
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DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



USEFUL CONSTRUCTION TO
WORK WITH NOT NEC. FINITE SS $\mathcal{C}$:
COENDS & ENDS

COENDS & ENDS

CONSIDER THE "NATURALITY"
OF EVALUATION
IN RIGID CATEGORIES

COENDS & ENDS

CONSIDER THE "NATURALITY"
OF EVALUATION
IN RIGID CATEGORIES



FOR $f: X \rightarrow y \in \mathcal{C}$

$$\begin{array}{c} y^* \\ \text{U-shape} \\ y \end{array} \begin{array}{c} X \\ \text{vertical line} \\ y \end{array} \boxed{f} = \begin{array}{c} y^* \\ \text{U-shape} \\ X \end{array} \begin{array}{c} y \\ \text{vertical line} \\ X^* \end{array} \boxed{f^*}$$

COENDS & ENDS

CONSIDER THE "NATURALITY"
OF EVALUATION
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FOR $f: X \rightarrow Y \in \mathcal{C}$

$$\begin{array}{c} y^* \\ \downarrow \\ \boxed{f} \\ \downarrow \\ y \end{array} \begin{array}{c} X \\ \downarrow \\ \boxed{f} \\ \downarrow \\ y \end{array} = \begin{array}{c} y^* \\ \downarrow \\ \boxed{f^*} \\ \downarrow \\ x^* \end{array} \begin{array}{c} X \\ \downarrow \\ \boxed{f^*} \\ \downarrow \\ x^* \end{array}$$

// DEF'N

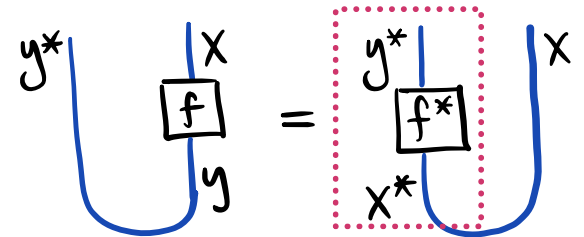
$$\begin{array}{c} y^* \\ \downarrow \\ \boxed{f} \\ \downarrow \\ y \end{array} \begin{array}{c} X \\ \downarrow \\ \boxed{f} \\ \downarrow \\ y \end{array} \begin{array}{c} X^* \\ \downarrow \\ \boxed{f} \\ \downarrow \\ y \end{array} \begin{array}{c} X \\ \downarrow \\ \boxed{f} \\ \downarrow \\ y \end{array}$$

COENDS & ENDS

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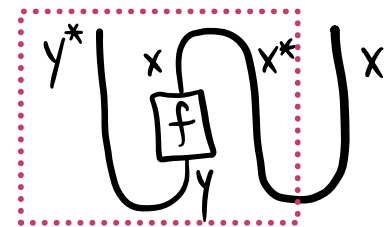


FOR $f: X \rightarrow y \in \mathcal{C}$



RIGIDITY //

// DEF'N

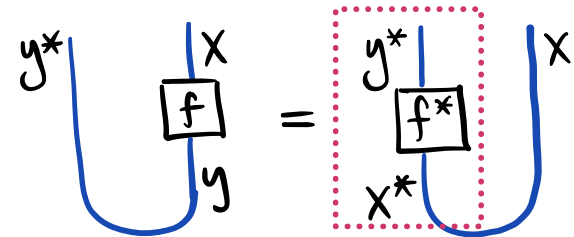


COENDS & ENDS

CONSIDER THE "NATURALITY"
OF EVALUATION
IN RIGID CATEGORIES

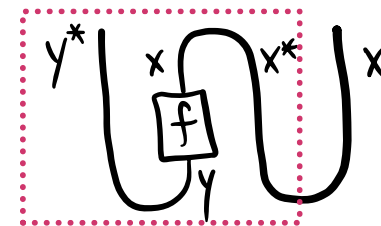
$$\begin{array}{ccc}
 y^* \otimes X & \xrightarrow{y^* \otimes f} & y^* \otimes y \\
 f^* \otimes X \downarrow & \cong & \downarrow \text{ev}_y^L \\
 X^* \otimes X & \xrightarrow{\text{ev}_X^L} & \mathbb{1}
 \end{array}$$

FOR $f: X \rightarrow y \in \mathcal{C}$



RIGIDITY //

// DEFIN

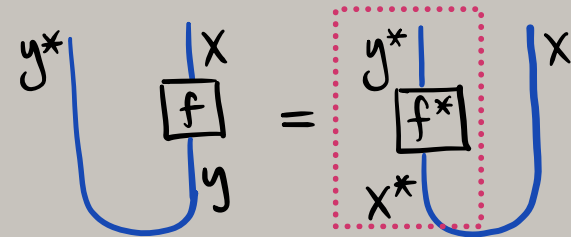


COENDS & ENDS

CONSIDER THE "NATURALITY"
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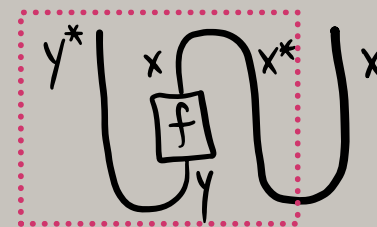
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 \end{array}$$

FOR $f: X \rightarrow y \in \mathcal{C}$



RIGIDITY $\parallel$

$\parallel$ DEF'N



FOR THE BIFUNCTOR:

$$\begin{array}{l}
 B: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \mathcal{C} \\
 (X, X') \longmapsto X^* \otimes X'
 \end{array}$$

GET "NATURALITY"

$$\begin{array}{ccc}
 B(y, X) & \xrightarrow{B(y, f)} & B(y, y) \\
 B(f, X) \downarrow & \cong & \downarrow \text{ev}_y^L \\
 B(X, X) & \xrightarrow{\text{ev}_X^L} & \mathbb{1}
 \end{array}$$

COENDS & ENDS

A COEND OF A BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \tilde{\mathcal{C}} \dots$

IS AN OBJECT $Z \in \tilde{\mathcal{C}}$ & COLLECTION OF MORPHISMS

$$\{ \iota_x: B(X, X) \longrightarrow Z \}_{X \in \mathcal{C}} \dots$$

FOR THE BIFUNCTOR:

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$$(X, X') \longmapsto X^* \otimes X'$$

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COENDS & ENDS

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$$\{ \iota_x: B(X, X) \longrightarrow z \}_{x \in \mathcal{C}} \dots$$

$$\exists. \forall f: X \longrightarrow y \in \mathcal{C}: \quad \begin{array}{ccccc} B(y, X) & \xrightarrow{B(y, f)} & B(y, y) \\ B(f, X) \downarrow & \cong & \downarrow \iota_y \\ B(X, X) & \xrightarrow{\iota_X} & z \end{array}$$

FOR THE BIFUNCTOR:

$$\begin{aligned} B: \mathcal{C}^{\text{op}} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (X, X') &\longmapsto X^* \otimes X' \end{aligned}$$

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THAT IS
UNIVERSAL...

COENDS & ENDS

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FOR ANY $w \in \tilde{\mathcal{C}}$ &

$\{ \theta_x: B(X, X) \rightarrow w \}_{x \in \mathcal{C}} \rightarrow$

$$\theta_x \circ B(f, X) = \theta_y \circ B(y, f)$$

COENDS & ENDS

A COEND OF A BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}} \dots$

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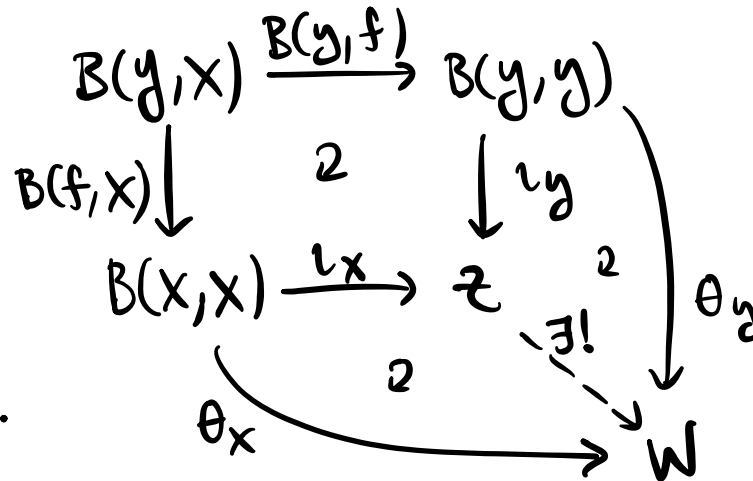
$$\{ \iota_x: B(X, X) \rightarrow Z \}_{X \in \mathcal{C}} \dots$$

$\exists. \forall f: X \rightarrow Y \in \mathcal{C}: \quad$

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COENDS & ENDS

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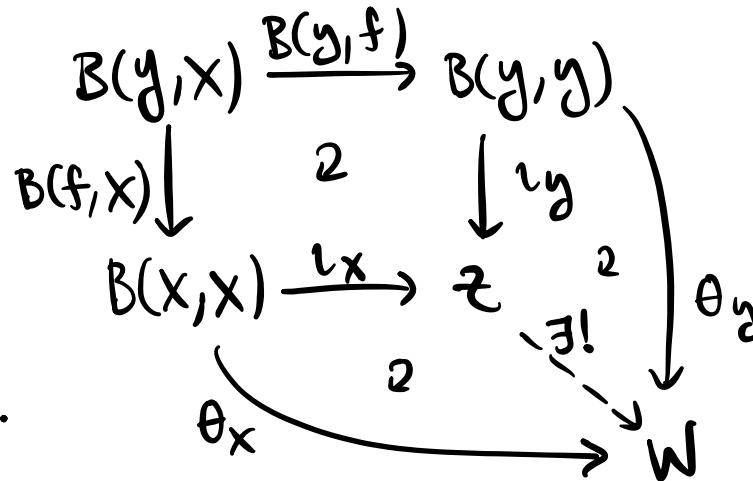
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$$\theta_x \circ B(f, X) = \theta_y \circ B(Y, f)$$



$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

$$\text{WITH } \iota_x: B(X, X) \rightarrow \text{Coend}(B)$$

COENDS & ENDS

A COEND OF A BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}} \dots$

IS AN OBJECT $Z \in \tilde{\mathcal{C}}$ & COLLECTION OF MORPHISMS

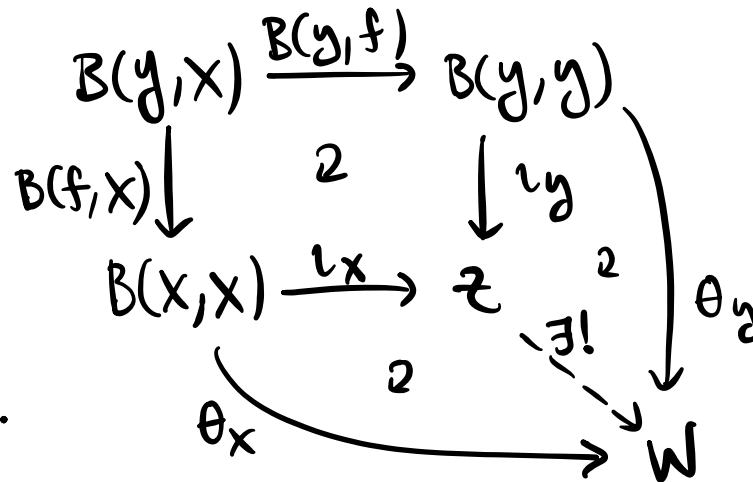
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$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

WITH $\iota_x: B(X, X) \rightarrow \text{Coend}(B)$

$$\cong \coprod_{X \in \text{Irr}(\mathcal{C})} B(X, X)$$

WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS $\leftarrow$ COEND IN $\mathcal{C}^{\text{op}} =: \int_{X \in \mathcal{C}} B(X, X)$

A COEND OF A BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}} \dots$

IS AN OBJECT $Z \in \tilde{\mathcal{C}}$ & COLLECTION OF MORPHISMS

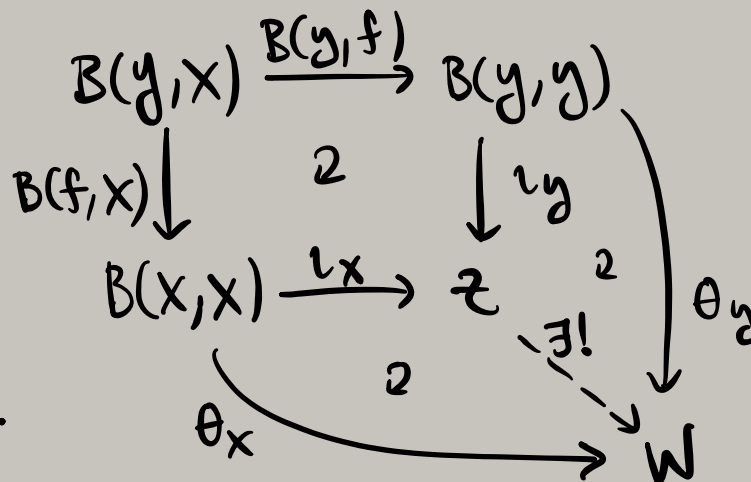
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$\exists. \forall f: X \rightarrow Y \in \mathcal{C}: \dots$

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$$\cong \coprod_{X \in \text{Irr}(\mathcal{C})} B(X, X)$$

WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS

$$B: \text{FdVec}^{\text{op}} \times \text{FdVec} \longrightarrow \text{FdVec}, (V, V') \longmapsto V^* \otimes_{\mathbb{K}} V'$$

$$\leadsto \int^{V \in \text{FdVec}} V^* \otimes_{\mathbb{K}} V$$

$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

$$\text{WITH } \iota_X: B(X, X) \longrightarrow \text{Coend}(B)$$

$$\cong \coprod_{X \in \text{Irr}(\mathcal{C})} B(X, X)$$

WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS

$$B: \text{FdVec}^{\text{op}} \times \text{FdVec} \longrightarrow \text{FdVec}, (V, V') \longmapsto V^* \otimes_{\mathbb{K}} V'$$

$$\rightarrow \int^{V \in \text{FdVec}} V^* \otimes_{\mathbb{K}} V \cong \bigoplus_{V \in \text{Irr}(\text{FdVec})} V^* \otimes_{\mathbb{K}} V \cong \mathbb{K}^* \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}$$

$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

$$\text{WITH } \iota_X: B(X, X) \longrightarrow \text{Coend}(B)$$

$$\cong \coprod_{X \in \text{Irr}(\mathcal{C})} B(X, X)$$

WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS

$$B: \mathbf{FdVec}^{\text{op}} \times \mathbf{FdVec} \longrightarrow \mathbf{FdVec}, (V, V') \longmapsto V^* \otimes_{\mathbb{K}} V'$$

$$\rightarrow \int^{V \in \mathbf{FdVec}} V^* \otimes_{\mathbb{K}} V \cong \bigoplus_{V \in \text{Irr}(\mathbf{FdVec})} V^* \otimes_{\mathbb{K}} V \cong \mathbb{K}^* \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}$$

$C := \int^{X \in \mathcal{C}} X^* \otimes X$ IS A COALGEBRA IN RIGID $\tilde{\mathcal{C}}$

$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

WITH $\iota_X: B(X, X) \longrightarrow \text{Coend}(B)$

$$\cong \coprod_{X \in \text{Irr}(\mathcal{C})} B(X, X)$$

WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS

$$B: \mathbf{FdVec}^{\text{op}} \times \mathbf{FdVec} \longrightarrow \mathbf{FdVec}, (V, V') \longmapsto V^* \otimes_{\mathbb{K}} V'$$

$$\rightarrow \int^{V \in \mathbf{FdVec}} V^* \otimes_{\mathbb{K}} V \cong \bigoplus_{V \in \text{Irr}(\mathbf{FdVec})} V^* \otimes_{\mathbb{K}} V \cong \mathbb{K}^* \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}$$

$C := \int^{X \in \mathcal{C}} X^* \otimes X$ IS A COALGEBRA IN RIGID $\tilde{\mathcal{C}}$

$$\begin{array}{ccc} X^* \otimes X & \xrightarrow{\text{id} \otimes \text{coev}_X^L \otimes \text{id}} & X^* \otimes X \otimes X^* \otimes X \\ \downarrow \wr_X & & \downarrow \wr_X \otimes \wr_X \\ C & \xrightarrow{\Delta} & C \otimes C \end{array}$$

$$\begin{array}{ccc} X^* \otimes X & \xrightarrow{\text{ev}_X^L} & 1 \\ \downarrow \wr_X & & \downarrow \varepsilon \\ C & \xrightarrow{\varepsilon} & 1 \end{array}$$

$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

$$\text{WITH } \wr_X: B(X, X) \longrightarrow \text{Coend}(B)$$

$$\cong \coprod_{X \in \text{Irr}(\mathcal{C})} B(X, X)$$

WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS

$$B: \text{FdVec}^{\text{op}} \times \text{FdVec} \longrightarrow \text{FdVec}, (V, V') \mapsto V^* \otimes_{\mathbb{K}} V'$$

$$\leadsto \int^{V \in \text{FdVec}} V^* \otimes_{\mathbb{K}} V \cong \bigoplus_{V \in \text{Irr}(\text{FdVec})} V^* \otimes_{\mathbb{K}} V \cong \mathbb{K}^* \otimes_{\mathbb{K}} \mathbb{K} \cong \mathbb{K}$$

$C := \int^{X \in \mathcal{C}} X^* \otimes X$ IS A COALGEBRA IN RIGID $\tilde{\mathcal{C}}$

$$\begin{array}{ccc} X^* \otimes X & \xrightarrow{\text{id} \otimes \text{coev}_X^L \otimes \text{id}} & X^* \otimes X \otimes X^* \otimes X \\ \downarrow \wr_X & & \downarrow \wr_X \otimes \wr_X \\ C & \xrightarrow{\Delta} & C \otimes C \end{array}$$

$$\begin{array}{ccc} X^* \otimes X & \xrightarrow{\text{ev}_X^L} & \mathbb{1} \\ \downarrow \wr_X & & \downarrow \varepsilon \\ C & \xrightarrow{\varepsilon} & \mathbb{1} \end{array}$$

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}\mathbb{C}$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \tilde{\mathcal{C}} \\ (V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int^{V \in \mathcal{C}} B(V, V) \cong C$$

$\uparrow$
 $\mathbb{K}$ -COALGS

$$Z := \text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X)$$

$$\text{WITH } \wr_X: B(X, X) \longrightarrow \text{Coend}(B)$$

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WHEN $\mathcal{C}$ IS
FINITE SS
&
 $\tilde{\mathcal{C}}$ IS ADDITIVE

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto A = C^* \cong \left(\int^{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$$

$$(V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int^{V \in \mathcal{C}} B(V, V) \cong C$$

$$\uparrow$$

$$\mathbb{K}\text{-COALGS}$$

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\begin{aligned} \leadsto A = C^* &\cong \left(\int^{V \in \text{FdComod-}C} V^* \otimes V \right)^* \\ &\cong \left(\int^{V \in A\text{-FdMod}} V^* \otimes V \right)^* \end{aligned}$$

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$\begin{aligned} B: \mathcal{C}^{\text{op}} \times \mathcal{C} &\longrightarrow \tilde{\mathcal{C}} \\ (V, V') &\longmapsto V^* \otimes V' \end{aligned}$$

$$\begin{aligned} \leadsto \int^{V \in \mathcal{C}} B(V, V) &\cong C \\ &\uparrow \\ &\mathbb{K}\text{-COALGS} \end{aligned}$$

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto A = C^* \cong \left(\int^{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

$$\cong \left(\int^{V \in A\text{-FdMod}} V^* \otimes V \right)^*$$

$$\cong \int_{V \in A\text{-FdMod}} V \otimes V^*$$

EX. $\mathbb{K}$ -COALG 2

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$$(V, V') \mapsto V^* \otimes V'$$

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$$\uparrow$$

$$\mathbb{K}\text{-COALGS}$$

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto A = C^* \cong \left(\int_{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

$$\cong \left(\int_{V \in A\text{-FdMod}} V^* \otimes V \right)^*$$

$$\cong \int_{V \in A\text{-FdMod}} V \otimes V^*$$

WHEN A IS
ALSO
SEMISIMPLE

$$\cong \left(\bigoplus_{V \in \text{Irr}(A\text{-FdMod})} V^* \otimes V \right)^*$$

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$$

$$(V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int_{V \in \mathcal{C}} B(V, V) \cong C$$

$\uparrow$
 $\mathbb{K}$ -COALGS

COENDS & ENDS

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$$\leadsto A = C^* \cong \left(\int_{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

$$\cong \left(\int_{V \in A\text{-FdMod}} V^* \otimes V \right)^*$$

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WHEN A IS
ALSO
SEMISIMPLE

$$\cong \left(\bigoplus_{V \in \text{Irr}(A\text{-FdMod})} V^* \otimes V \right)^*$$

$$\cong \prod_{V \in \text{Irr}(A\text{-FdMod})} V \otimes V^*$$

$$\cong \prod_{i=1}^r \text{Mat}_{n_i}(\mathbb{K}) \text{ AS } \mathbb{K}\text{-ALGEBRAS}$$

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$$

$$(V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int_{V \in \mathcal{C}} B(V, V) \cong C$$

$\uparrow$
 $\mathbb{K}$ -COALGS

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto A = C^* \cong \left(\int^{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

$$\cong \left(\int^{V \in A\text{-FdMod}} V^* \otimes V \right)^*$$

$$\cong \int^{V \in A\text{-FdMod}} V \otimes V^*$$

WHEN A IS
ALSO
SEMISIMPLE

$$\cong \left(\bigoplus_{V \in \text{Irr}(A\text{-FdMod})} V^* \otimes V \right)^*$$

ARTIN-
WEDDERBURN
THEOREM

$$\cong \prod_{V \in \text{Irr}(A\text{-FdMod})} V \otimes V^*$$

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EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$$

$$(V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int^{V \in \mathcal{C}} B(V, V) \cong C$$

$\uparrow$
 $\mathbb{K}$ -COALGS

$$r = \text{rk}(A\text{-FdMod})$$

$$n = \dim V \leadsto$$

$$V \otimes V^* \cong \text{Mat}_n(\mathbb{K})$$

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto \underline{A} = C^* \cong \left(\int_{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

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WHEN A IS
ALSO
SEMISIMPLE

$$\cong \left(\bigoplus_{V \in \text{Irr}(A\text{-FdMod})} V^* \otimes V \right)^*$$

ARTIN-
WEDDERBURN
THEOREM

$$\cong \prod_{V \in \text{Irr}(A\text{-FdMod})} V \otimes V^*$$

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RECOVER A
FROM ITS CATEG.
OF REPS

EX. $\mathbb{K}$ -COALG 2

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$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$$

$$(V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int_{V \in \mathcal{C}} B(V, V) \cong C$$

$$\uparrow$$

$$\mathbb{K}\text{-COALGS}$$

$$r = \text{rk}(A\text{-FdMod})$$

$$n = \dim V \leadsto$$

$$V \otimes V^* \cong \text{Mat}_n(\mathbb{K})$$

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto \underline{A} = C^* \cong \left(\int^{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

$$\cong \left(\int^{V \in A\text{-FdMod}} V^* \otimes V \right)^*$$

$$\cong \underline{\int^{V \in A\text{-FdMod}} V \otimes V^*}$$

WHEN A IS
ALSO
SEMISIMPLE

$$\cong \left(\bigoplus_{V \in \text{Irr}(A\text{-FdMod})} V^* \otimes V \right)^*$$

ARTIN-
WEDDERBURN
THEOREM

$$\cong \prod_{V \in \text{Irr}(A\text{-FdMod})} V \otimes V^*$$

$$\cong \prod_{i=1}^r \text{Mat}_{n_i}(\mathbb{K}) \text{ AS } \mathbb{K}\text{-ALGEBRAS}$$

RECOVER A
FROM ITS CATEG.
OF REPS

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$$

$$(V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int^{V \in \mathcal{C}} B(V, V) \cong C$$

$\uparrow$
 $\mathbb{K}$ -COALGS

$$r = \text{rk}(A\text{-FdMod})$$

$$n = \dim V \leadsto$$

$$V \otimes V^* \cong \text{Mat}_n(\mathbb{K})$$

COENDS & ENDS

Let $A := C^*$ A F.D. $\mathbb{K}$ -ALGEBRA

$$\leadsto A = C^* \cong \left(\int_{V \in \text{FdComod-}C} V^* \otimes V \right)^*$$

$$\cong \left(\int_{V \in A\text{-FdMod}} V^* \otimes V \right)^*$$

$$\cong \int_{V \in A\text{-FdMod}} V \otimes V^*$$

WHEN A IS
ALSO
SEMISIMPLE

$$\cong \left(\bigoplus_{V \in \text{Irr}(A\text{-FdMod})} V^* \otimes V \right)^*$$

ARTIN-
WEDDERBURN
THEOREM

$$\cong \prod_{V \in \text{Irr}(A\text{-FdMod})} V \otimes V^* \cong \prod_{i=1}^r \text{Mat}_{n_i}(\mathbb{K}) \text{ AS } \mathbb{K}\text{-ALGEBRAS}$$

RECOVER C
FROM ITS CATEG.
OF COREPS

EX. $\mathbb{K}$ -COALG 2

$$\mathcal{C} = \text{FdComod-}C$$

$$\tilde{\mathcal{C}} = \text{FdVec}$$

$$B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}} \\ (V, V') \mapsto V^* \otimes V'$$

$$\leadsto \int_{V \in \mathcal{C}} \underline{B(V, V)} \cong \underline{C} \\ \uparrow \\ \mathbb{K}\text{-COALGS}$$

$$r = \text{rk}(A\text{-FdMod})$$

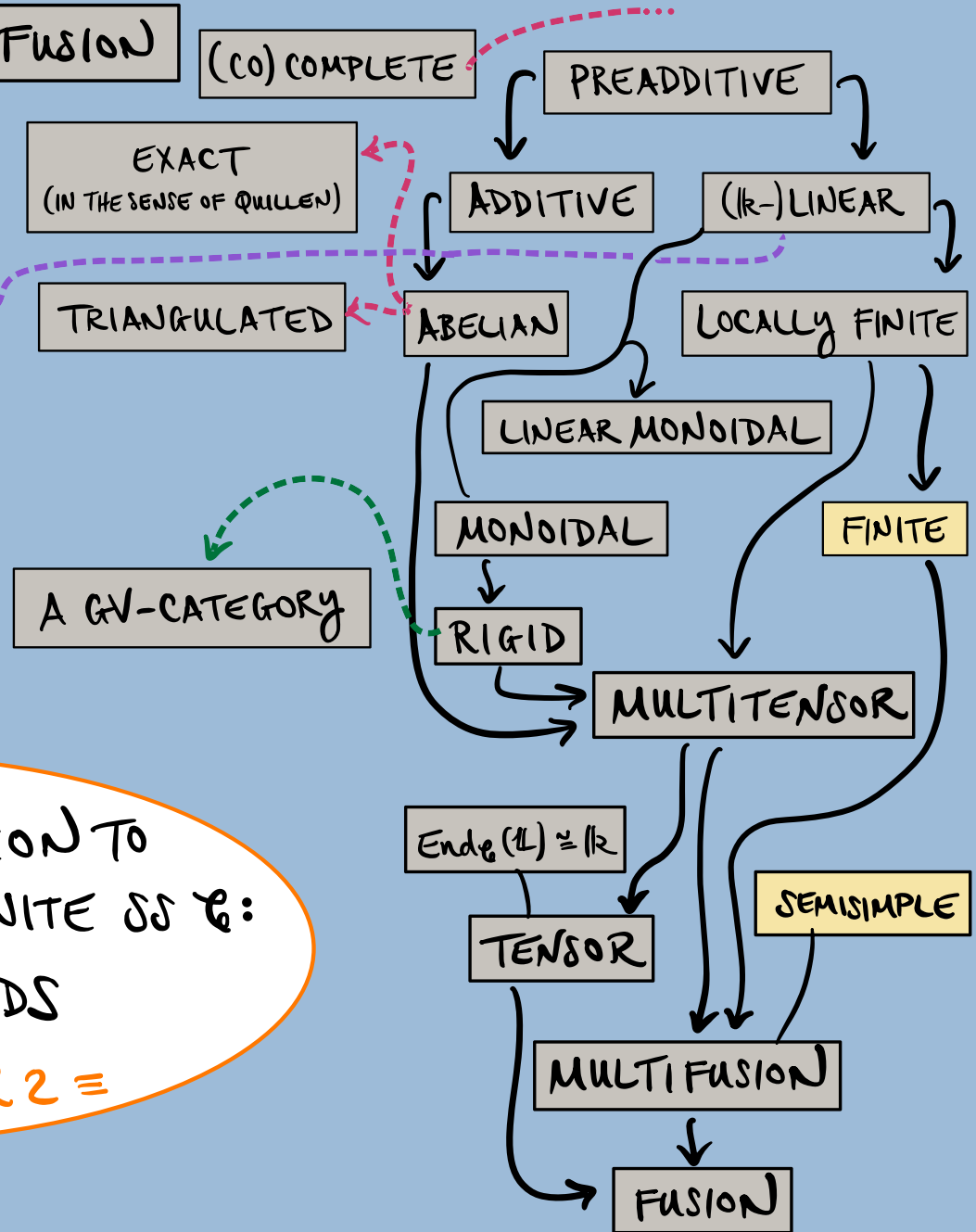
$$n = \dim V \leadsto$$

$$V \otimes V^* \cong \text{Mat}_n(\mathbb{K})$$

DETOURS OFF THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



USEFUL CONSTRUCTION TO
WORK WITH NOT NEC. FINITE SS $\mathcal{C}$:
COENDS & ENDS
≡ MORE IN HOUR 2 ≡

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