

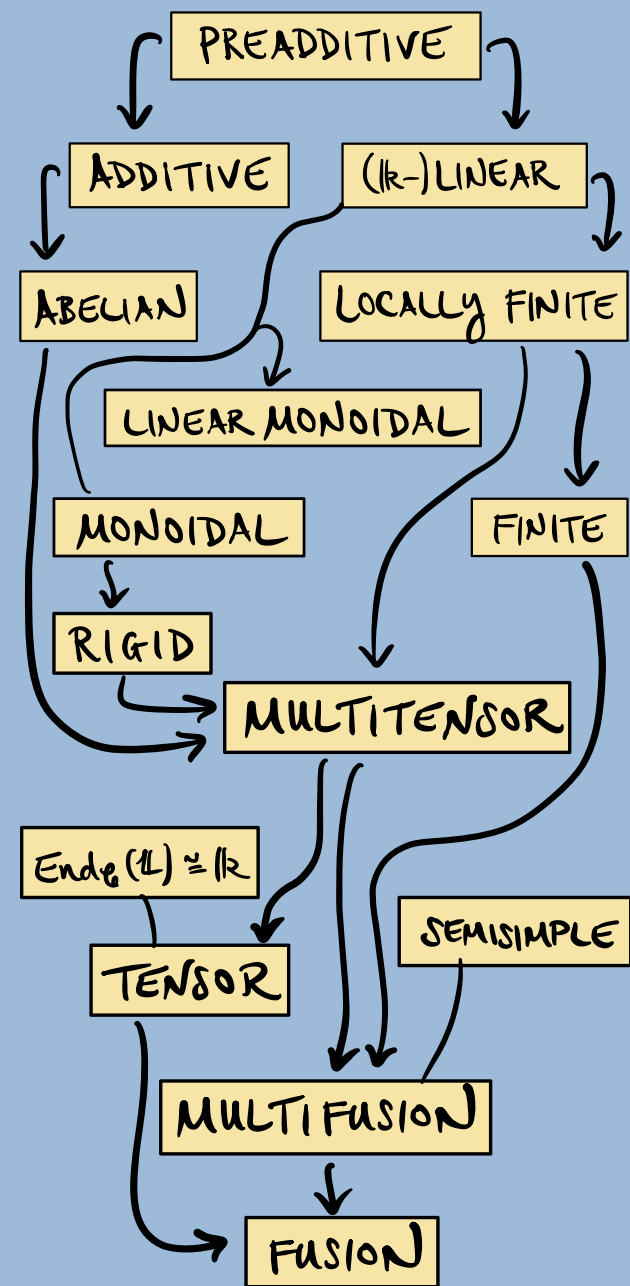
AN INTRODUCTION TO TENSOR CATEGORIES

Hour 2: ON THE ROAD TO MODULARITY

CHELSEA WALTON
RICE UNIVERSITY
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LAST TIME: THE ROAD TO FUSION

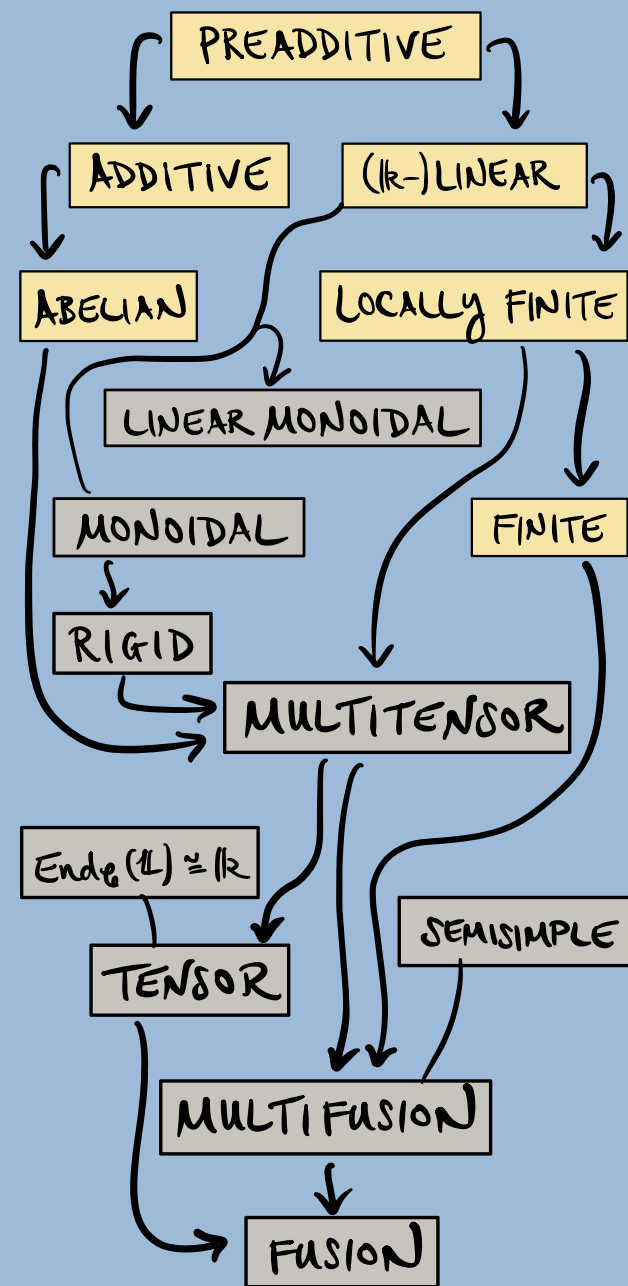
A CATEGORY $\mathcal{C}$ IS ...



LAST TIME: THE ROAD TO FUSION

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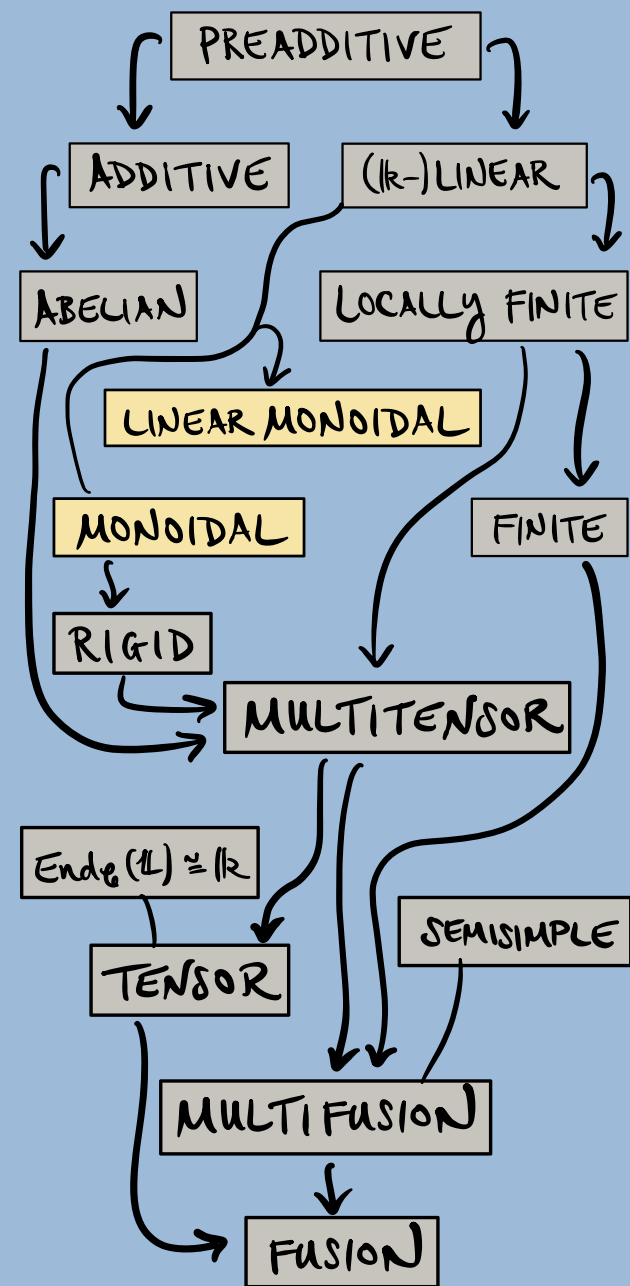
HOMOLOGICAL
CONDITIONS



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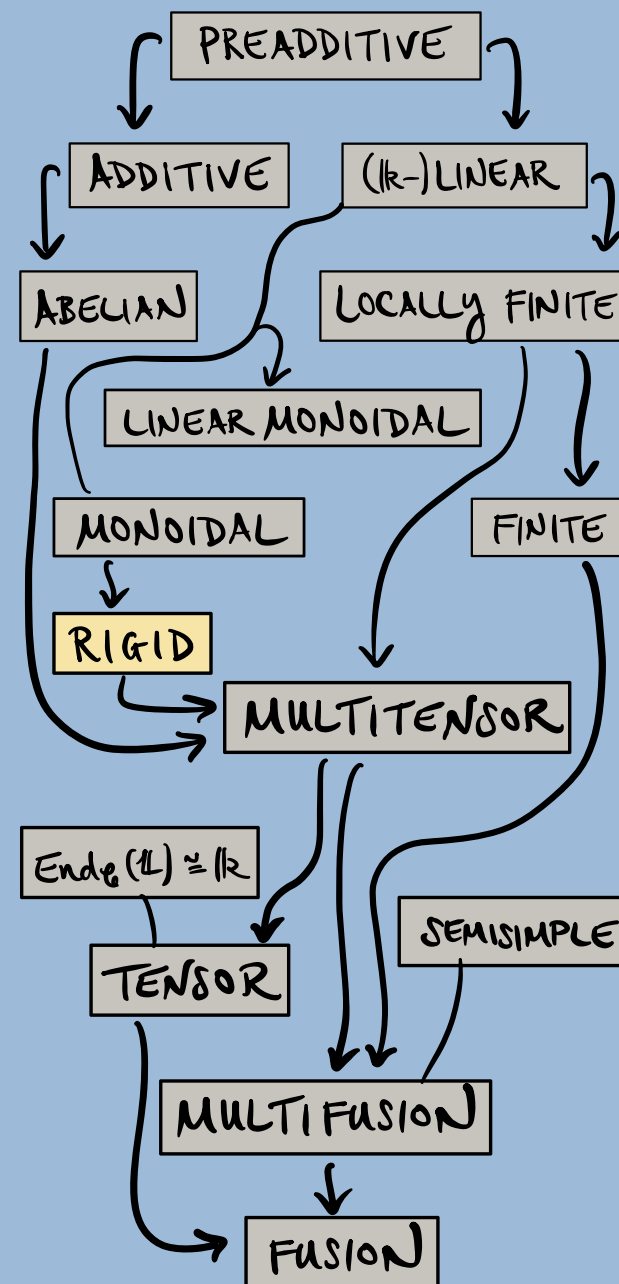
AN OPERATION $\otimes$



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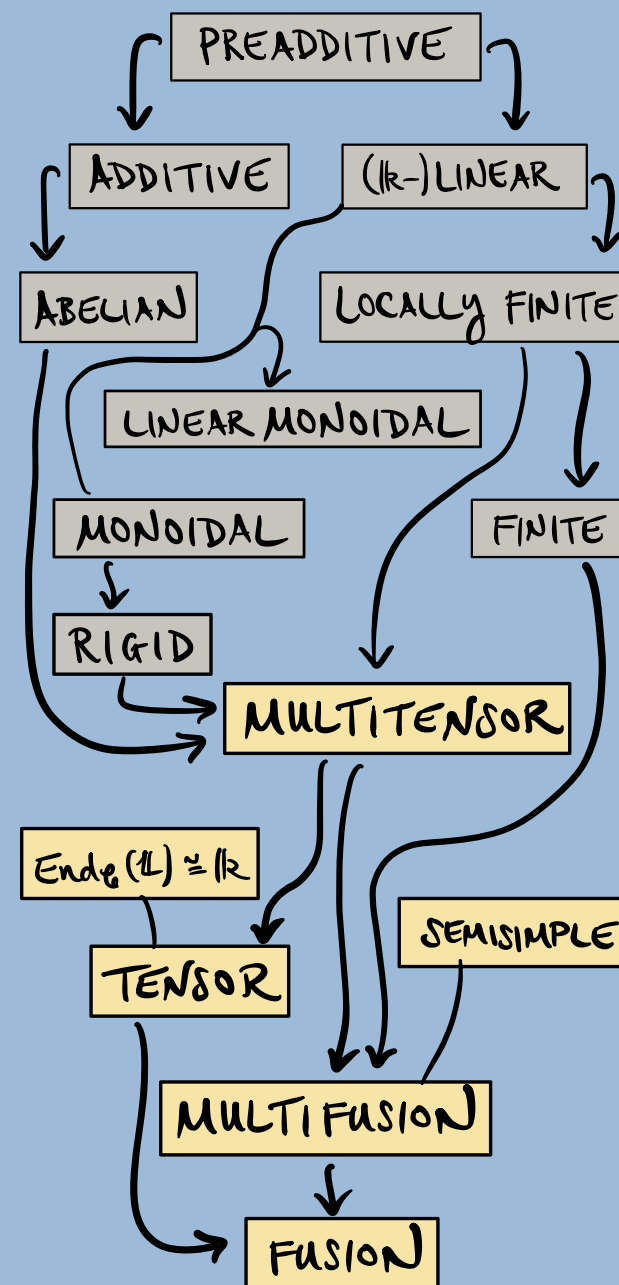
DUALITY WITH RESP TO $\otimes$



LAST TIME: THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

COMBINATIONS OF
THE ABOVE WITH SOME
EXTRA FEATURES



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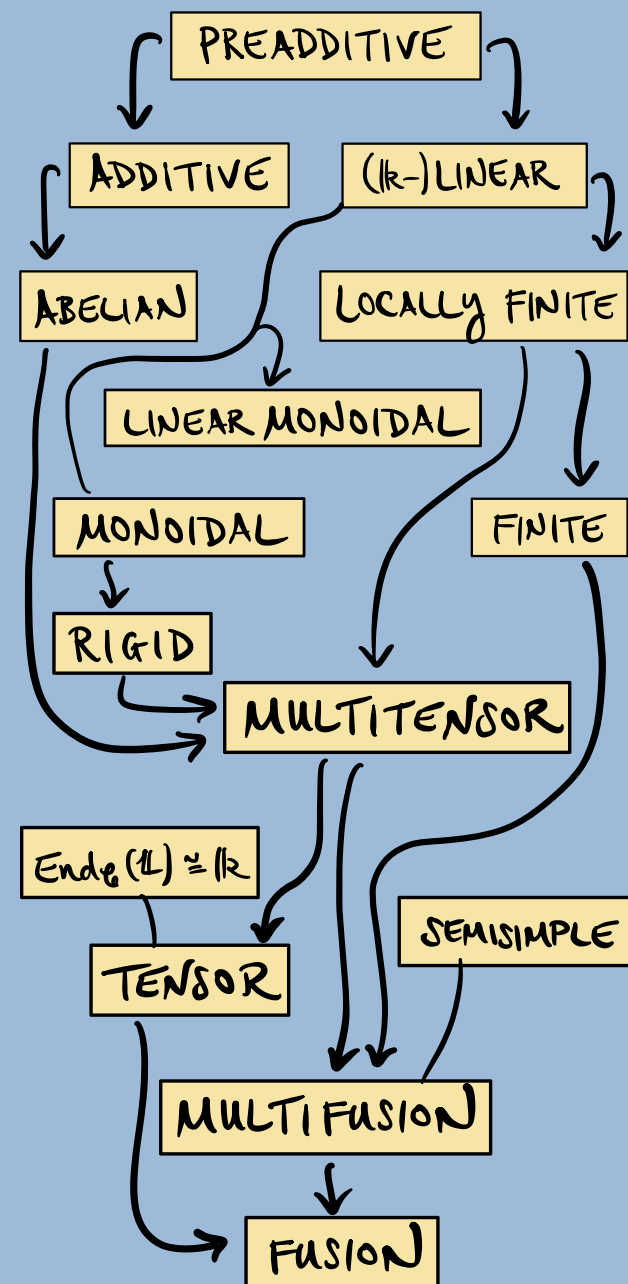
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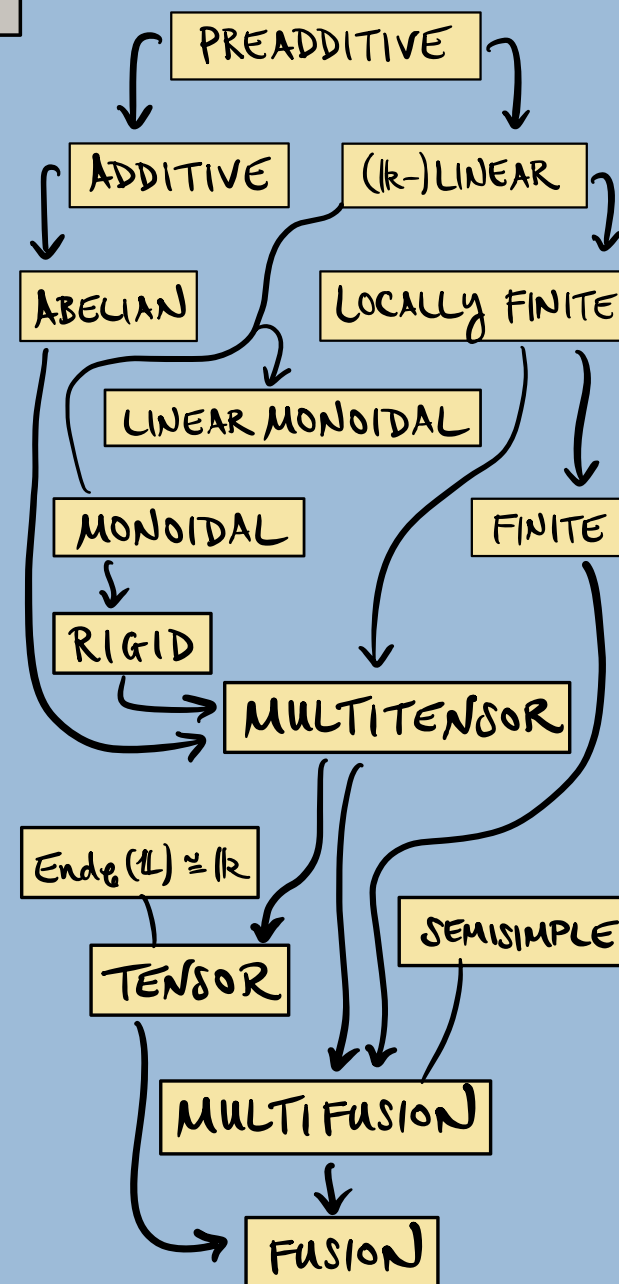
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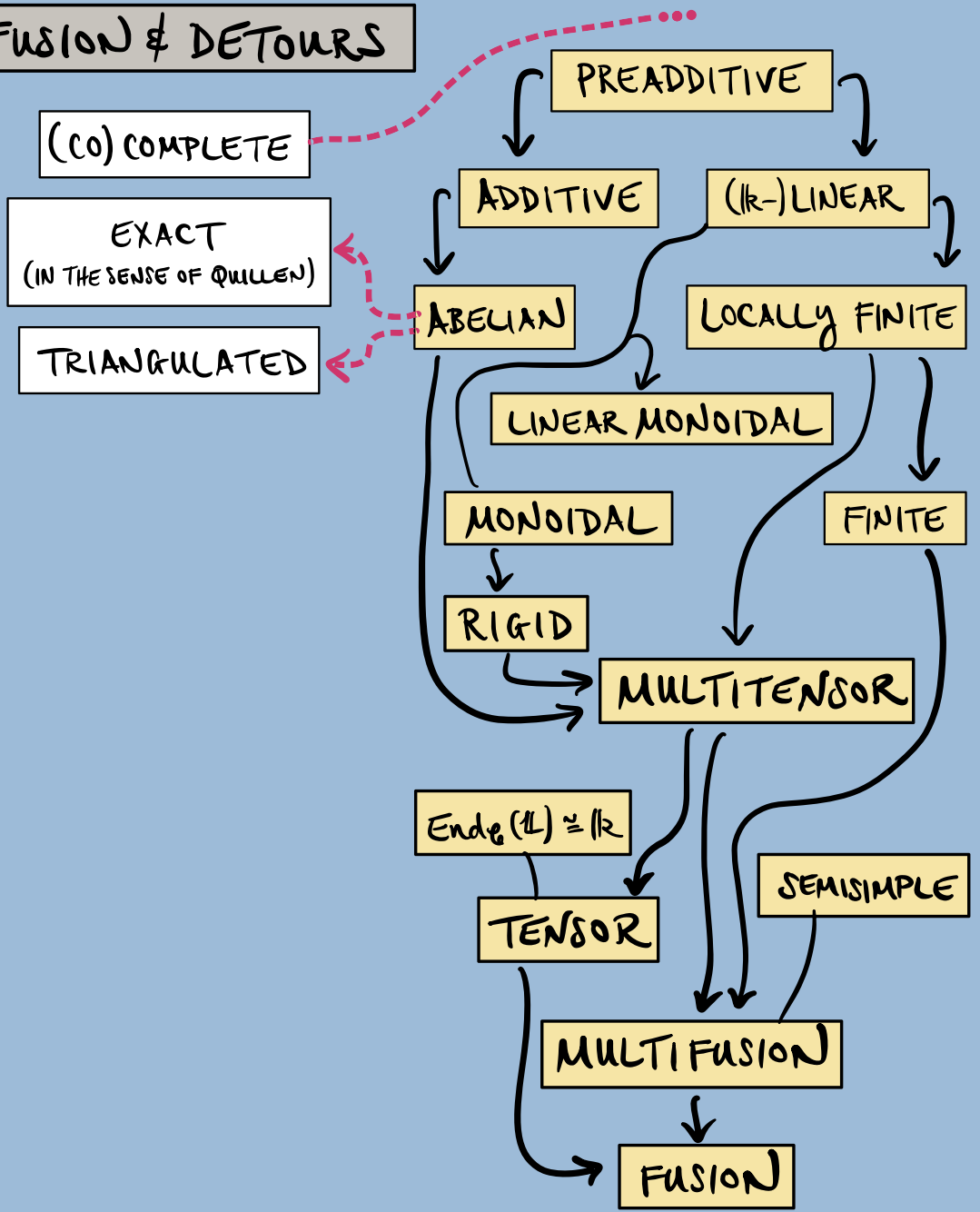
LAST TIME: THE ROAD TO FUSION & DETOURS

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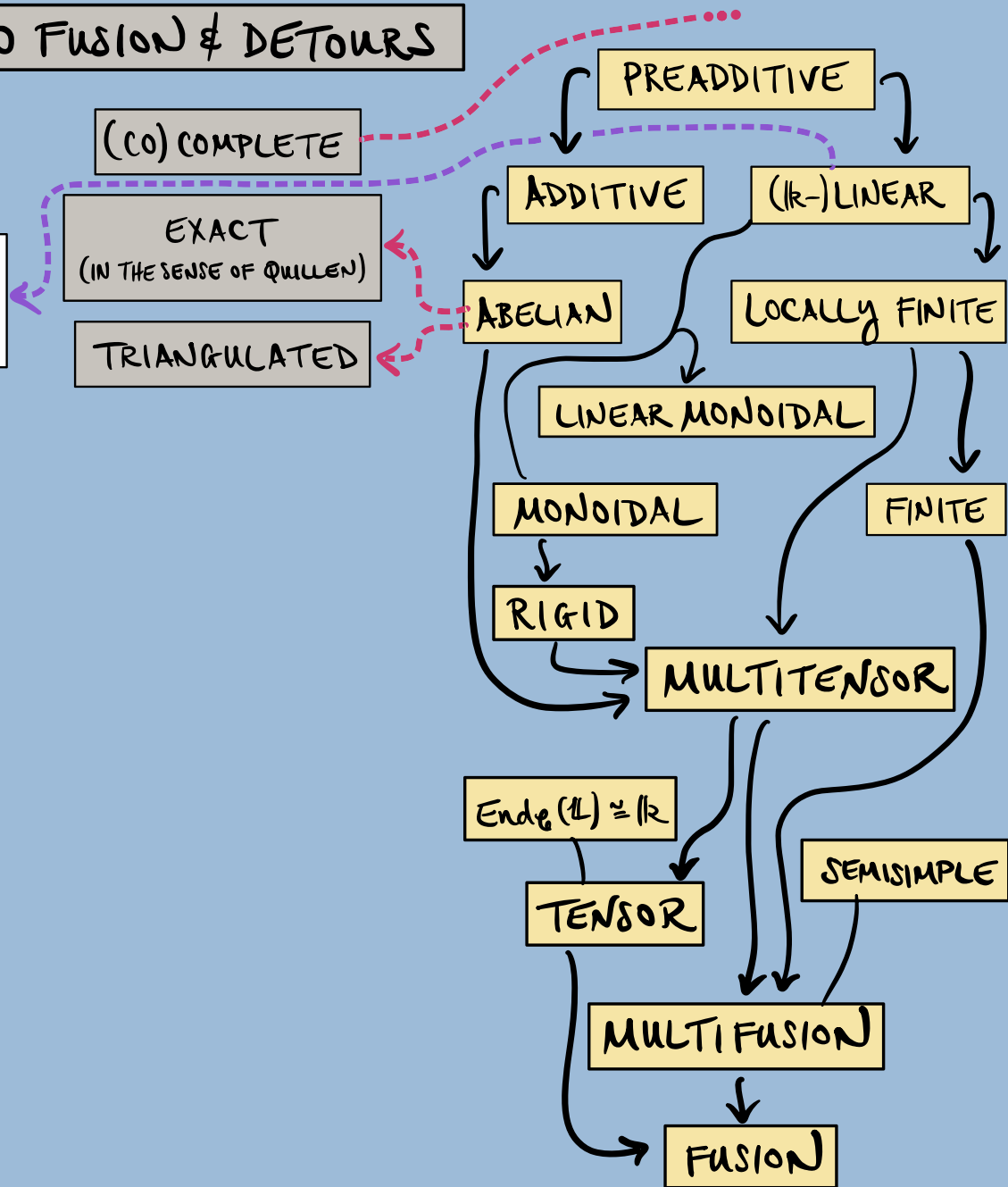
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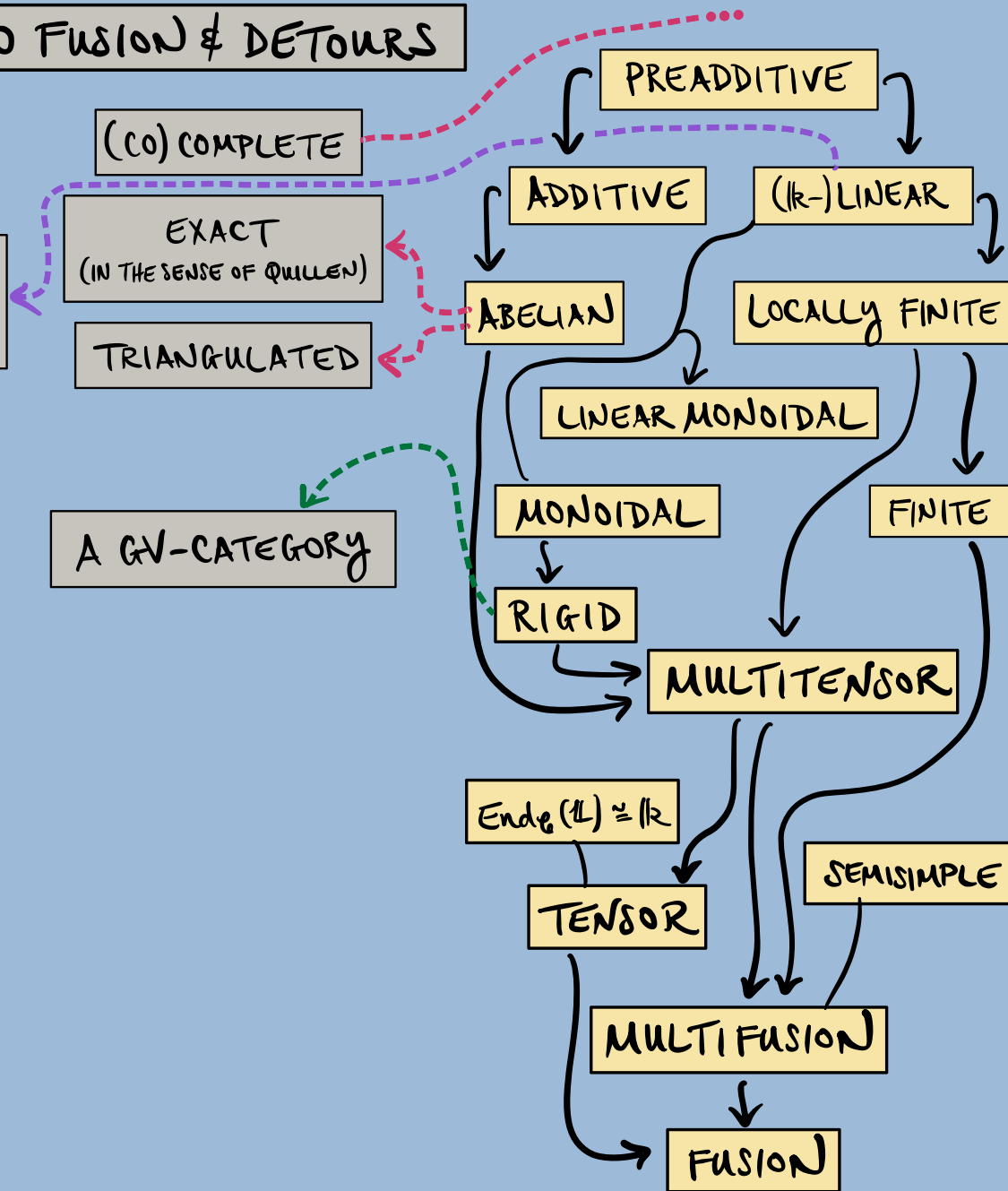
CATEGORY $\mathcal{A}$ ENRICHED OVER
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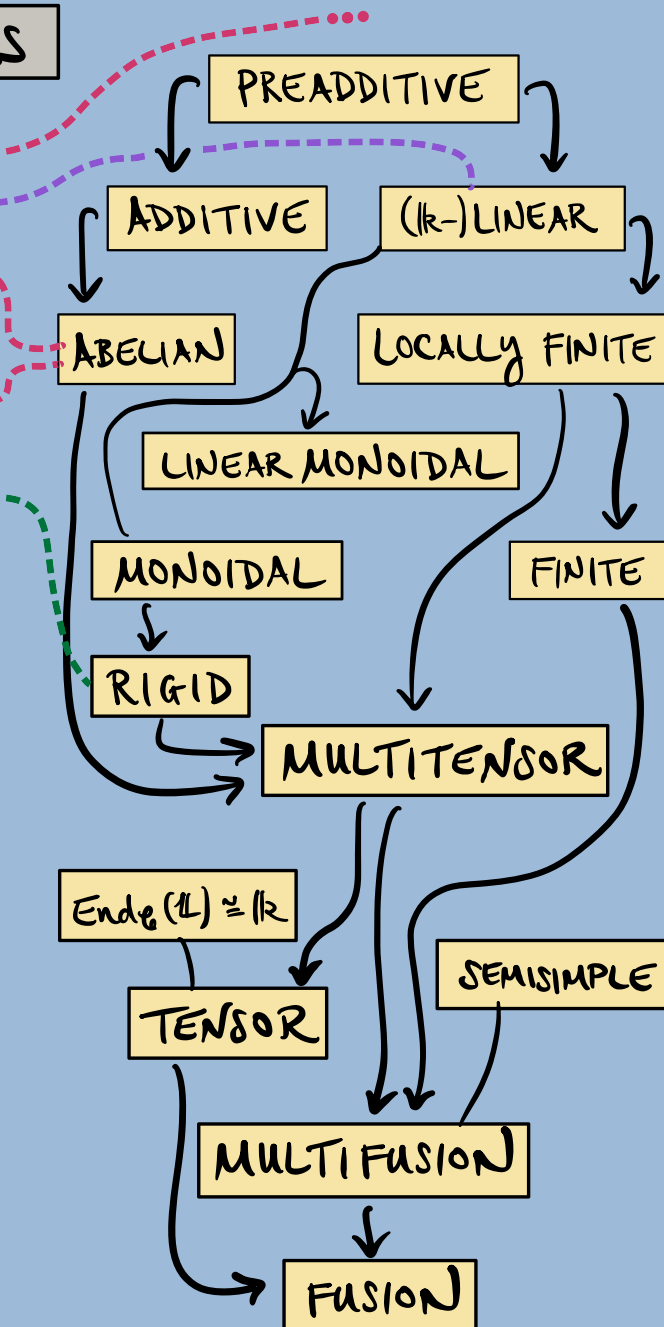
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USEFUL CONSTRUCTION:
COENDS & ENDS

(CO) COMPLETE
EXACT
(IN THE SENSE OF QUILLÉN)
TRIANGULATED

A GV-CATEGORY



LAST TIME: THE ROAD TO FUSION & DETOURS

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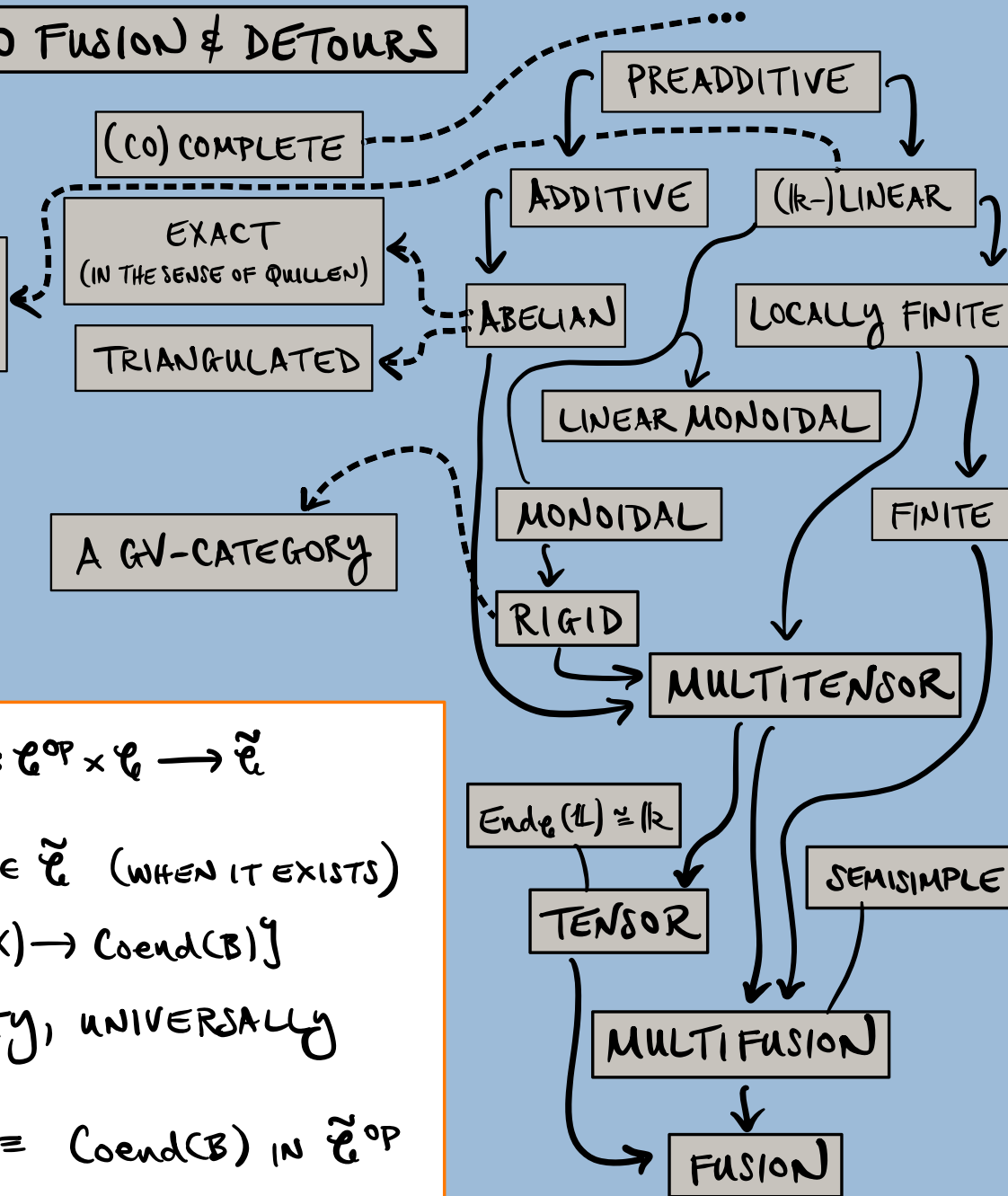
TAKE BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$

$\text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X) \in \tilde{\mathcal{C}}$ (WHEN IT EXISTS)

EQUIPPED W/ $\{\iota_X: B(X, X) \rightarrow \text{Coend}(B)\}$

SATISFYING DINATURALITY, UNIVERSALLY

$\text{End}(B) := \int_{X \in \mathcal{C}} B(X, X) = \text{Coend}(B) \text{ IN } \mathcal{C}^{\text{op}}$



LAST TIME: THE ROAD TO FUSION & DETOURS

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

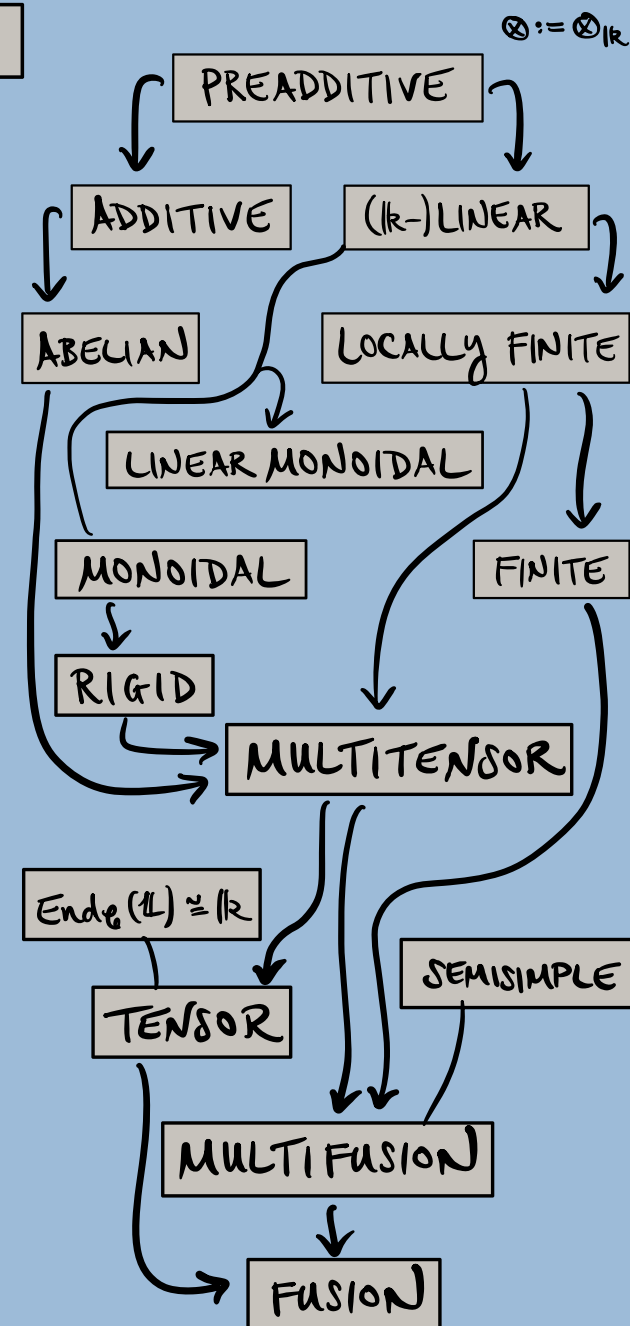
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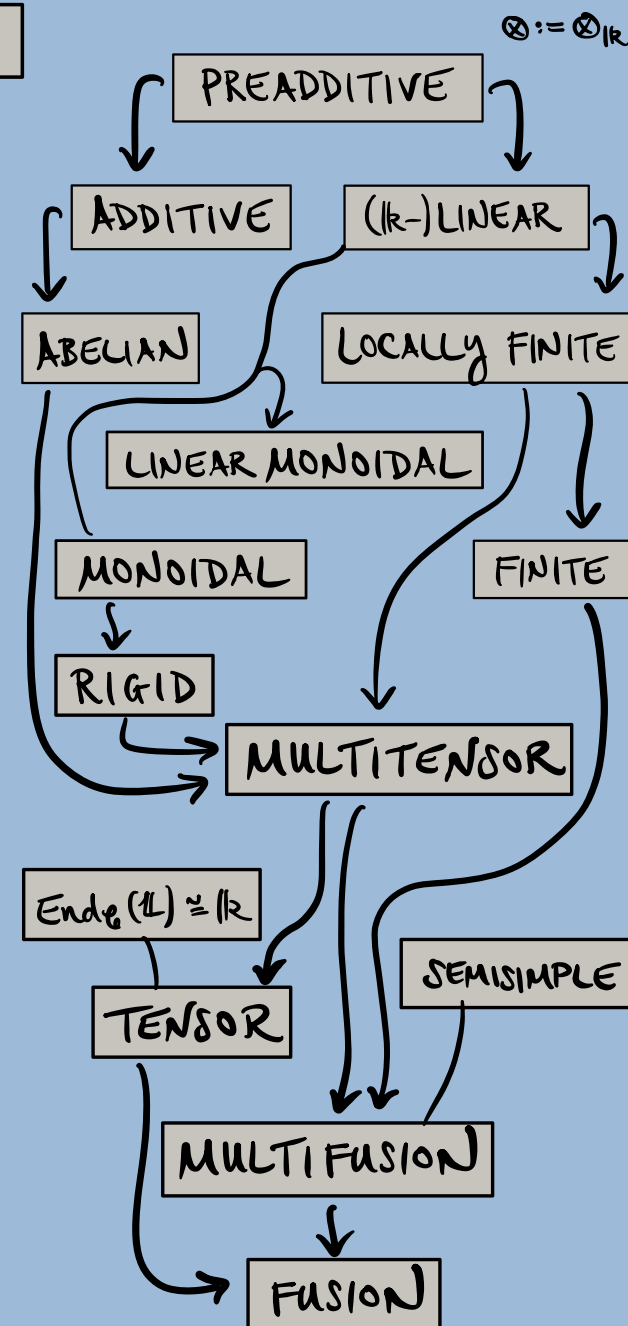
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RECOVER (CO)ALG.
FROM ITS CATEG.
OF (CO)MODS.

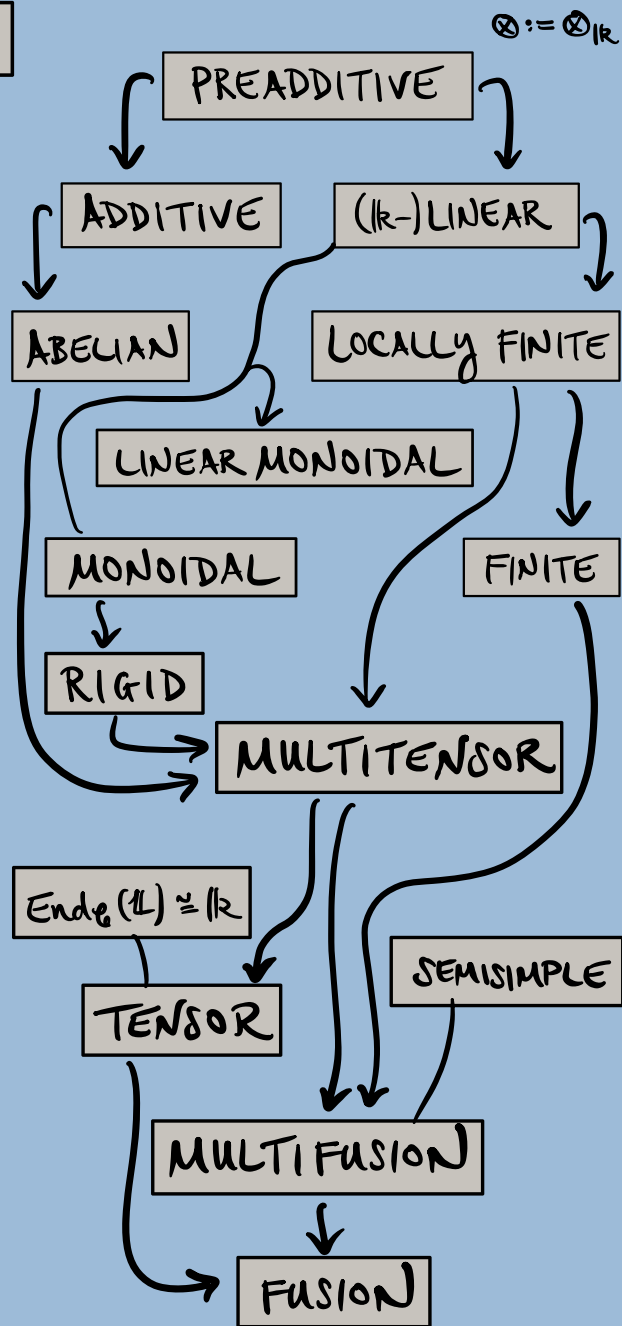
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RECONSTRUCTION THEORY

EX. TAKE $\tilde{c} = Fd \text{Vec.}$

FOR A k -COALGEBRA C :

$$\int_{V \in \text{FdComod-C}} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALG}$$

FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

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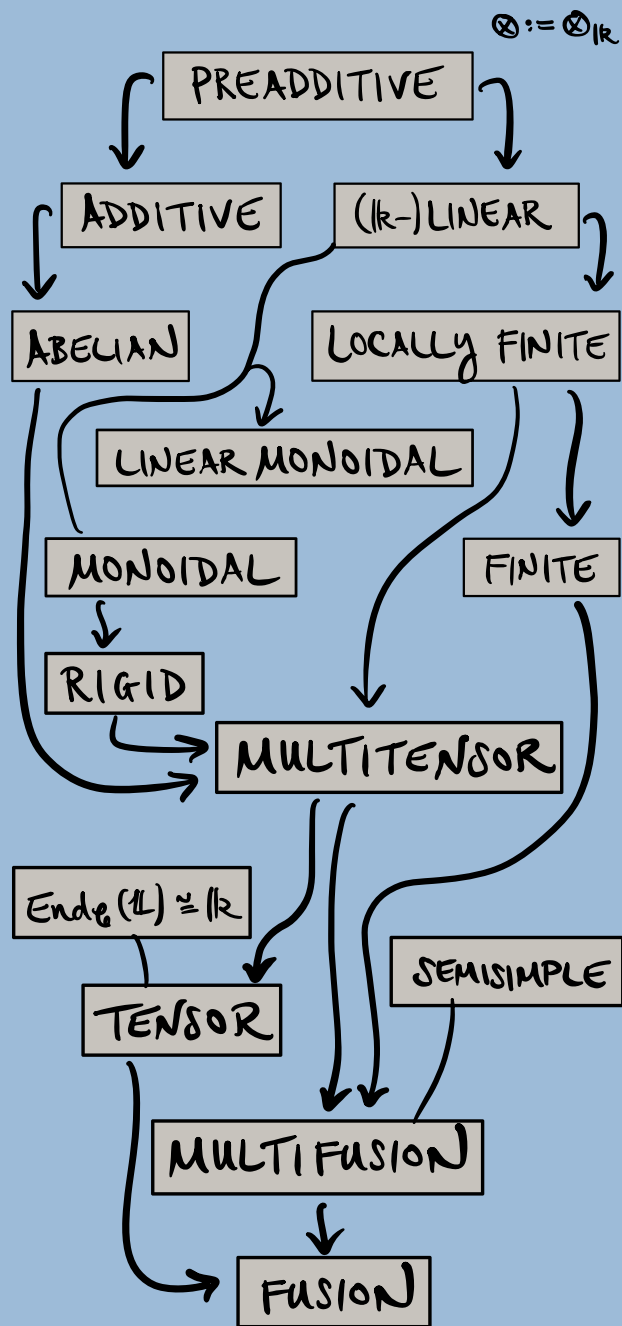
TAKE BIFUNCTOR $B: \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}_2$

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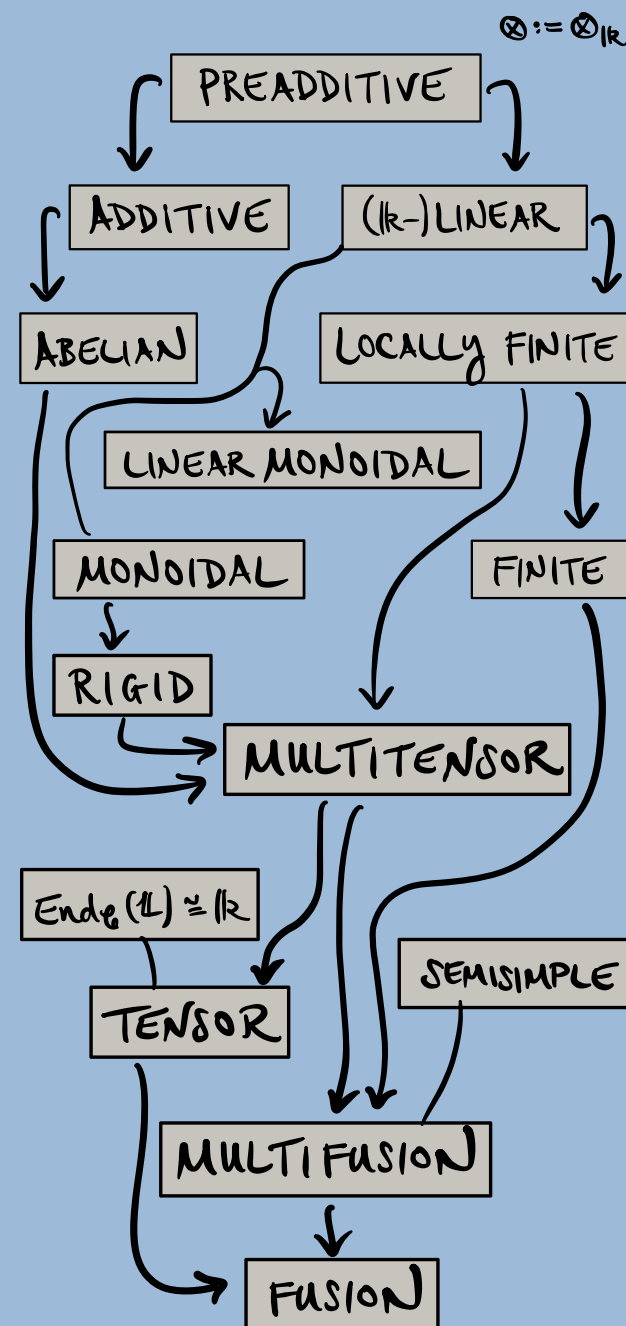
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QUESTION:

TAKE A (CO)ALGEBRAIC STRUCTURE $\mathcal{S}$
 $\rightarrow$ A CATEGORY OF (CO)MODULES OVER $\mathcal{S}$
 WITH NICE FEATURES ★



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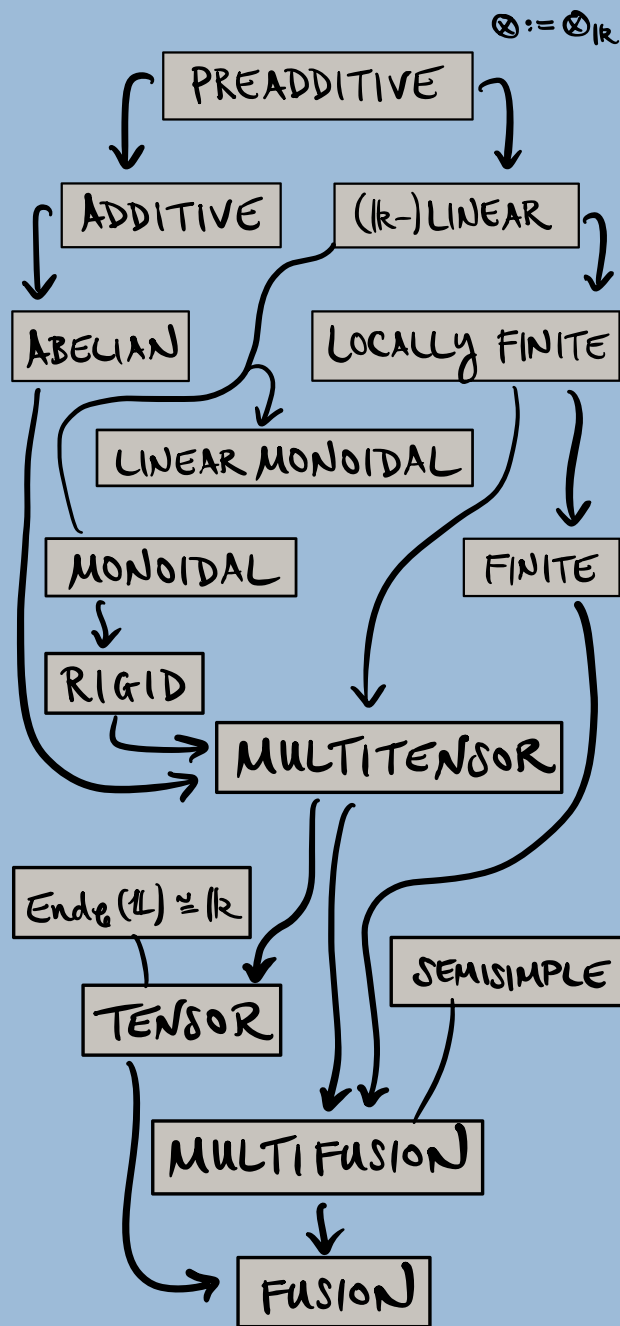
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DOES A CATEG. $\mathcal{C}$ SATISFYING ★
 $\Rightarrow \mathcal{C} \cong (\text{CO})\text{MODS OVER } \mathcal{S}$?



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

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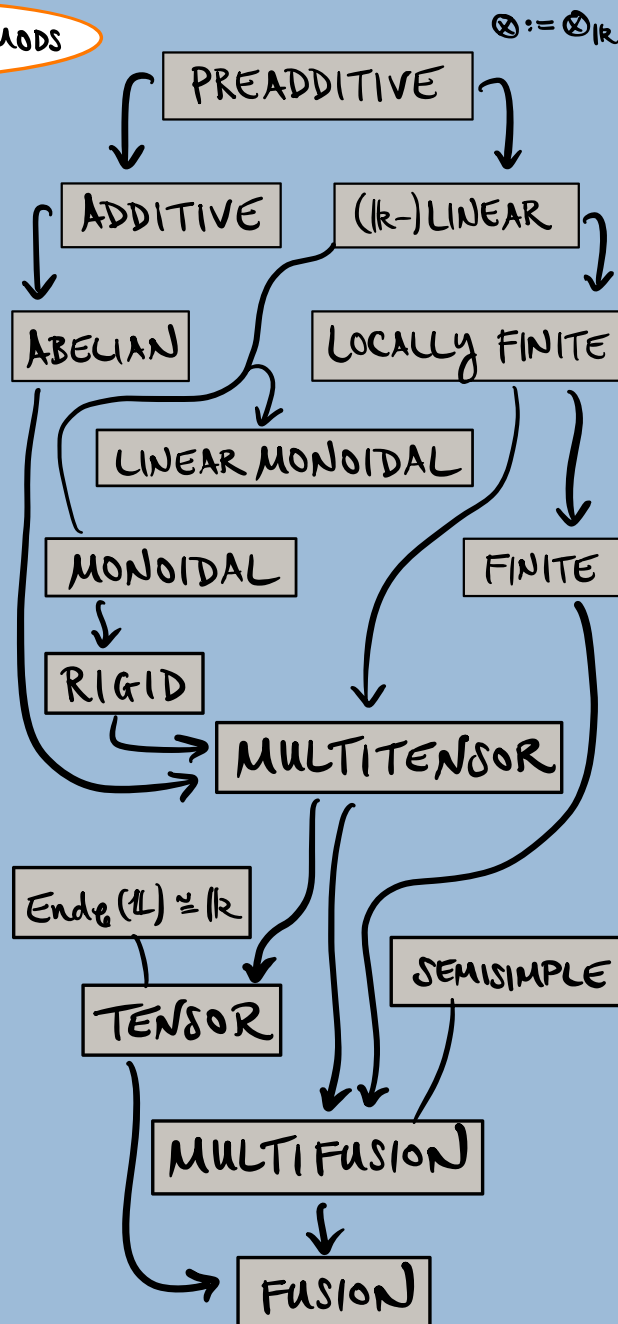
$$\int_{V \in A\text{-FdMod}} V \otimes V^* \simeq A \text{ AS } \mathbb{K}\text{-ALGS}$$

RECOVER $(\text{co})\text{ALG.}$
FROM ITS CATEG.
OF $(\text{co})\text{MODS.}$

QUESTION:

TAKE A $(\text{co})\text{ALGEBRAIC STRUCTURE } \mathcal{S}$
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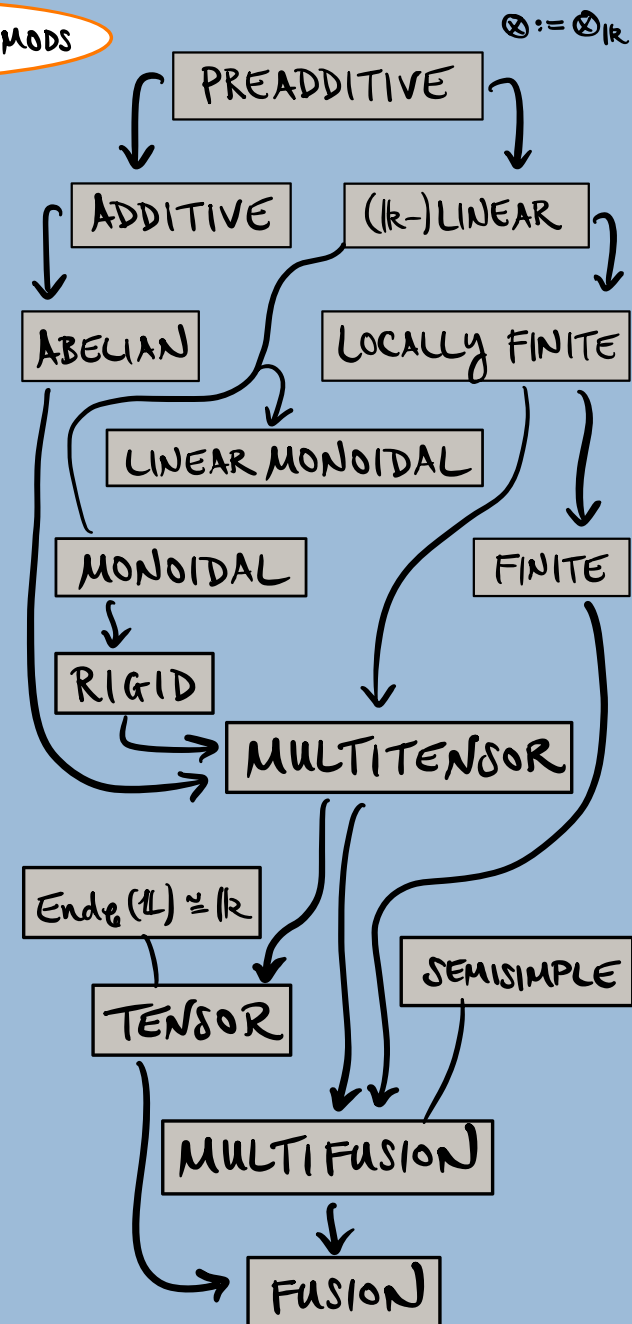
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RECONSTRUCTION THEORY

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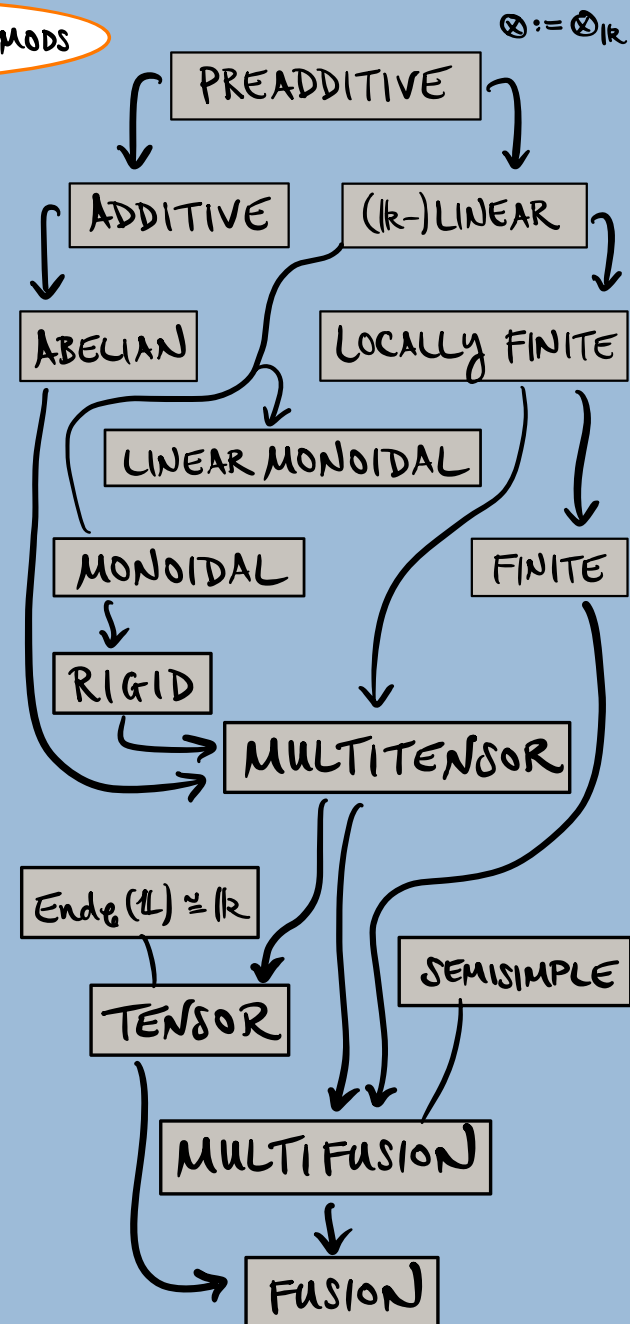


RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

RECIPE -

- TAKE A CATEGORY $\mathcal{C}$ W/ NICE FEATURES



RECONSTRUCTION THEORY

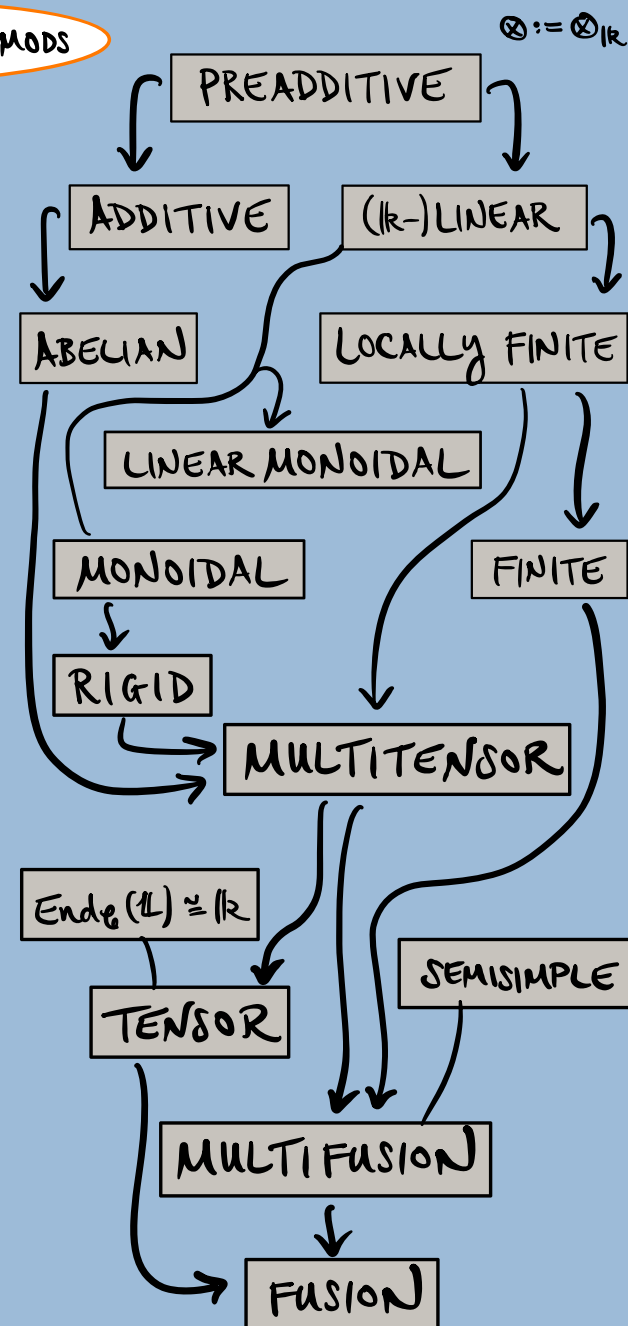
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- EQUIP IT W/ A NICE FUNCTOR:

$$U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$$



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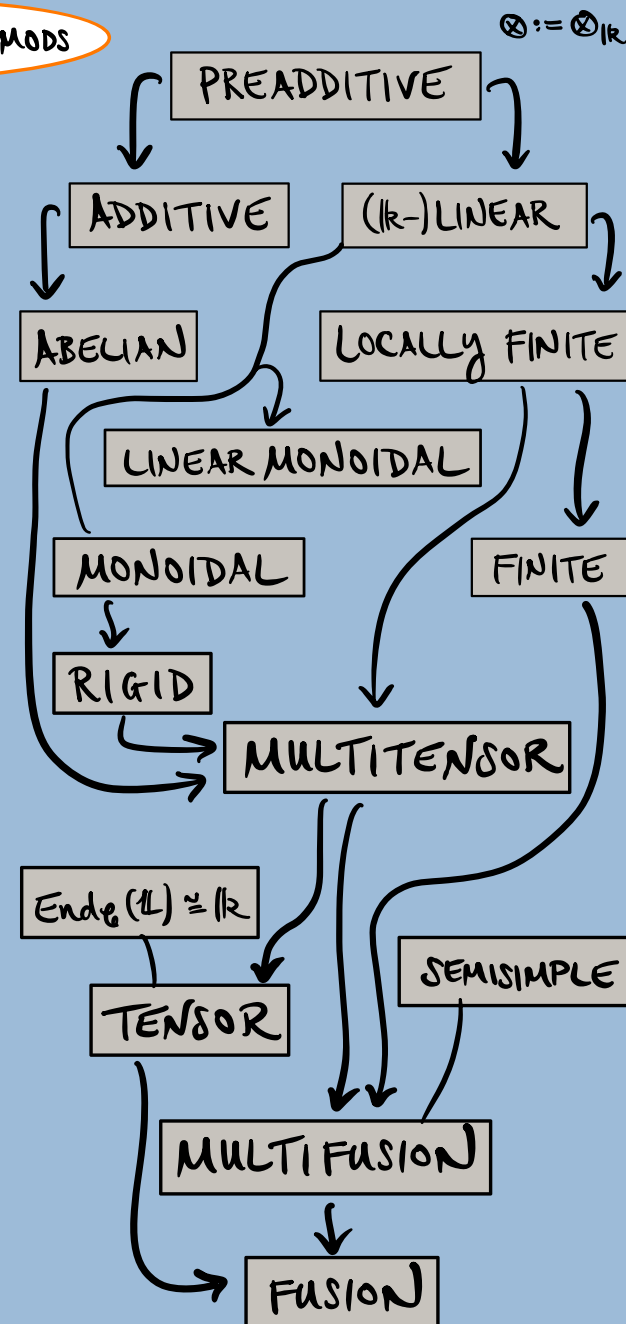
$$U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$$

- EITHER: FORM THE COEND -

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X) \in (\text{Fd})\text{Vec}$$

(WHEN IT EXISTS, E.G. $\mathcal{C} \simeq \text{SMALL CATEG.}$)

‡ GET THAT IT IS A NICE COALG. $_{/\mathbb{K}}$...



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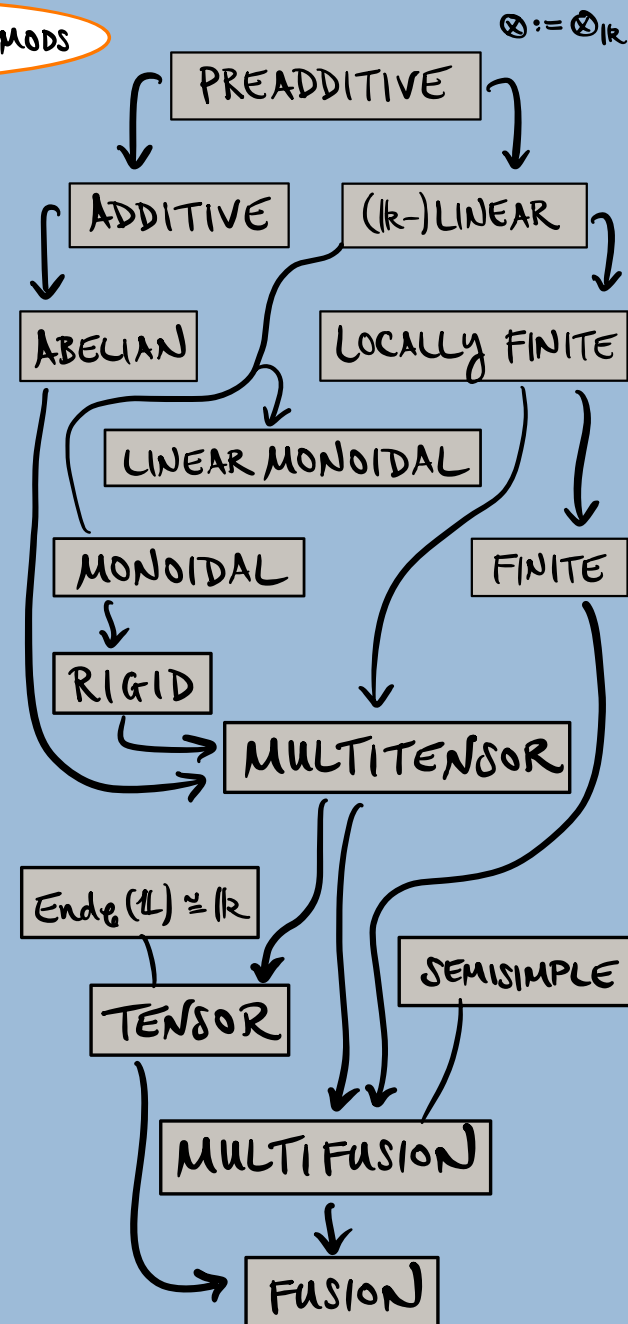
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OR FORM THE END -

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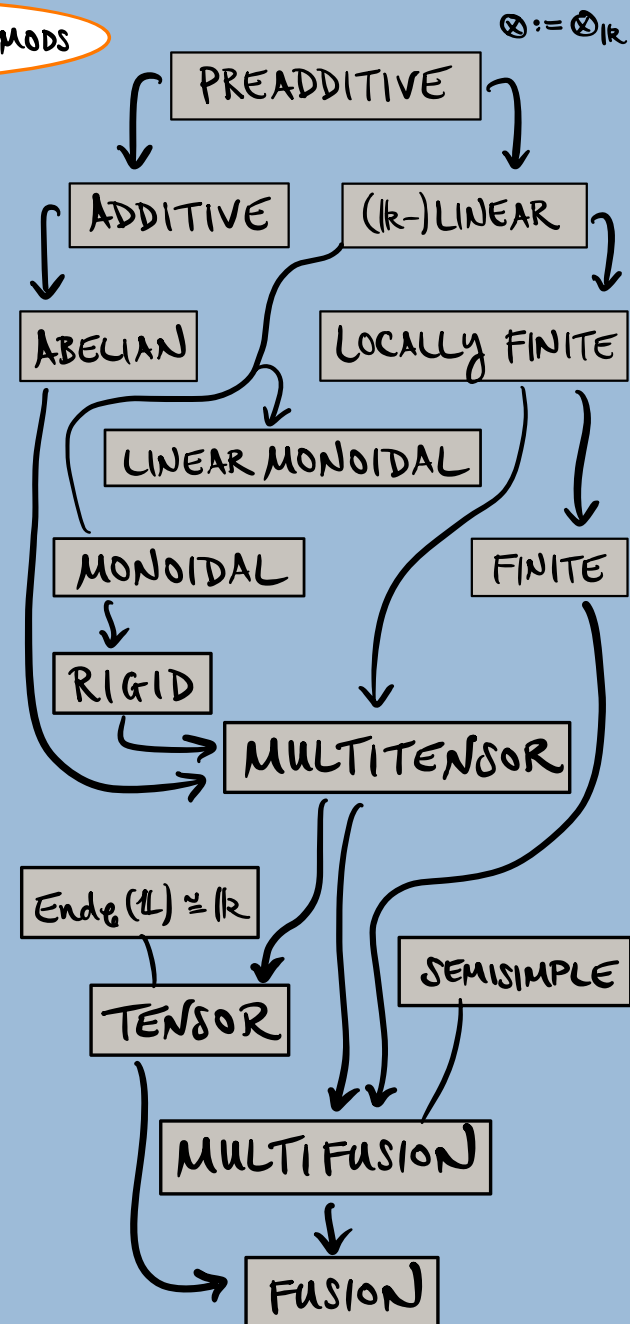
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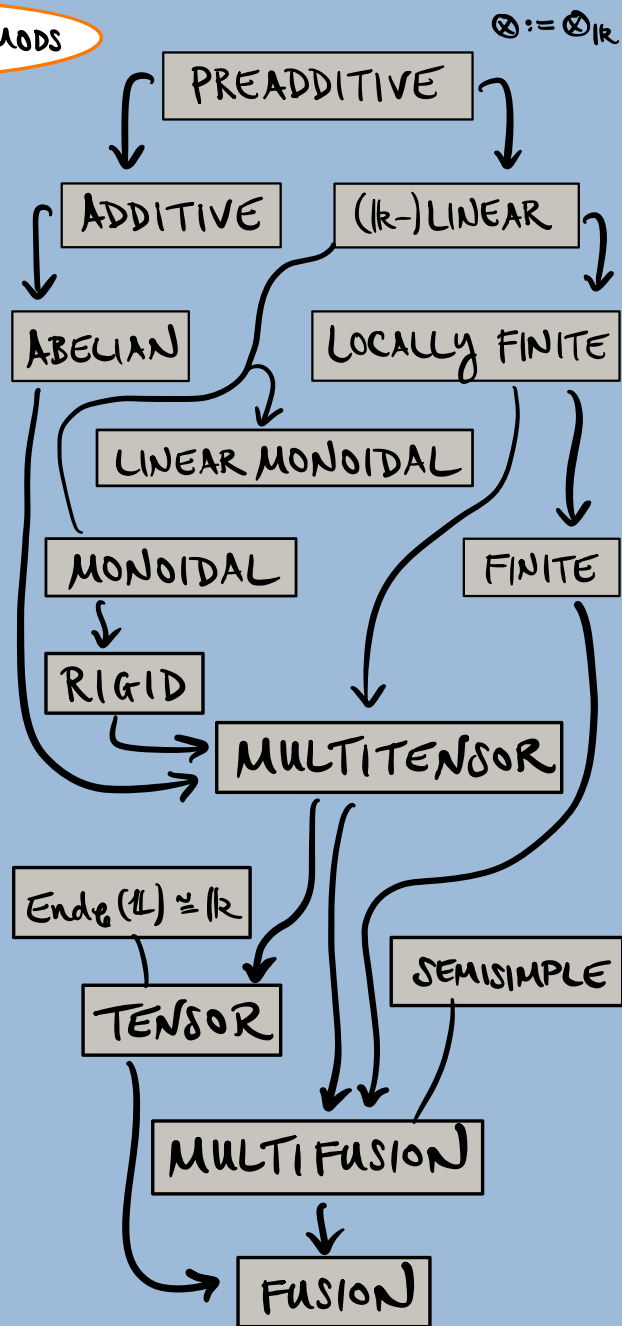
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RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF STRUCTURES/IR"
Struc

$$(\mathcal{C}, U) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(U)$$

$$((\text{Co})\text{Mod}/s, \text{Forg}) \longleftarrow \mathcal{S}$$



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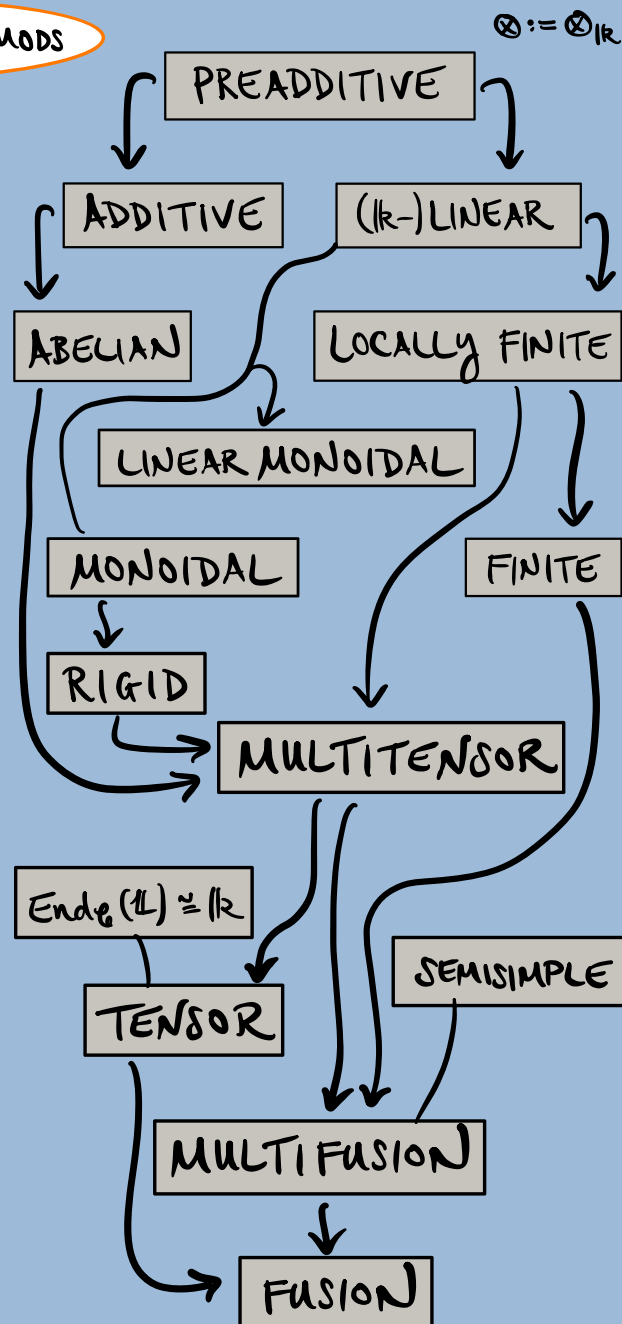
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$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod}/_{(\text{Co})\text{End}_{\mathcal{C}}(U)}$ AS CATEGORIES

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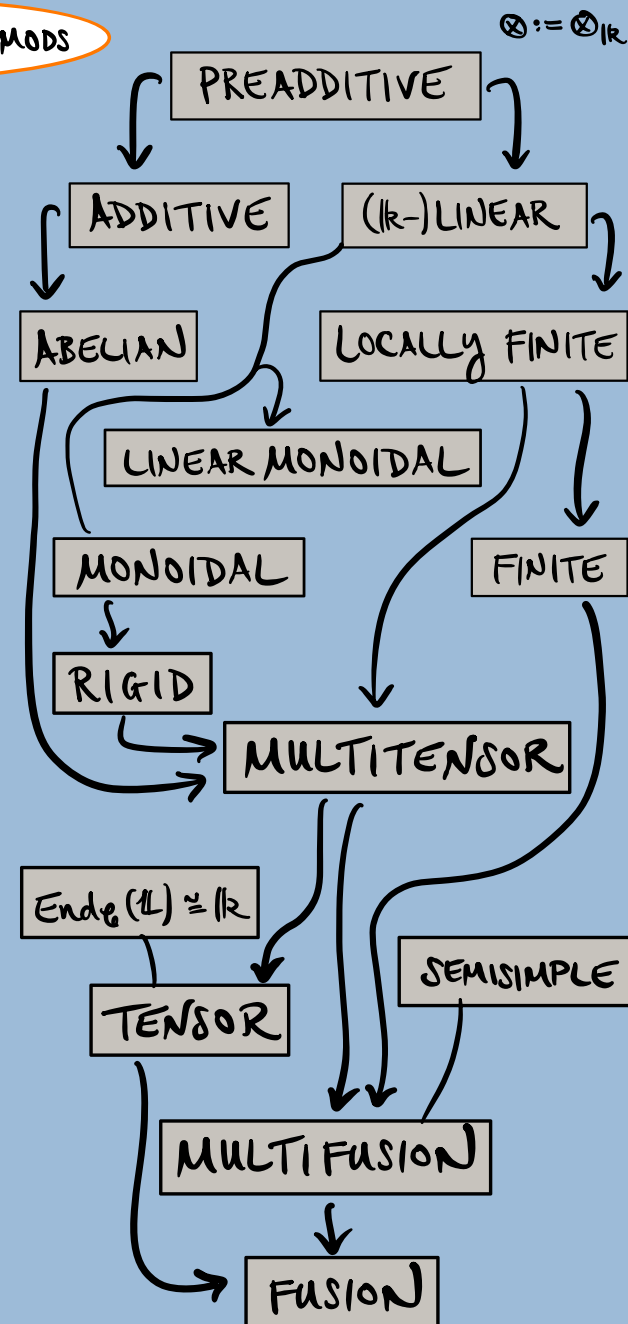
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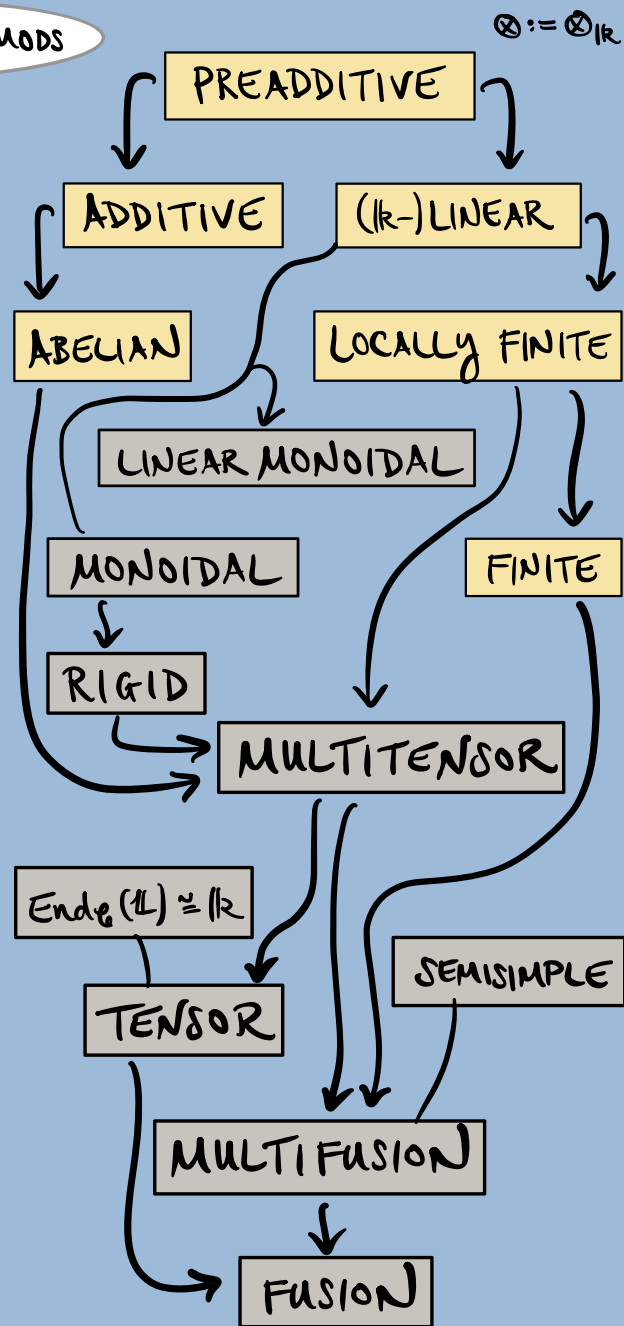
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 (SEE HIGHLIGHTED FEATURES)
 • LINEAR • EXACT • FAITHFUL

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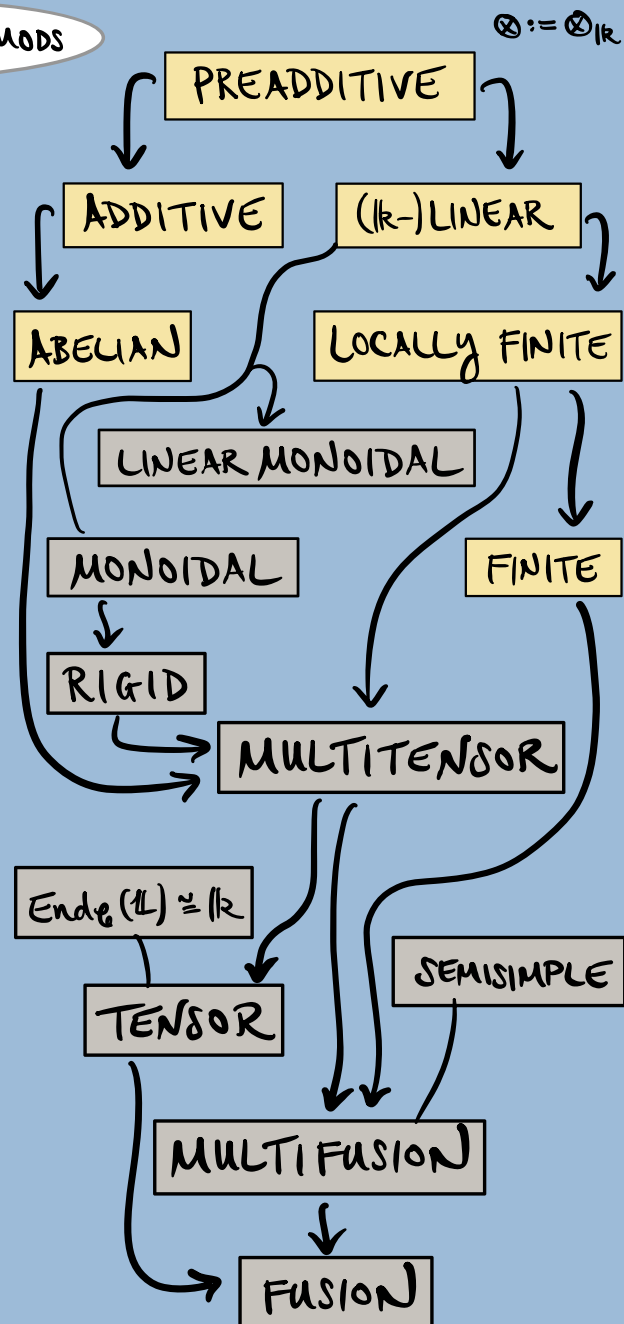
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(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

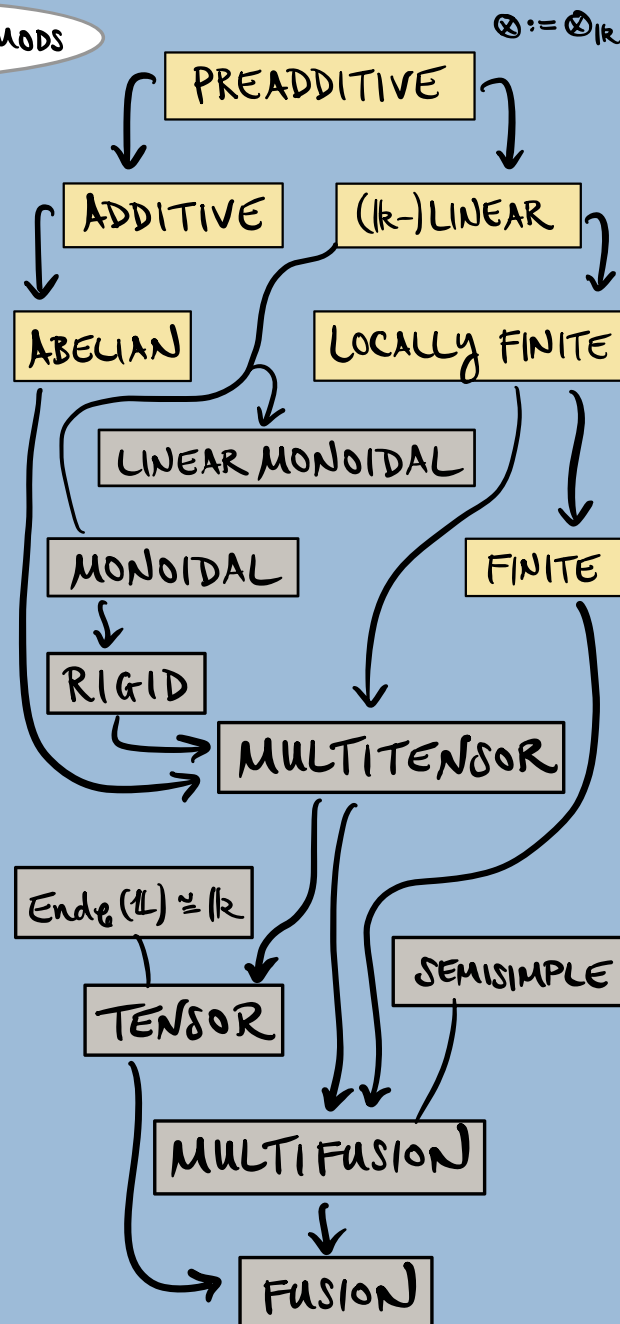
"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ ^{F.D. ALGEBRAS} / $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ FdAlg

$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$

$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$

$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod} / (\text{Co})\text{End}_{\mathcal{C}}(U)$ AS CATEGORIES

$\therefore \mathcal{S} \simeq (\text{Co})\text{End}_{(\text{Co})\text{Mod}/\mathcal{S}}(U)$ AS STRUCTURES



RECONSTRUCTION THEORY

$$\mathcal{C} \text{ SAT. ??} \Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Fd Vec}$

(SEE HIGHLIGHTED
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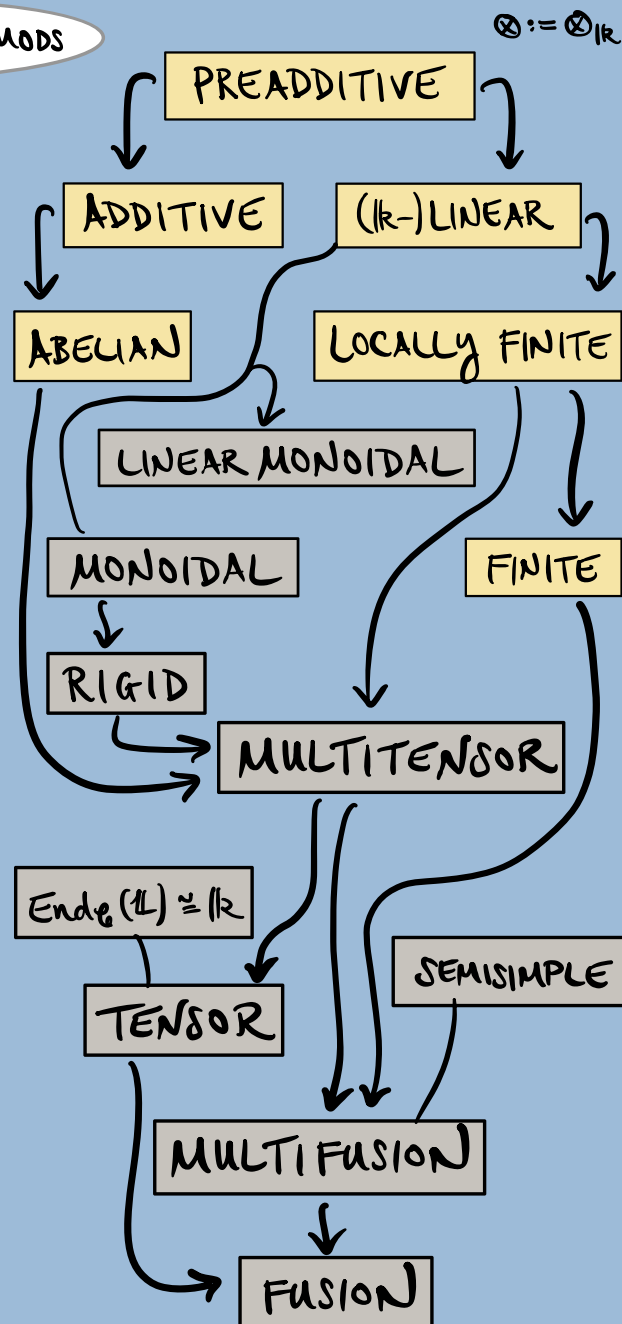
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EVERY FINITE LINEAR ABELIAN CATEGORY
EQUIPPED W/ A LINEAR, EXACT, FAITHFUL U
IS EQUIVALENT TO $A\text{-FdMod}$
FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

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(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

$$U = \text{Hom}_{\mathcal{C}}(\bigoplus_{i \in \text{Irr}(\mathcal{C})} \text{Proj Cov}(X_i), -)$$

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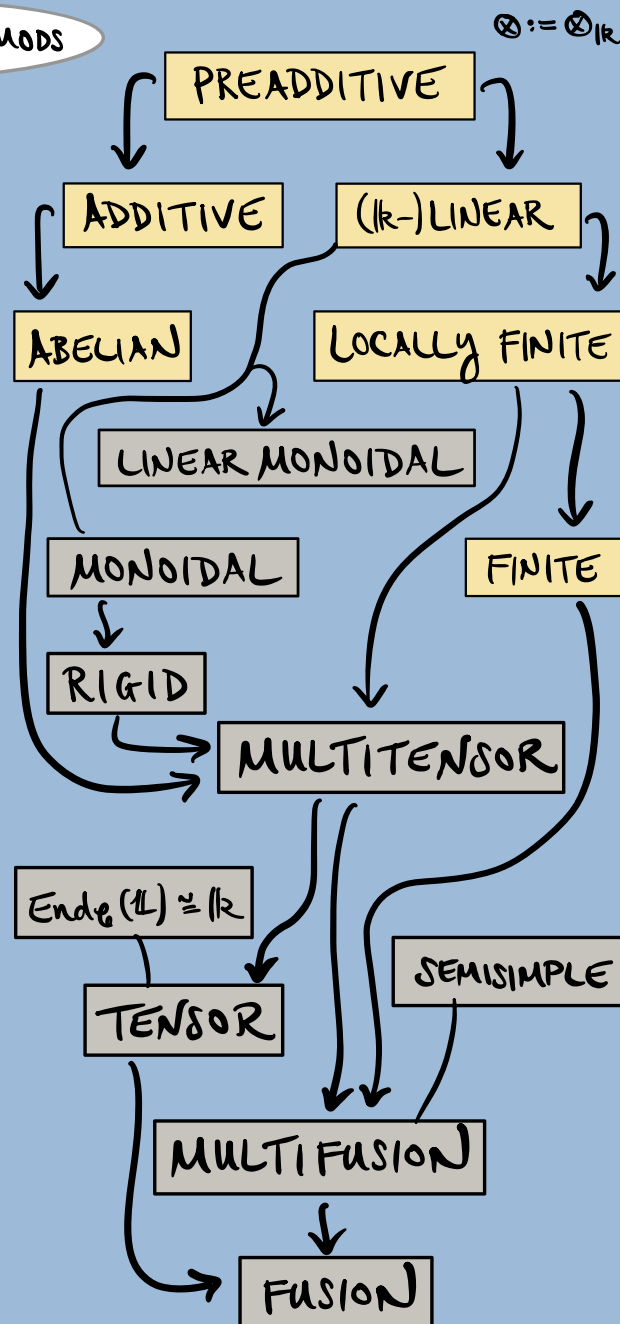
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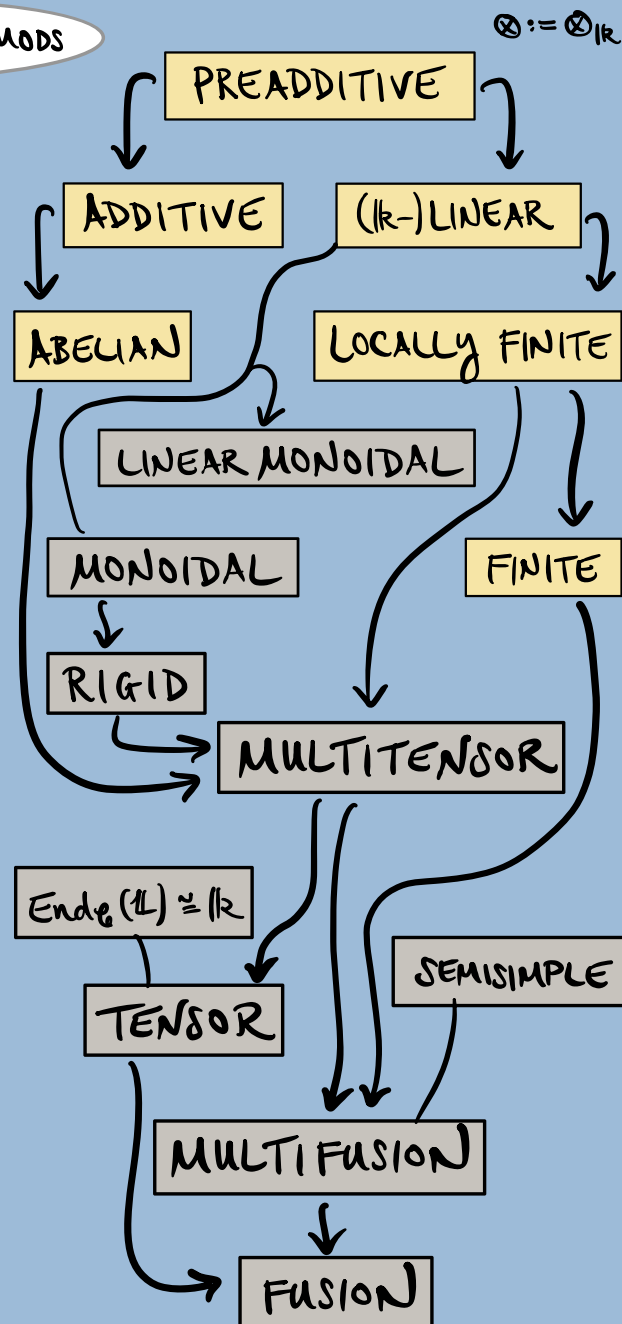
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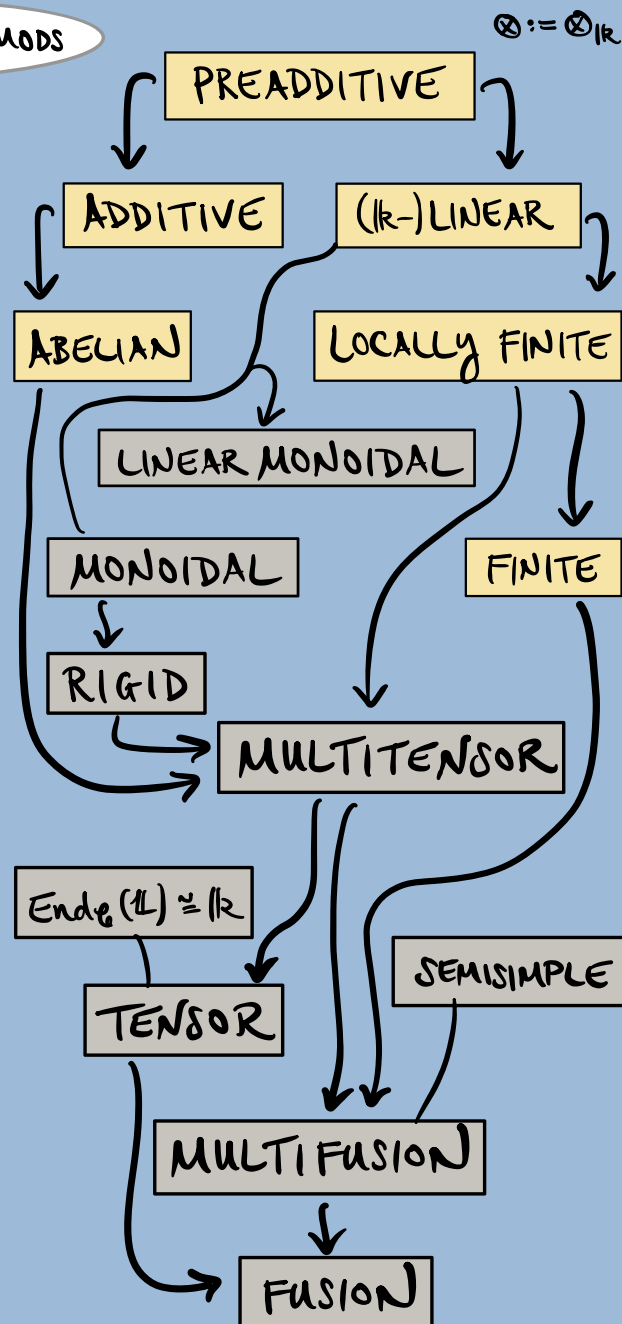
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RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (co)MODS$

TAKE ~~NICE~~ $\mathcal{C}$ & $U: \mathcal{C} \rightarrow Fd Vec$
 (SEE HIGHLIGHTED FEATURES)
 • LINEAR • EXACT • FAITHFUL

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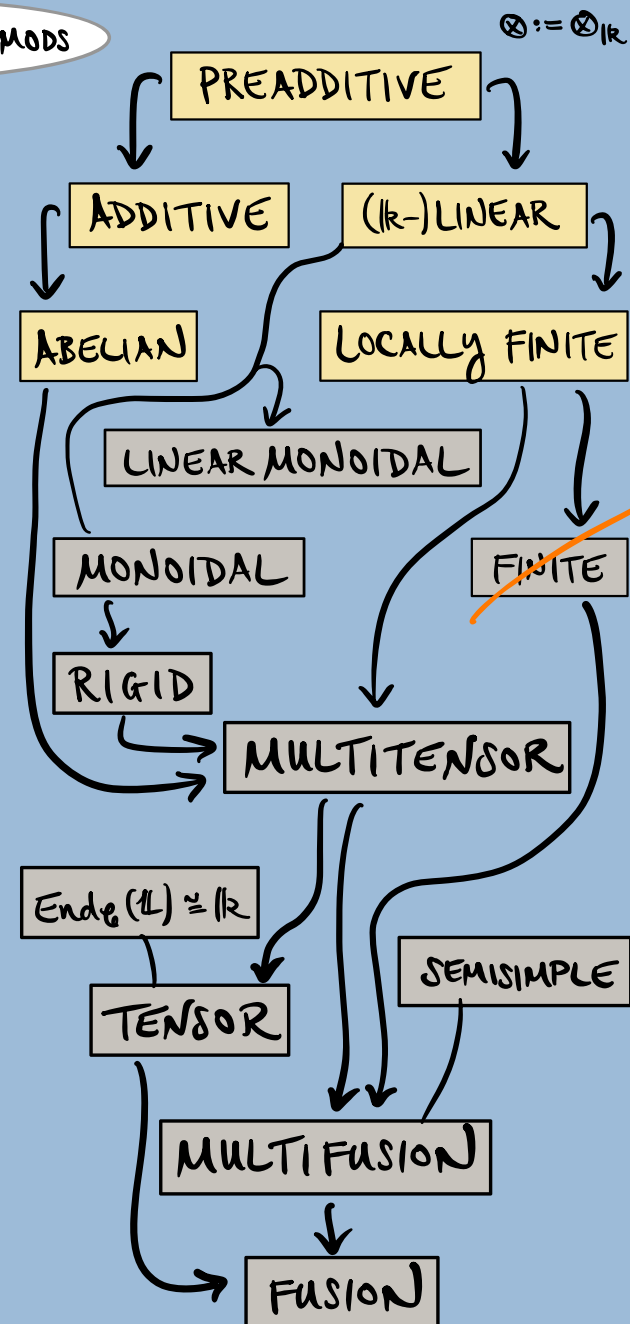
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RECONSTRUCTION THEORY

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(SEE HIGHLIGHTED
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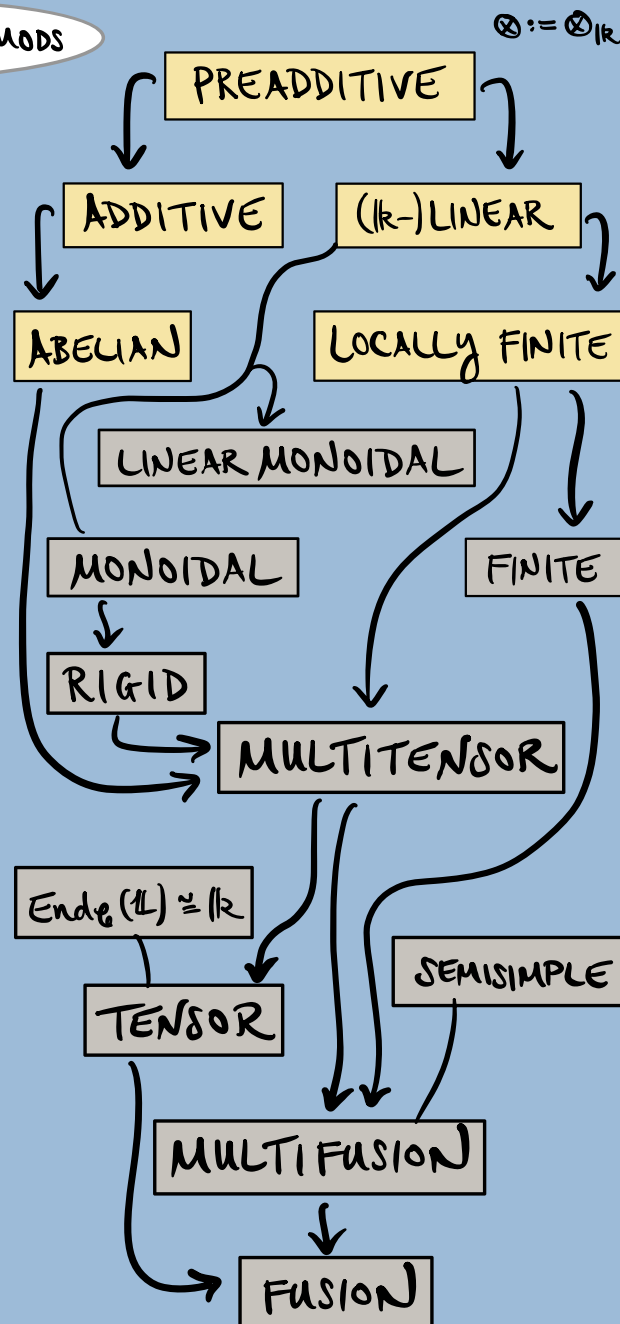
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(SEE HIGHLIGHTED
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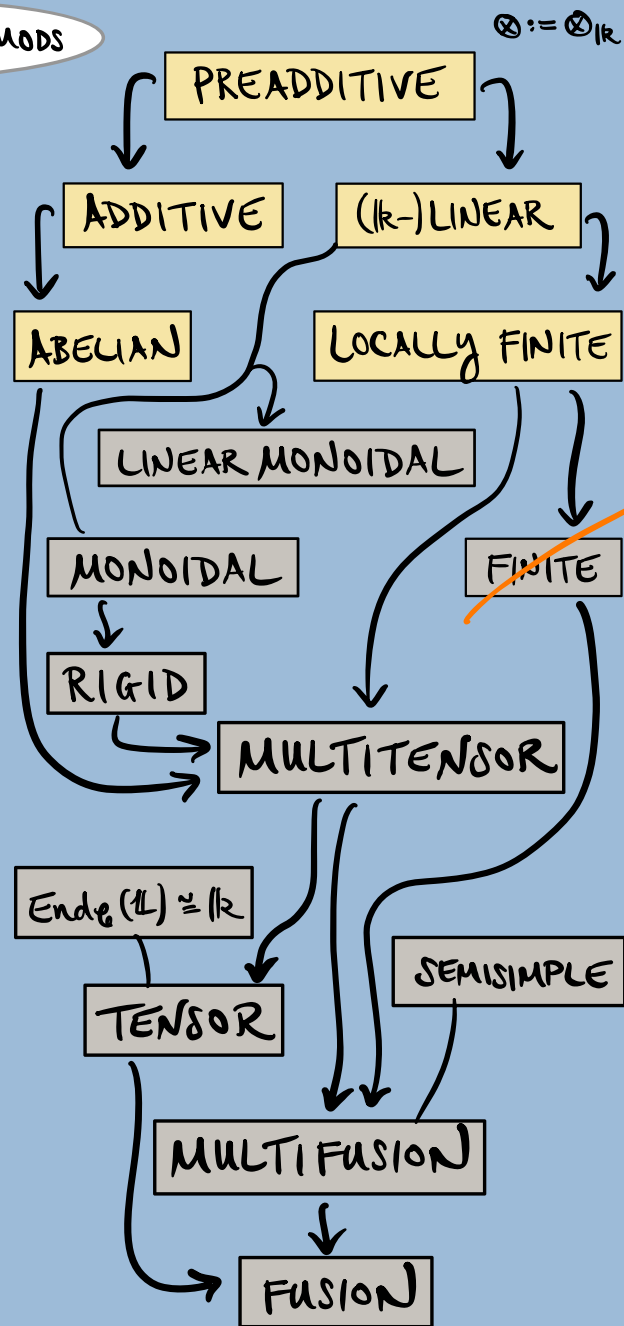
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"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ ^{COALGEBRAS} / $\mathbb{K}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ $\{\text{Coalg}\}$

$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$

$(\text{CoMod}/S, \text{Forg}) \longleftarrow S$

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RECONSTRUCTION THEORY

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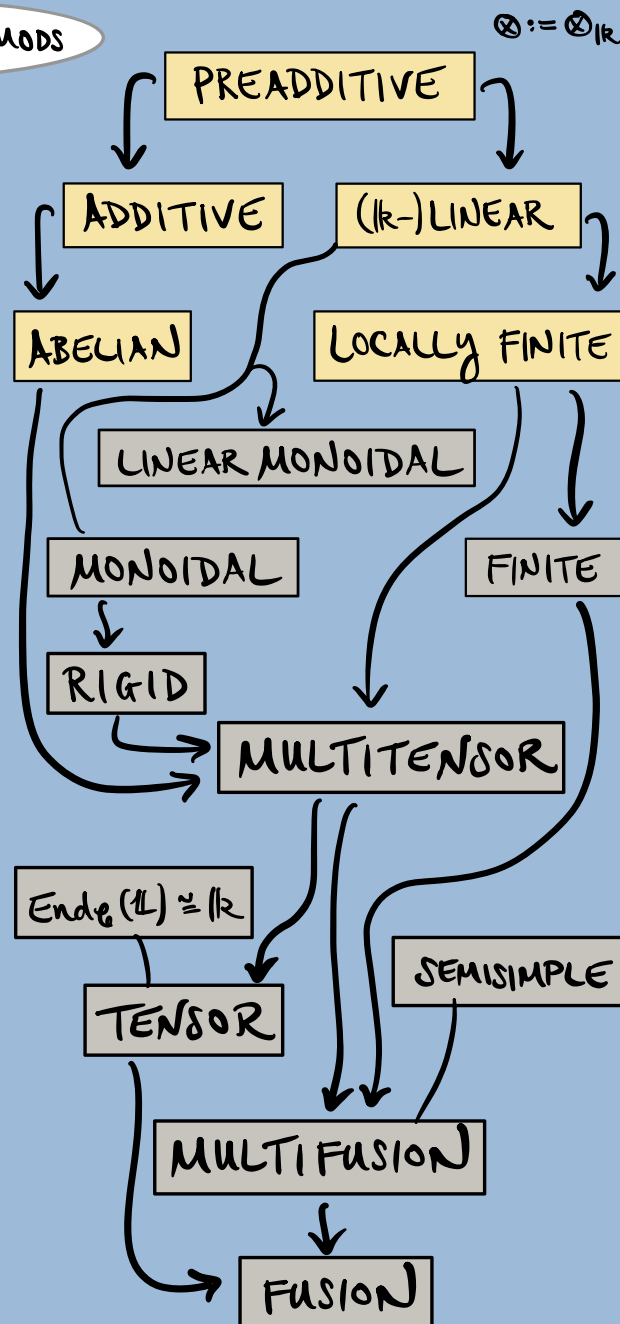
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EVERY LOCALLY FINITE, LINEAR ABELIAN CATEG.
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FOR SOME COALGEBRA $\mathcal{C}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & $\text{HCS } U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED
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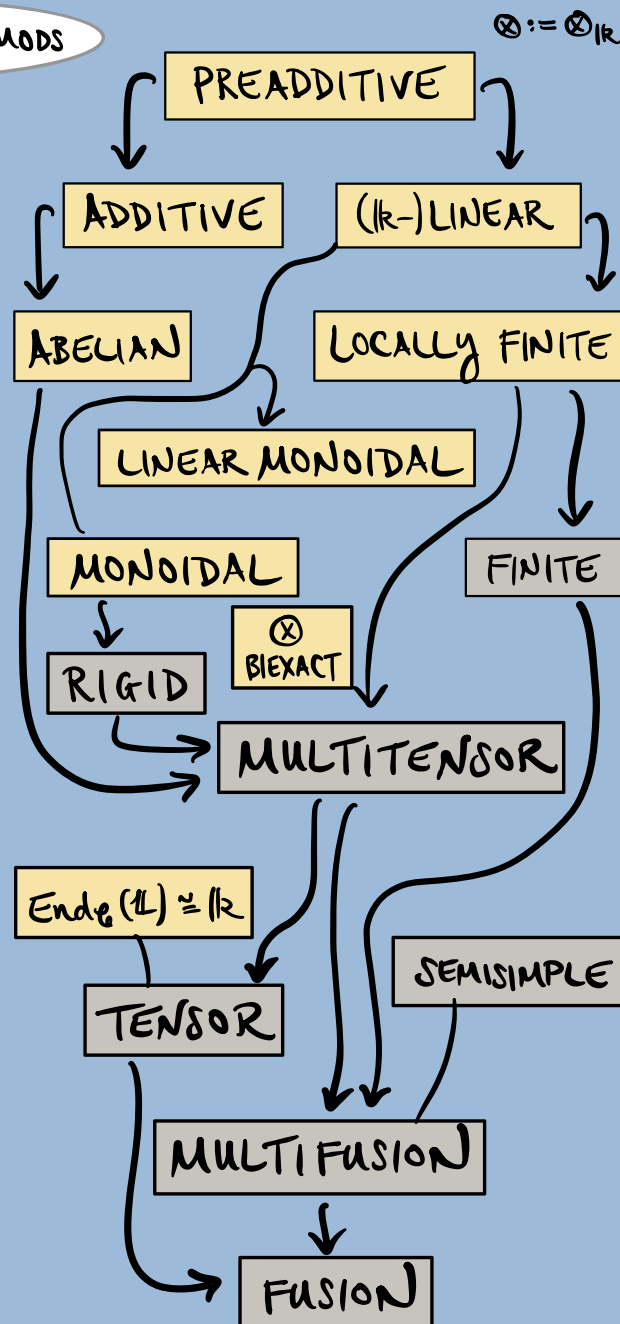
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TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{k}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{k}$$

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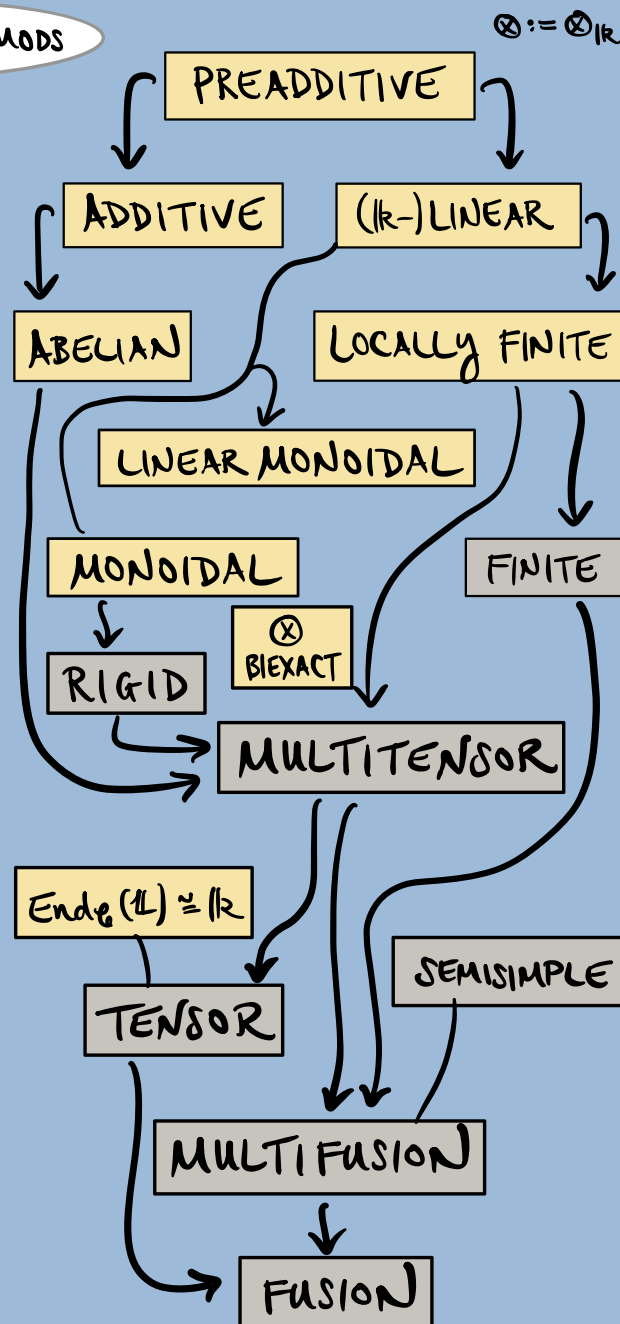
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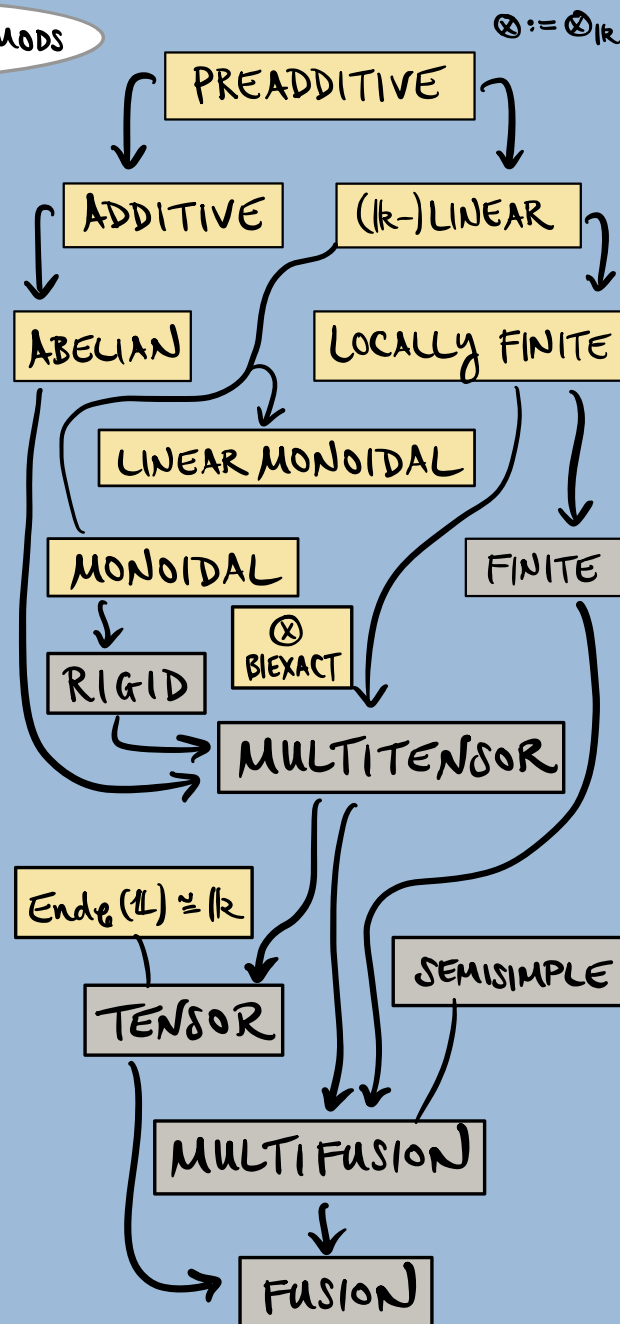
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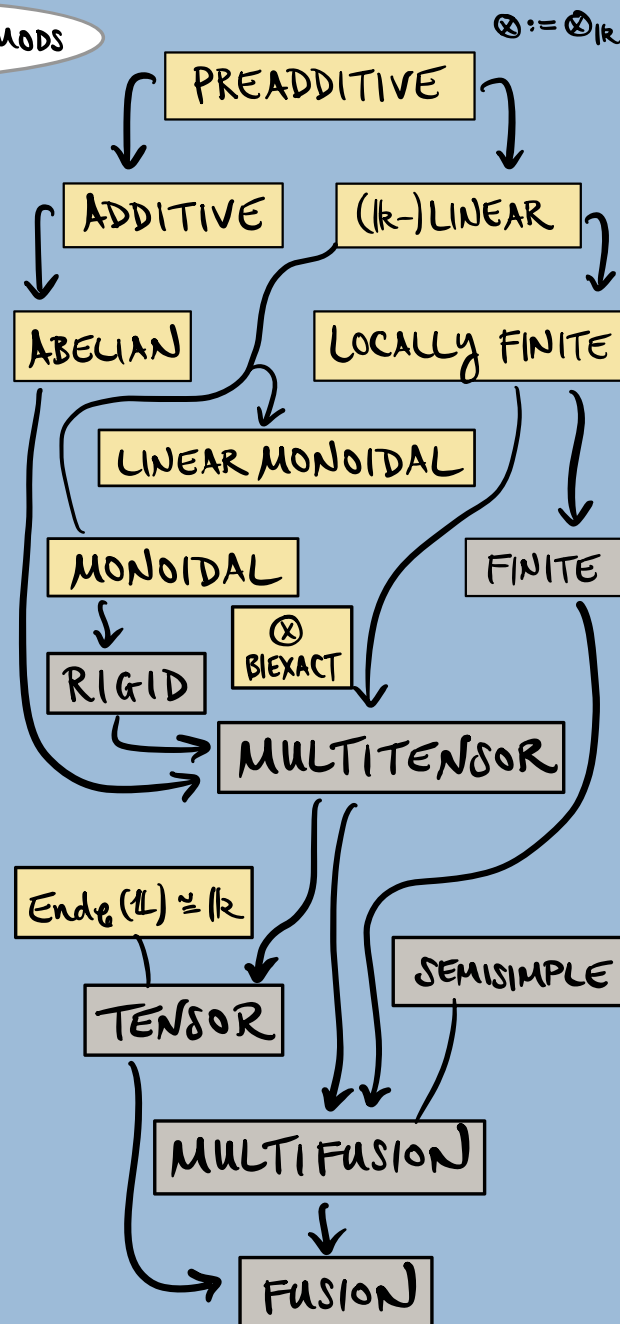
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EVERY "RING" CAT. EQUIPPED W/ A "FIBER" FUNCTOR
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 FOR SOME BIALGEBRA $\mathcal{B}$



RECONSTRUCTION THEORY

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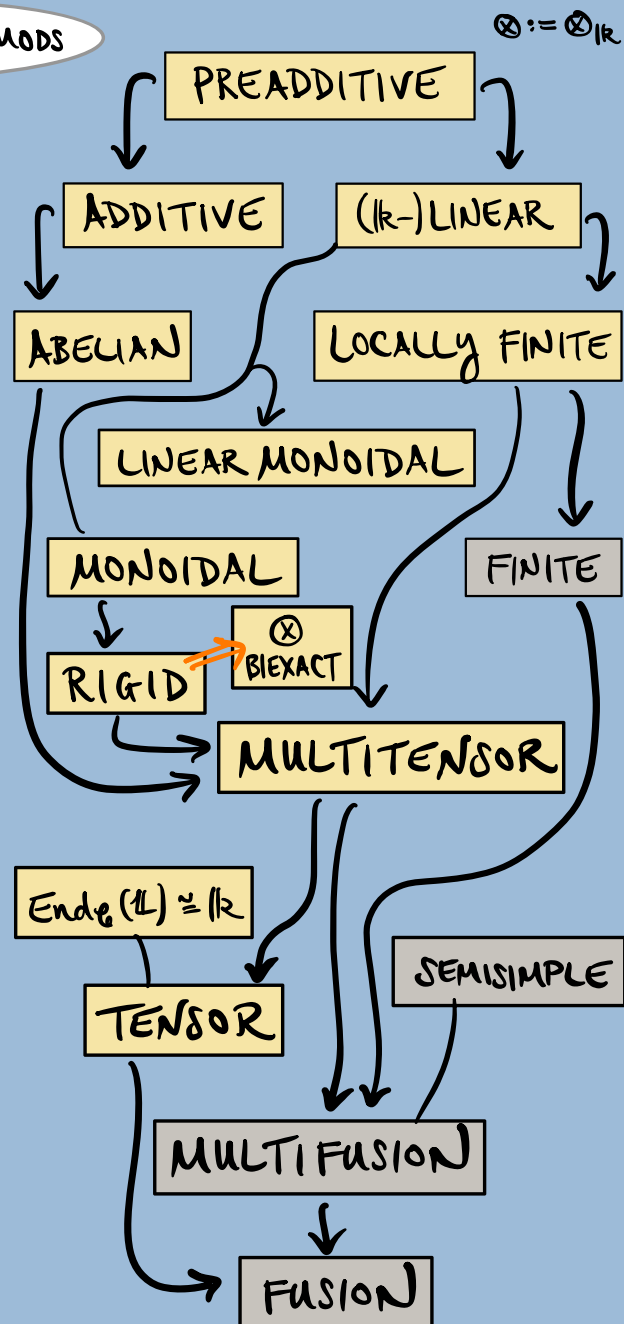
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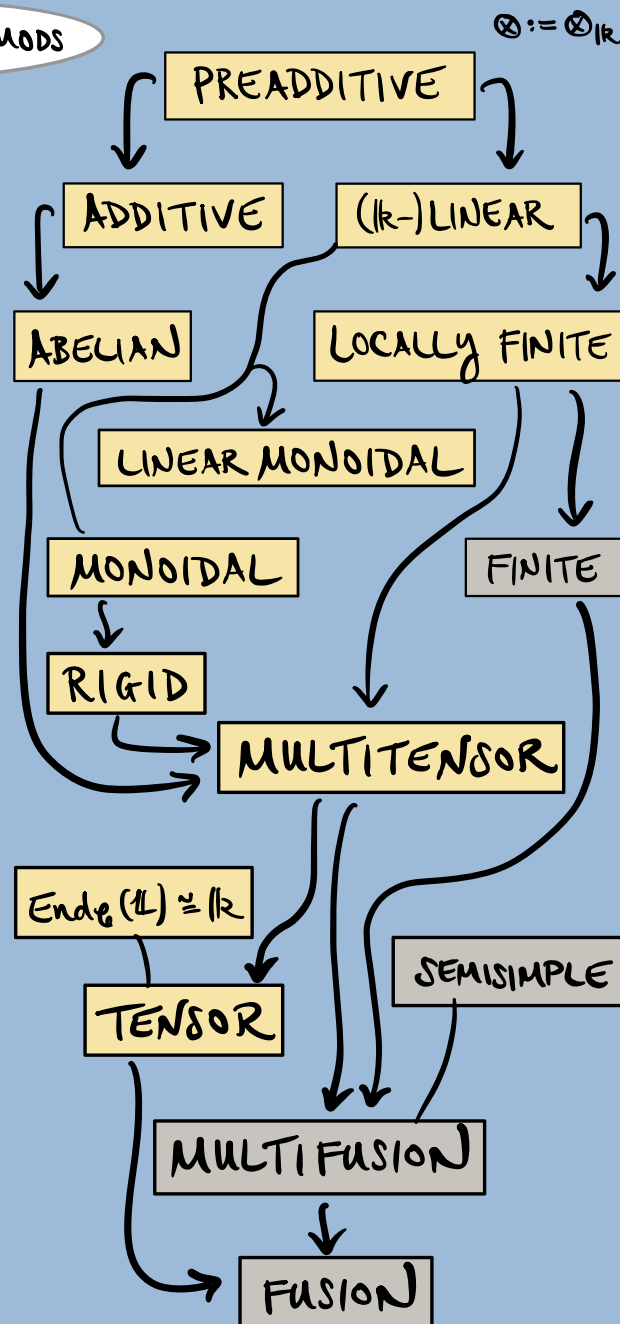
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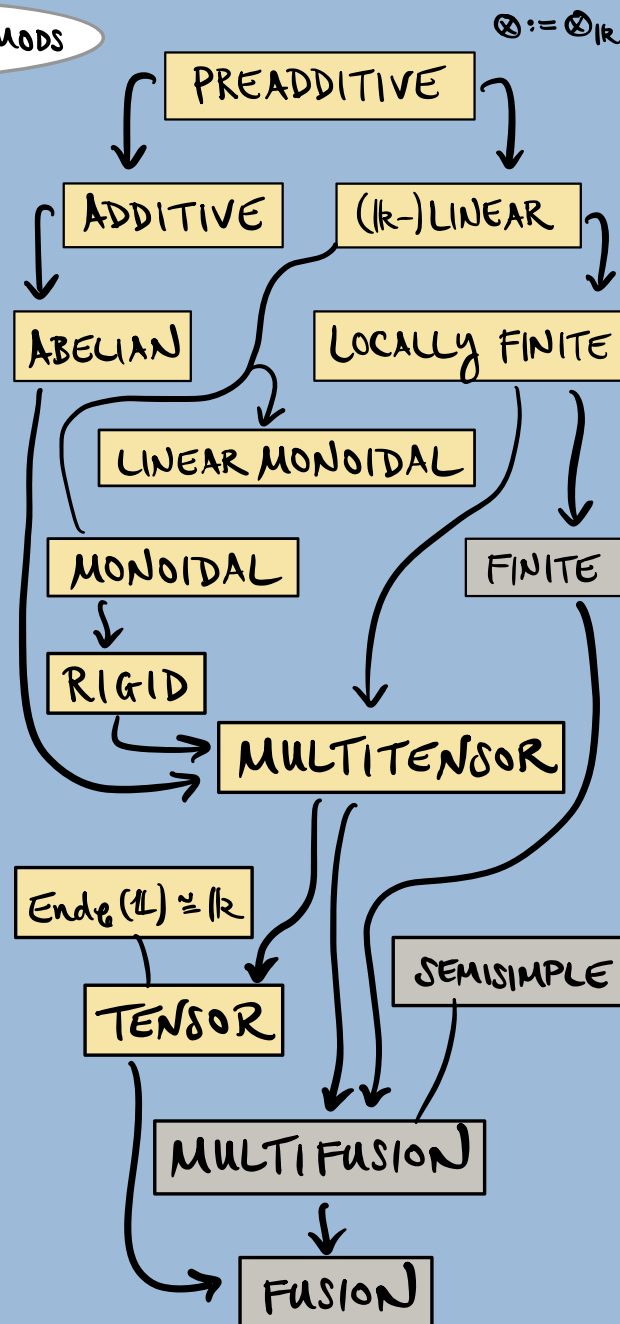
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 $\uparrow$ HOPE ALGEBRAS
Hopf Alg

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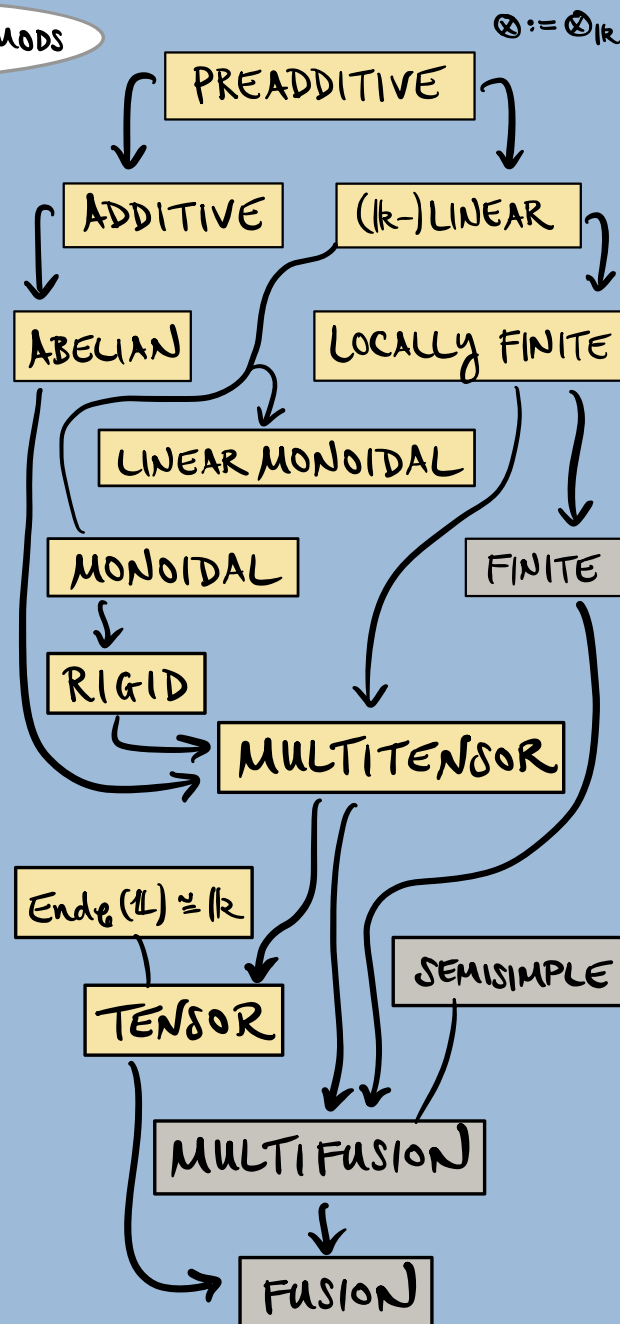
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EVERY TENSOR CAT. EQUIPPED W/ A "FIBER" FUNCTOR
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FOR SOME HOPF ALGEBRA H



RECONSTRUCTION THEORY

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TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

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RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\xrightarrow{\sim}$ "CAT. OF STRUCTURES" / $\mathbb{K}$

$\{(\mathcal{C}, U)\}$

AND SO ON...

$(\mathcal{C}, U)$

$\longleftrightarrow$

$\text{Coend}_{\mathcal{C}}(U)$

$(\text{CoMod}/\mathcal{S}, \text{Forg})$

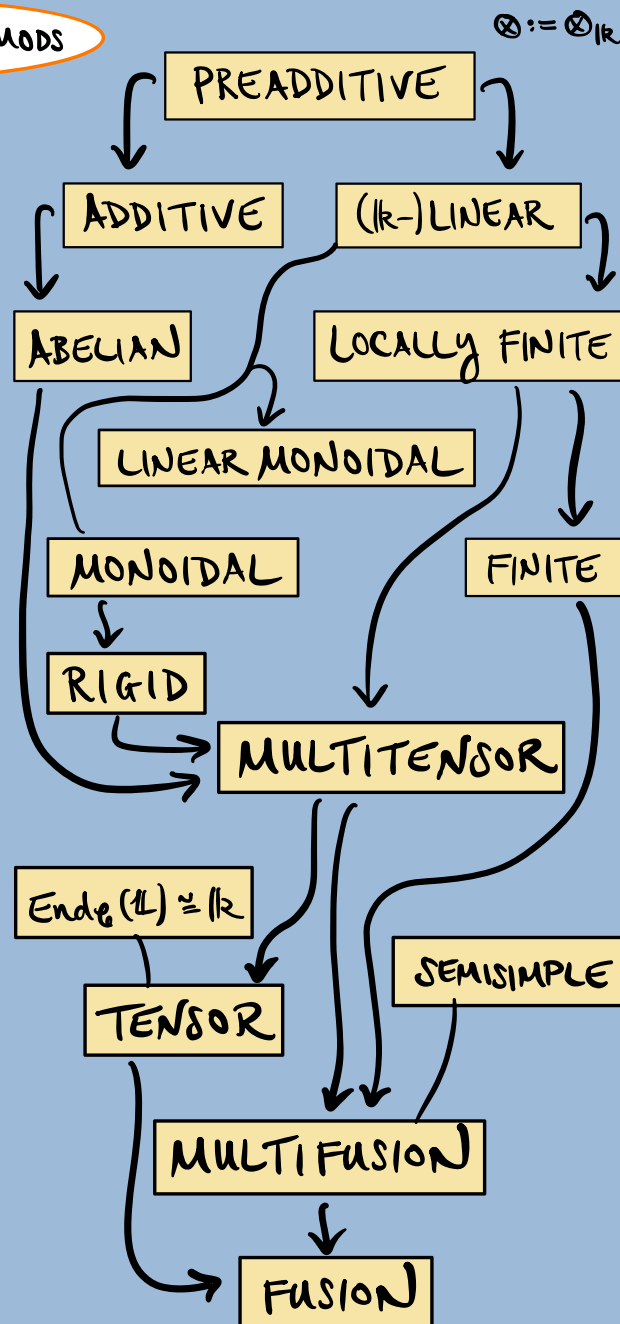
$\longleftarrow$

$\mathcal{S}$

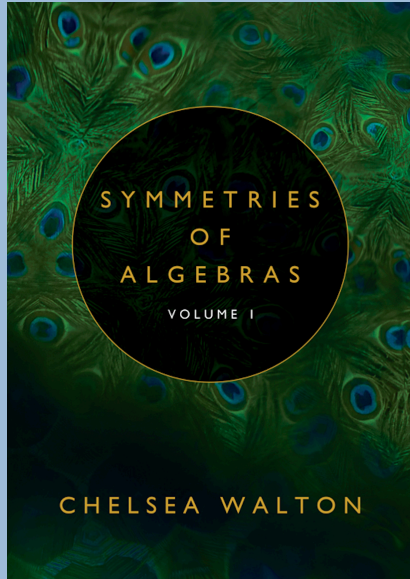
EVERY TENSOR CAT. EQUIPPED W/ A "FIBER" FUNCTOR

IS EQUIVALENT TO $\text{CoMod-}H$

FOR SOME HOPF ALGEBRA H



COMMERCIAL BREAK



"SYMMETRIES OF ALGEBRAS"
3-VOLUME BOOK SERIES
GEARED FOR NEWCOMERS

VOL 1: ALGS & MONOIDAL CATS
VOL 2: COALGS & FROBENIUS ALGS
VOL 3: HOPF ALGS & BRAIDINGS

GOAL: TO HAVE PUBLISHED BY
THE END OF 2026

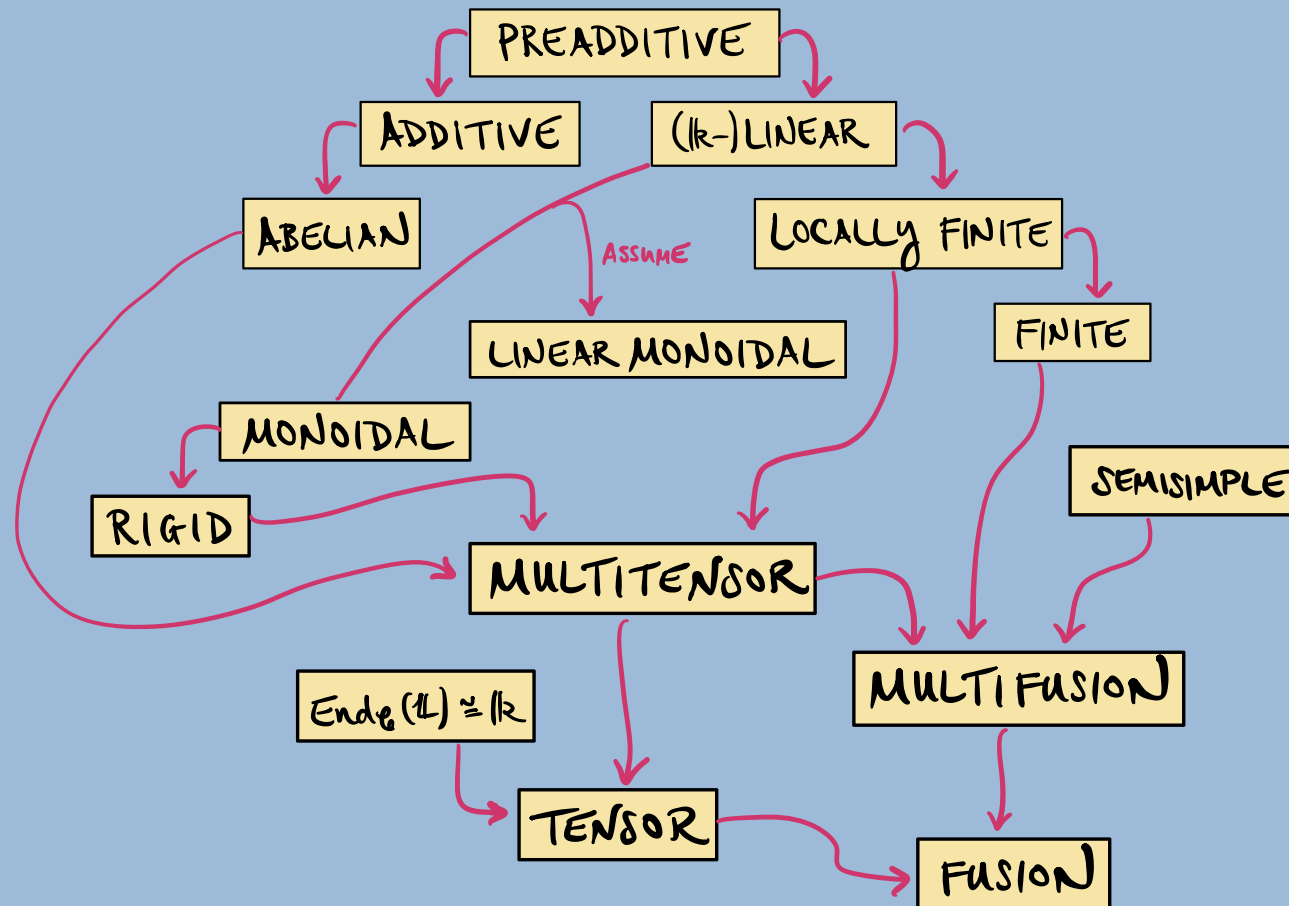
- CHAP 5: COALGS OVER $\mathbb{k}$
- CHAP 6: COALGS IN $\otimes$ CATS
INCLUDING (CO)ENDS
& RECONSTRUCTION THEORY
- CHAP 7: FROB. ALGS. OVER $\mathbb{k}$
- CHAP 8: FROB. ALGS IN $\otimes$ CATS

NEARLY DONE WITH
DRAFTING CHAPS. 5 & 6

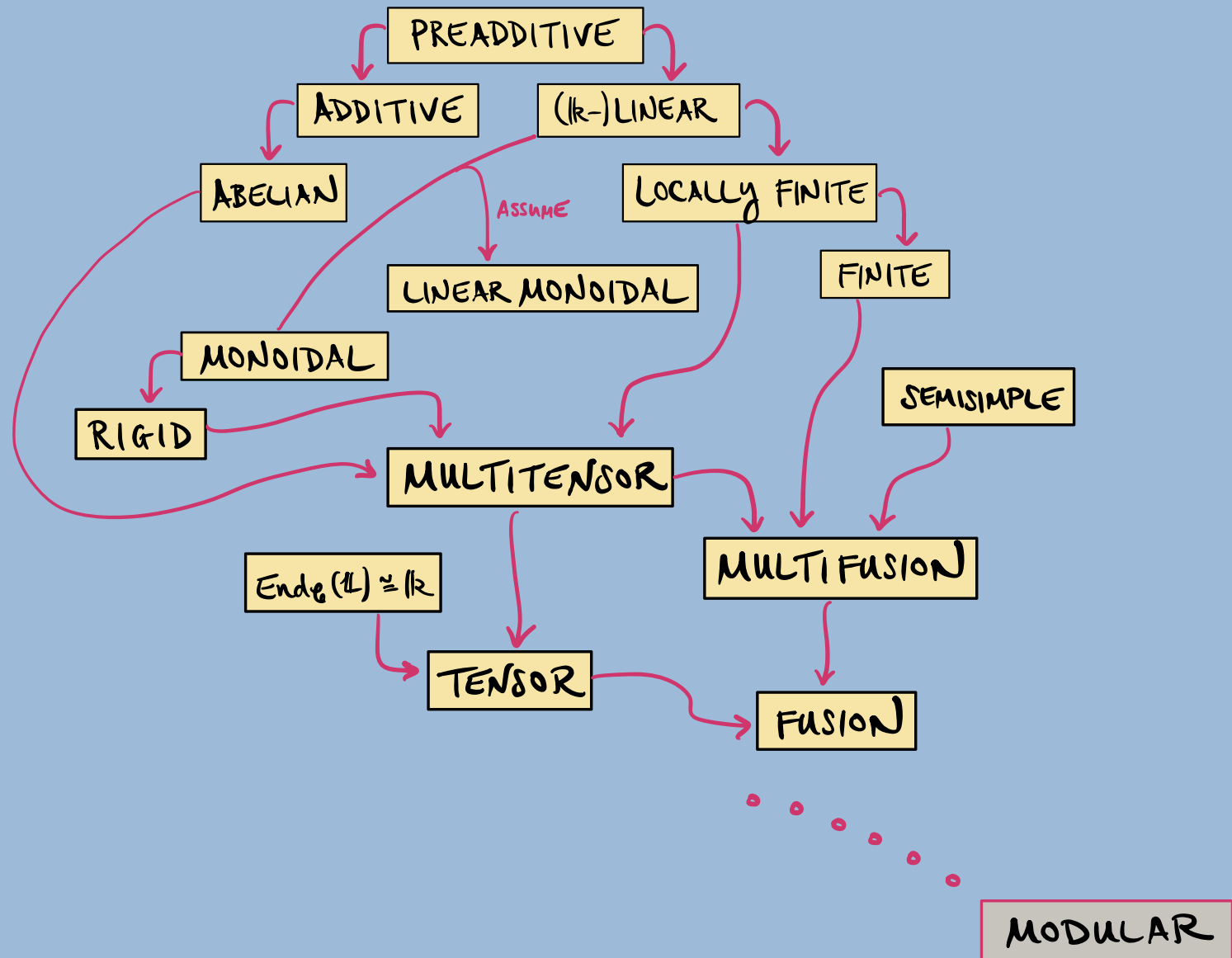
78 EXERCISES & 67 EXERCISES
SO FAR

READING "COURSE" FEEDBACK
THIS FALL ??

THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



THE ROAD TO MODULARITY

MODULAR FUSION / TENSOR CATEGORIES
ARE AT THE HEART OF
"QUANTUM SYMMETRIES"

THE ROAD TO MODULARITY

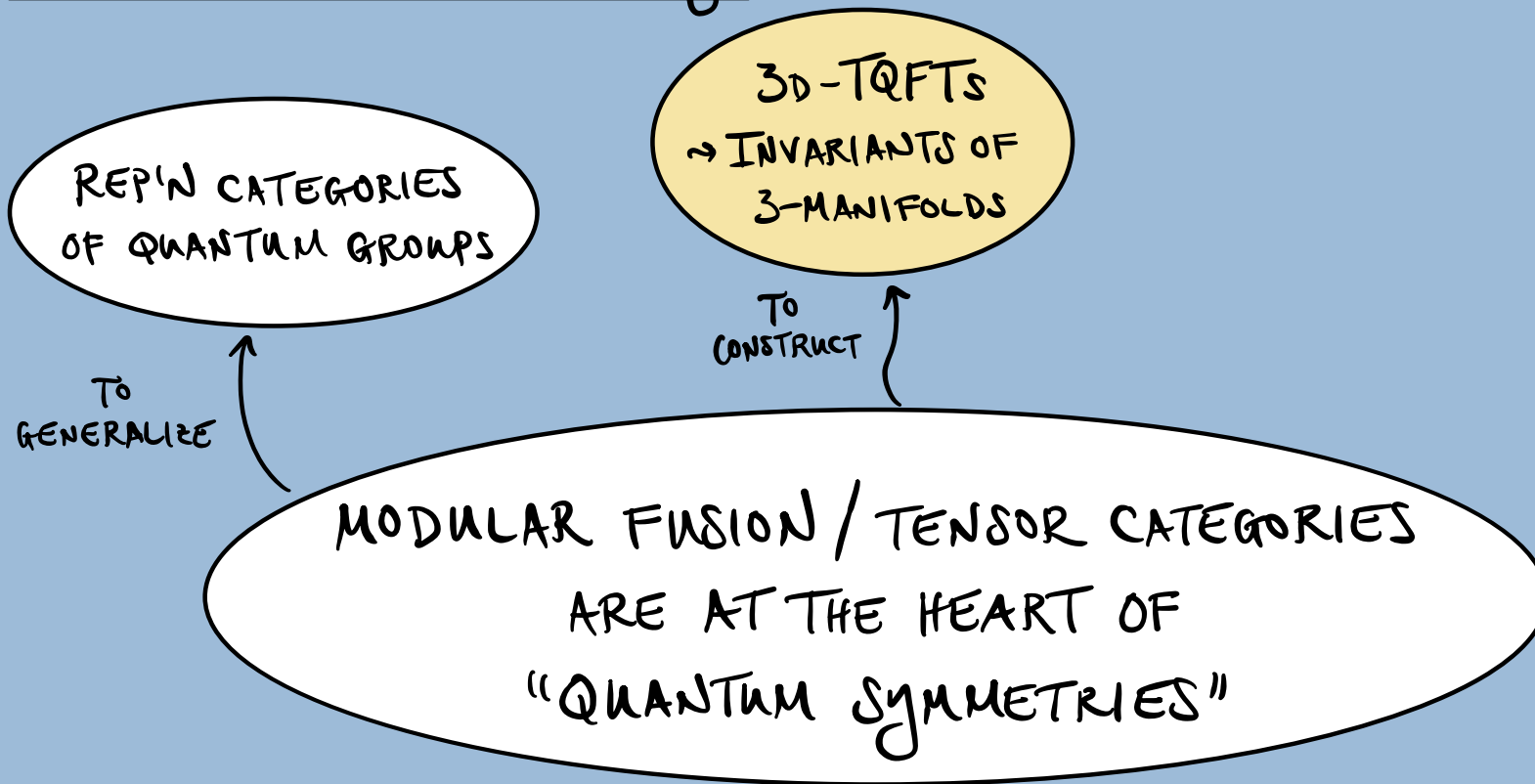
REP'N CATEGORIES
OF QUANTUM GROUPS

TO
GENERALIZE

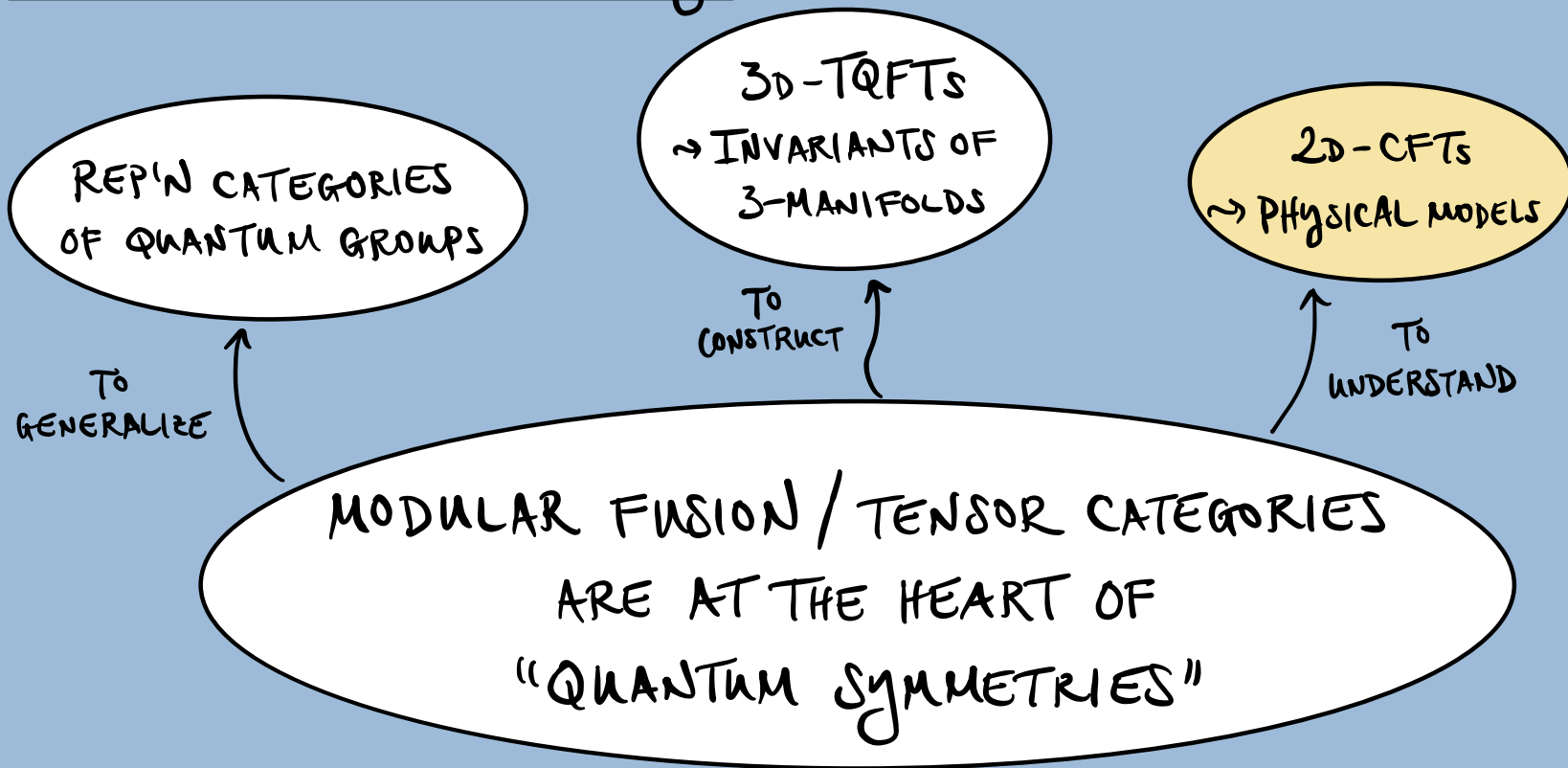


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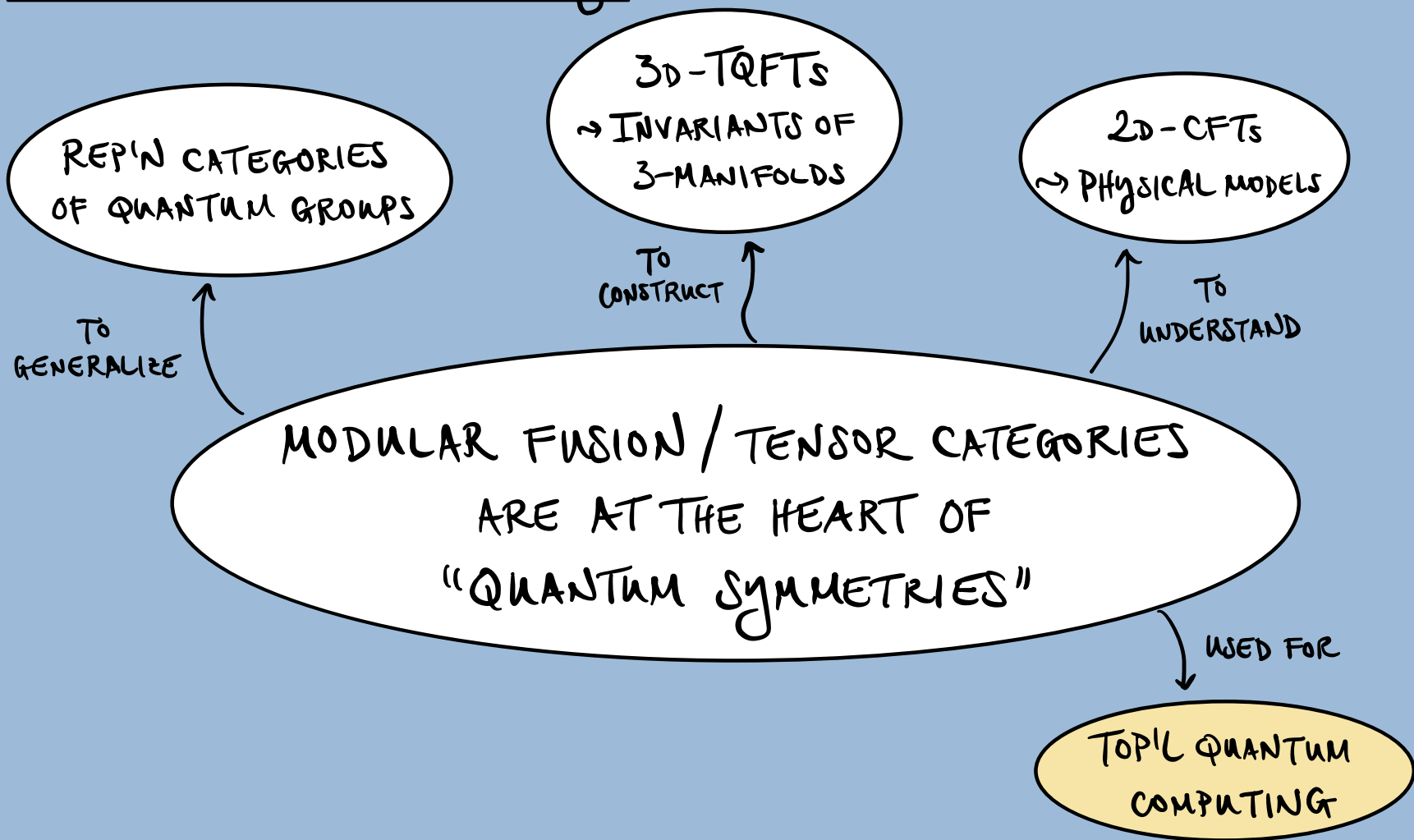
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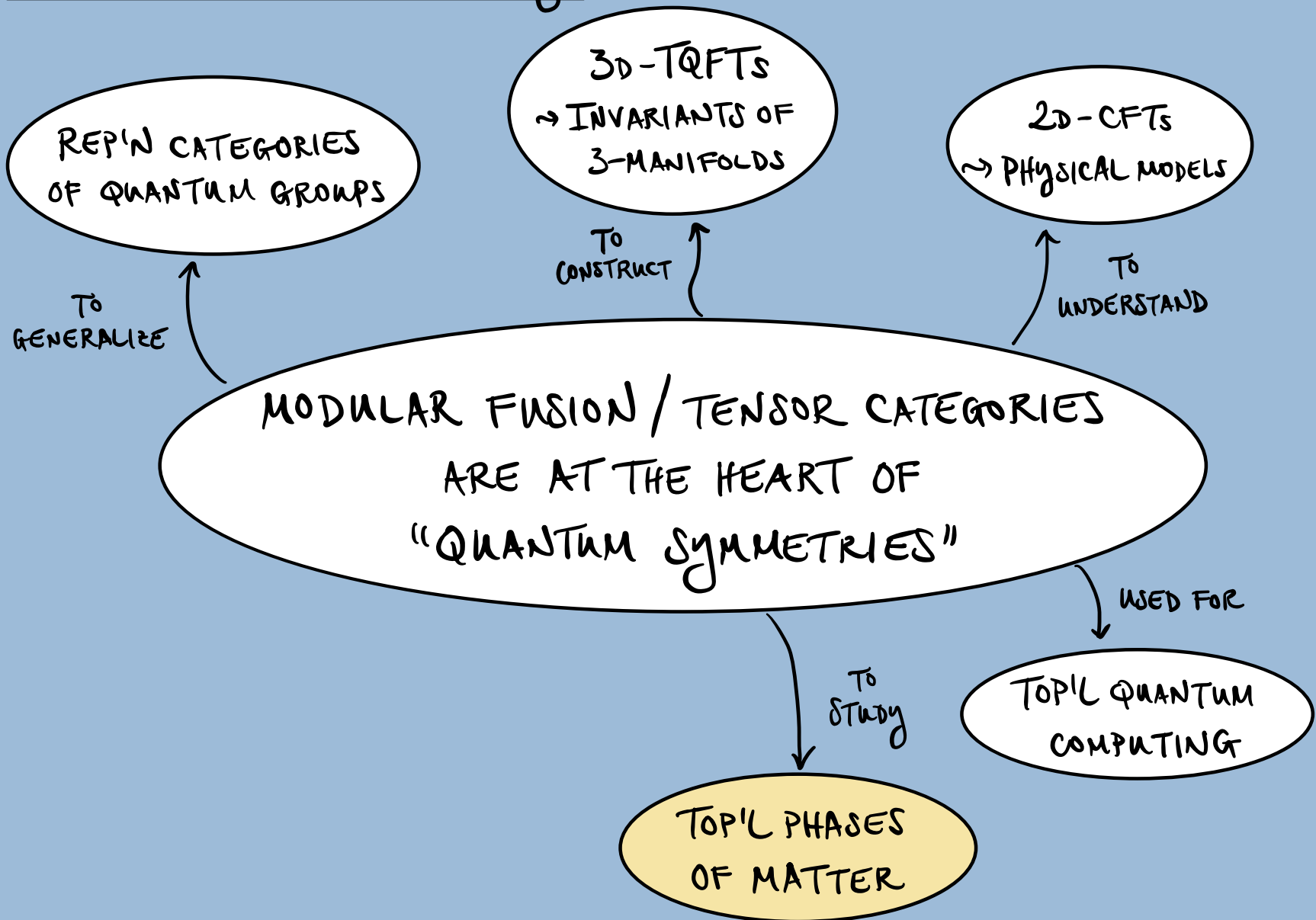
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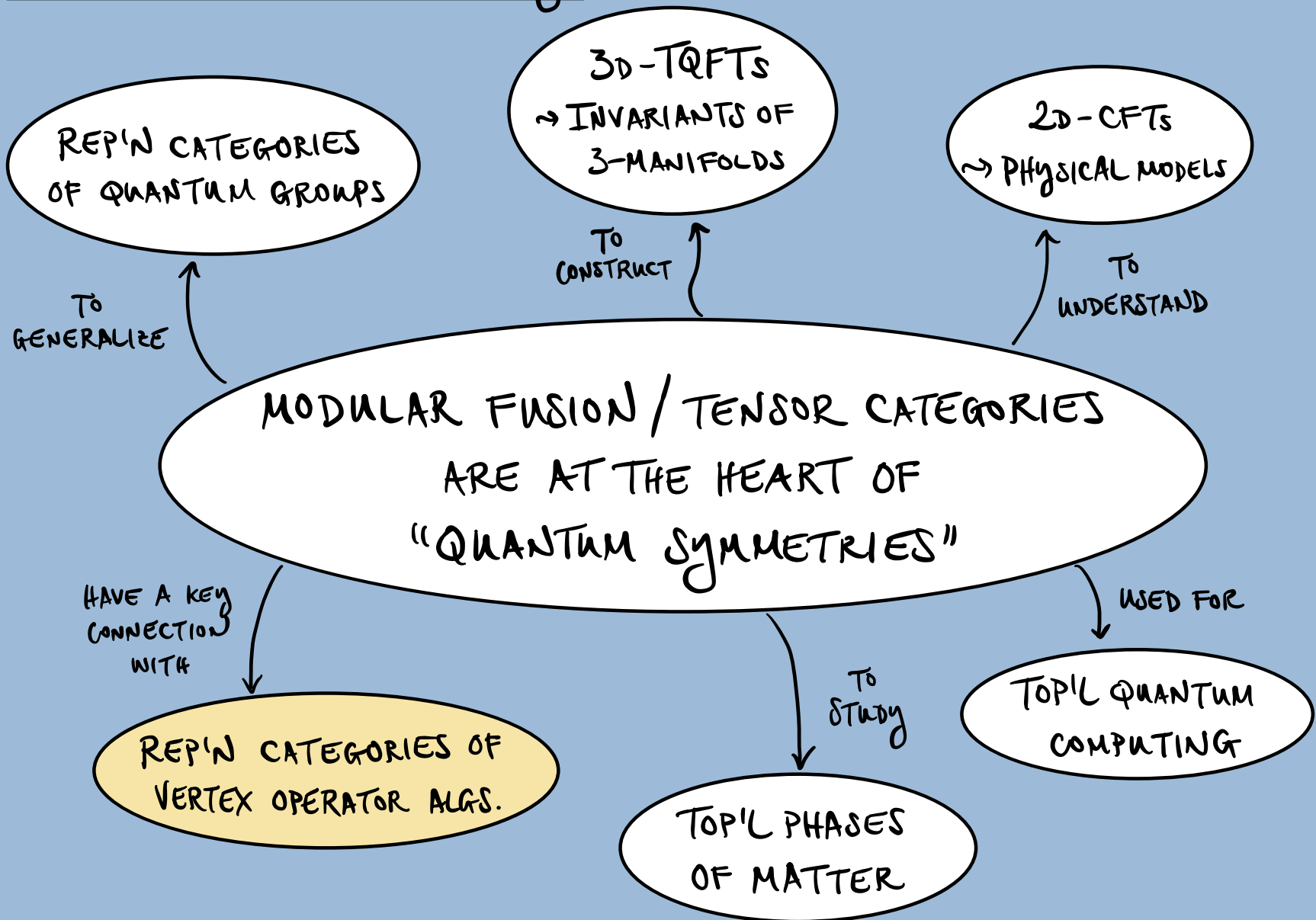
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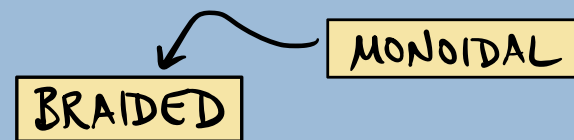
THE ROAD TO MODULARITY



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BRAIDINGS



BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$

$\#$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

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A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM
$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

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VIA $\neq$
GRAPHICAL CALC.
(SAW A BIT LAST TIME)

$$X \otimes Y \equiv \begin{array}{c} X \\ | \\ | \\ Y \end{array}$$

$$f: X \rightarrow Y \equiv \begin{array}{c} X \\ \boxed{f} \\ Y \end{array}$$

$$c_{X,Y} \equiv \begin{array}{c} X \quad Y \\ \swarrow \quad \searrow \\ \searrow \quad \swarrow \end{array}$$

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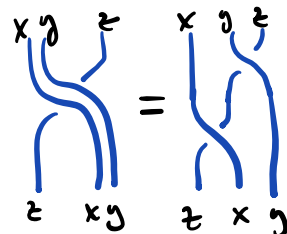
$$C := \{c_{x,y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{x,y \in \mathcal{C}}$$



SUCH THAT

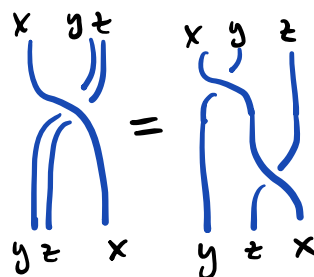
$$X \otimes Y \otimes Z \xrightarrow{c_{X \otimes Y, Z}} Z \otimes X \otimes Y$$

Diagram showing the naturality of the braiding. The top path goes from $X \otimes Y \otimes Z$ to $Z \otimes X \otimes Y$ via $c_{X \otimes Y, Z}$. The bottom path goes from $X \otimes Y \otimes Z$ to $X \otimes Z \otimes Y$ via $\text{id} \otimes c_{Y, Z}$, and then to $Z \otimes X \otimes Y$ via $c_{X, Z} \otimes \text{id}$. The two paths are equal.



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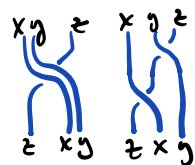
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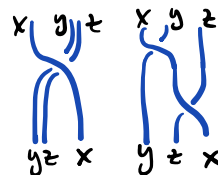
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SUCH
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$\neq$



BRAIDED

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REP'NS OF
BRAID GROUPS

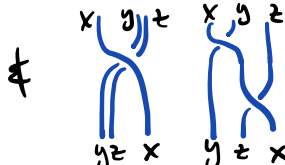
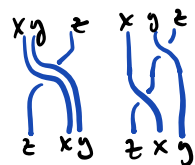
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REP'NS OF
BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

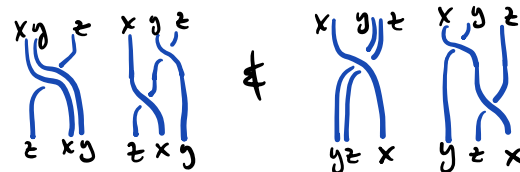
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TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

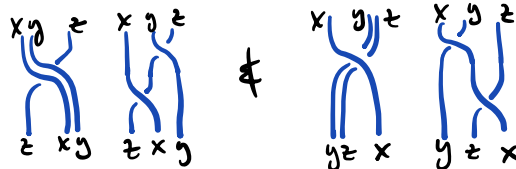
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REPRESENTATIONS OF
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 $\equiv$
 GROUP HOMOM

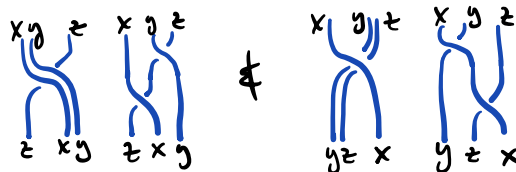
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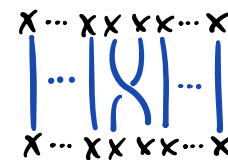
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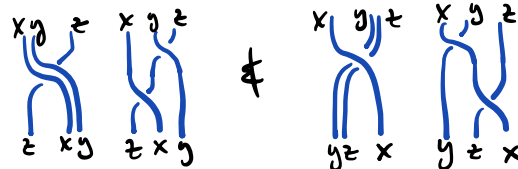
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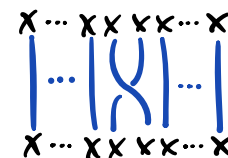
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EX.

$$\pi(\sigma_1 \sigma_2 \sigma_1) = \text{[Braid Diagram]} = \dots$$

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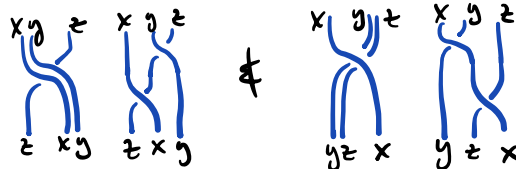
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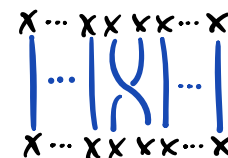
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BRAID
AXIOM
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= ...



BRAID
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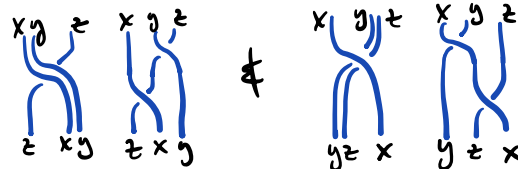
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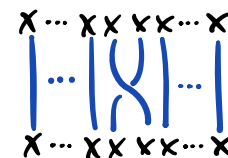
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BRAID
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NATURALITY
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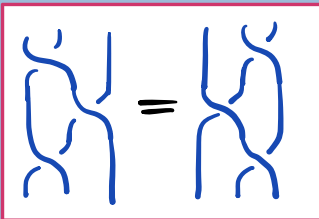


BRAID
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$$= \pi(\sigma_2 \sigma_1 \sigma_2)$$

BRAIDINGS

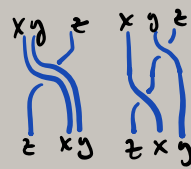


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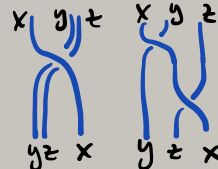
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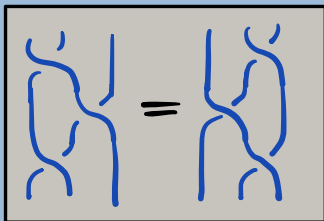


SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

BRAIDINGS

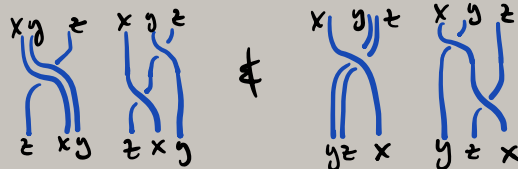


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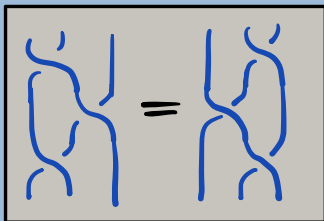
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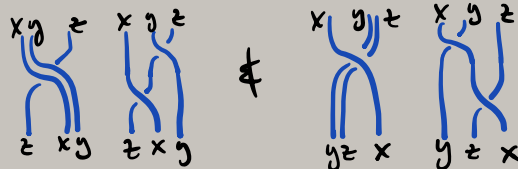


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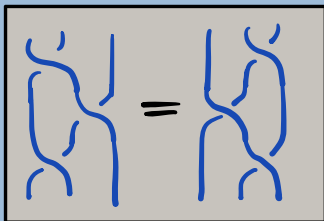
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$$\left\{ \begin{array}{l} R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{array} \right.$$

BRAIDINGS

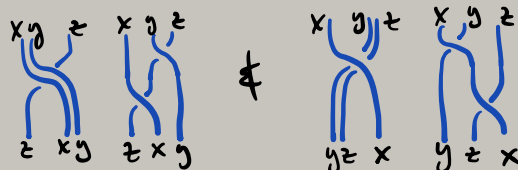


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BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

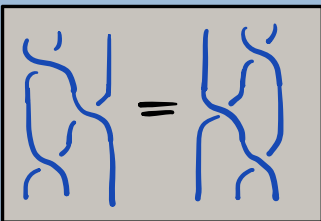
A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

BRAIDINGS

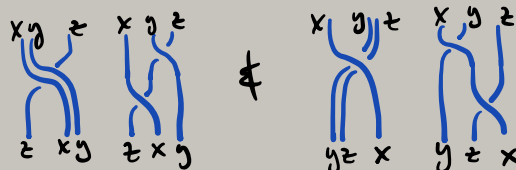


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \mid x, y \in \mathcal{C}\}$$



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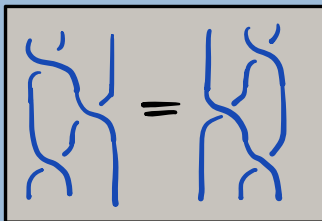
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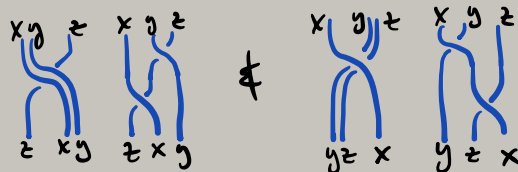


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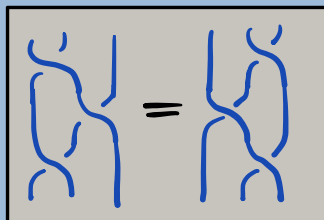
$$c_{x,y}(x \otimes y) = \dots$$

FOR $X, Y \in H\text{-Mod}$, $x \in X$, $y \in Y$



...

BRAIDINGS

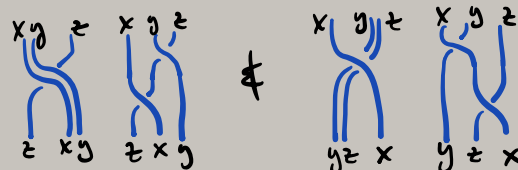


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SOLUTIONS TO
THE YBE

III

"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS
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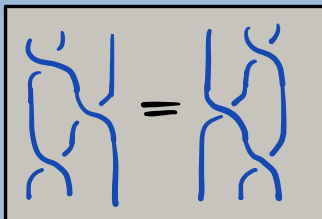
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$R = R' \otimes R'' \in H \otimes H$ IS
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BRAIDINGS

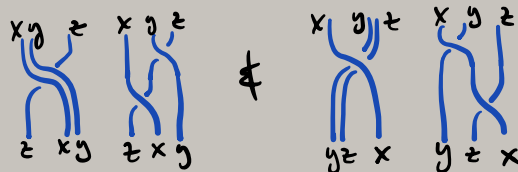


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SOLUTIONS TO
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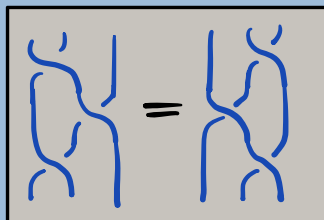


$R = R' \otimes R'' \in H \otimes H$ IS
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EXAMPLES:

H COCOMMUTATIVE,
 $R = 1 \otimes 1$

BRAIDINGS

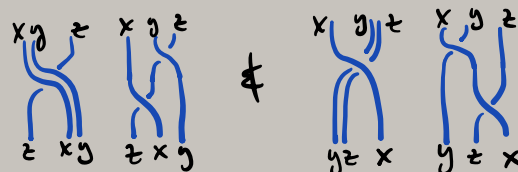


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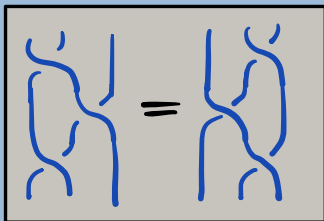
H COCOMMUTATIVE,
 $R = 1 \otimes 1$

$\mathbb{K}G$

FOR $u \in Z(G)$ w/ $u^2 = e$

$$R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$$

BRAIDINGS

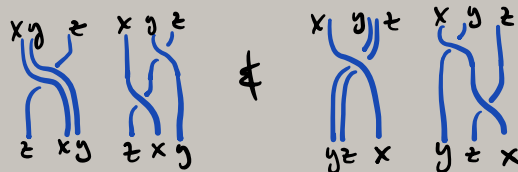


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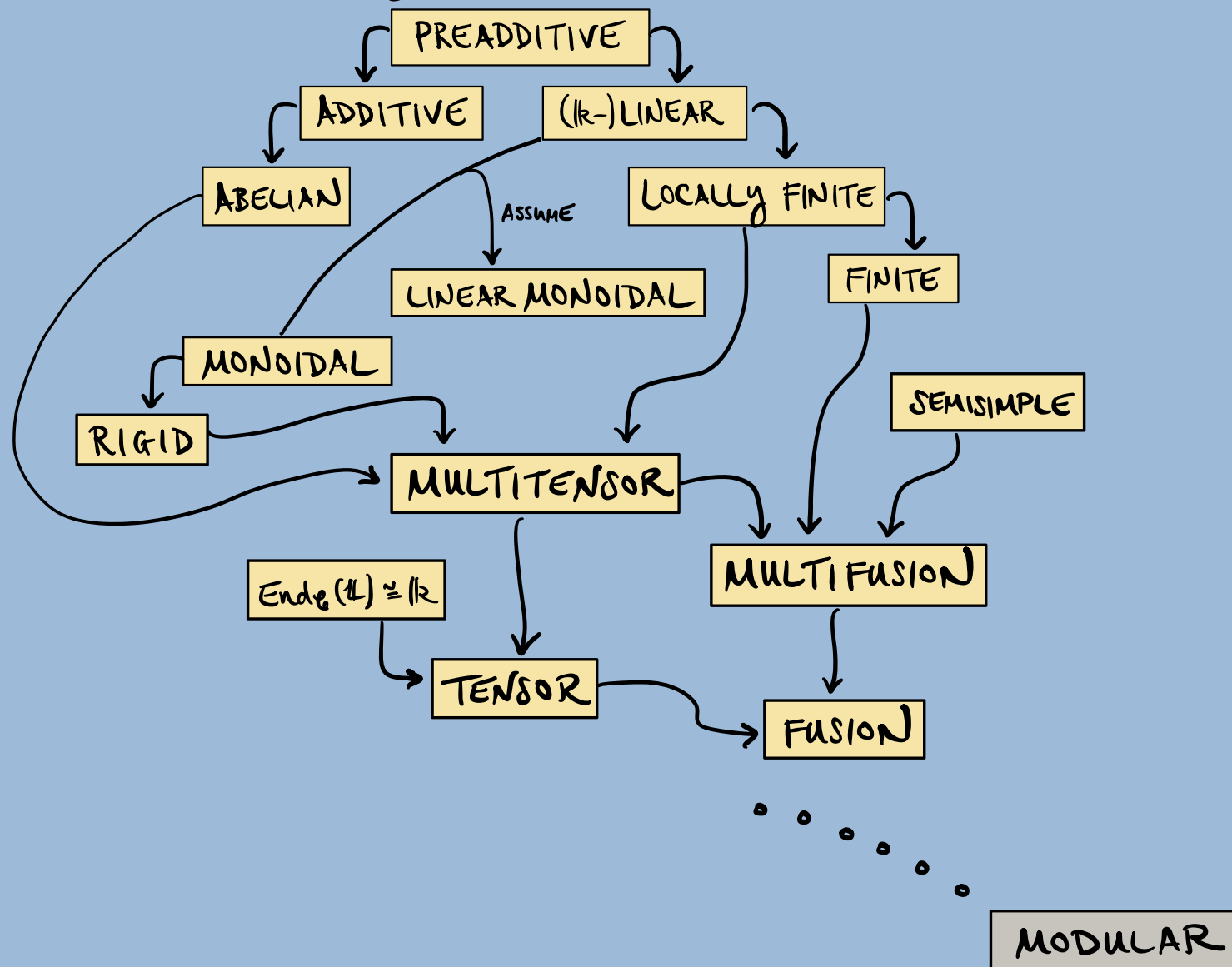
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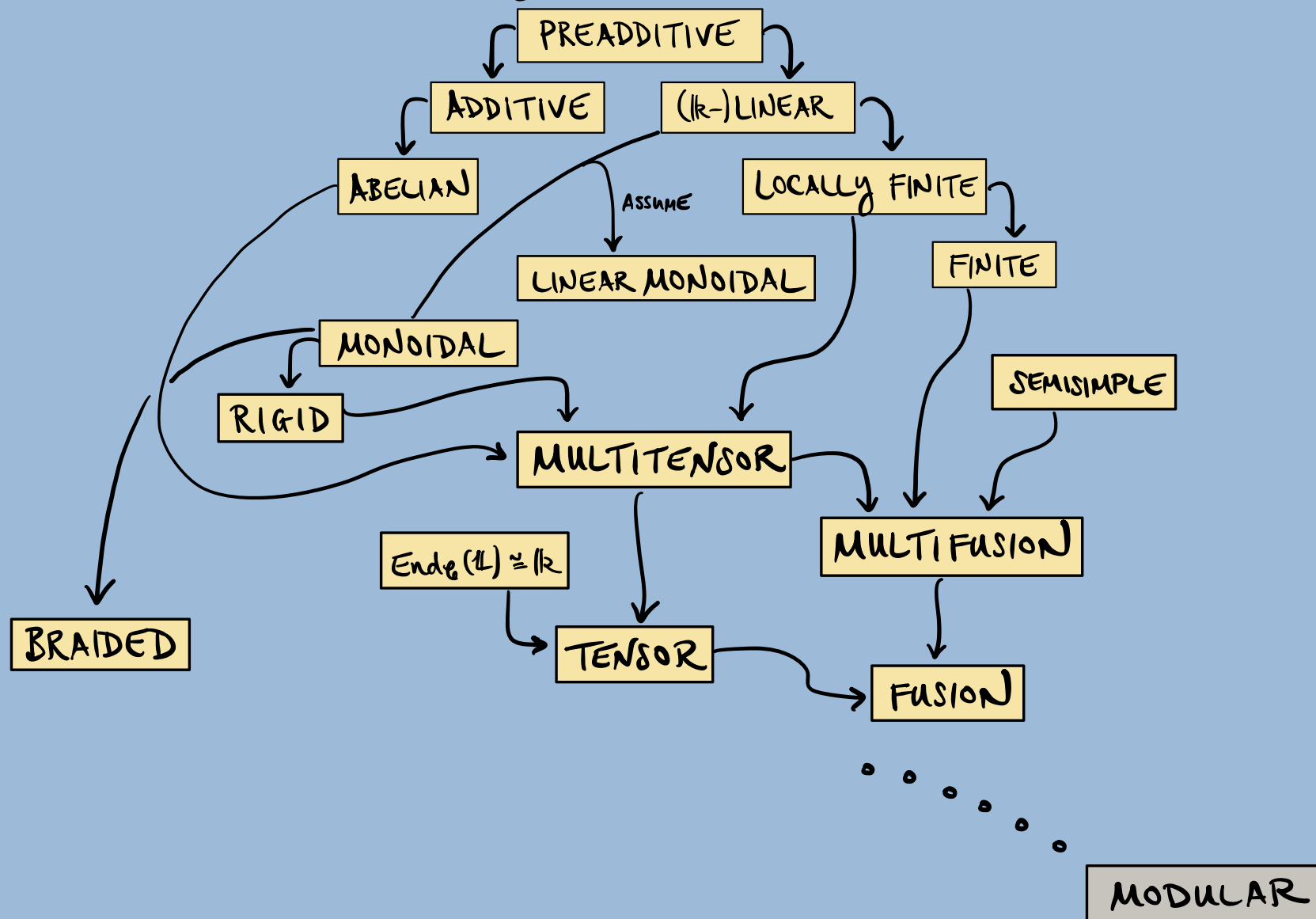
SMALL Q.GROUP

$$u_q(\mathfrak{sl}_2)$$

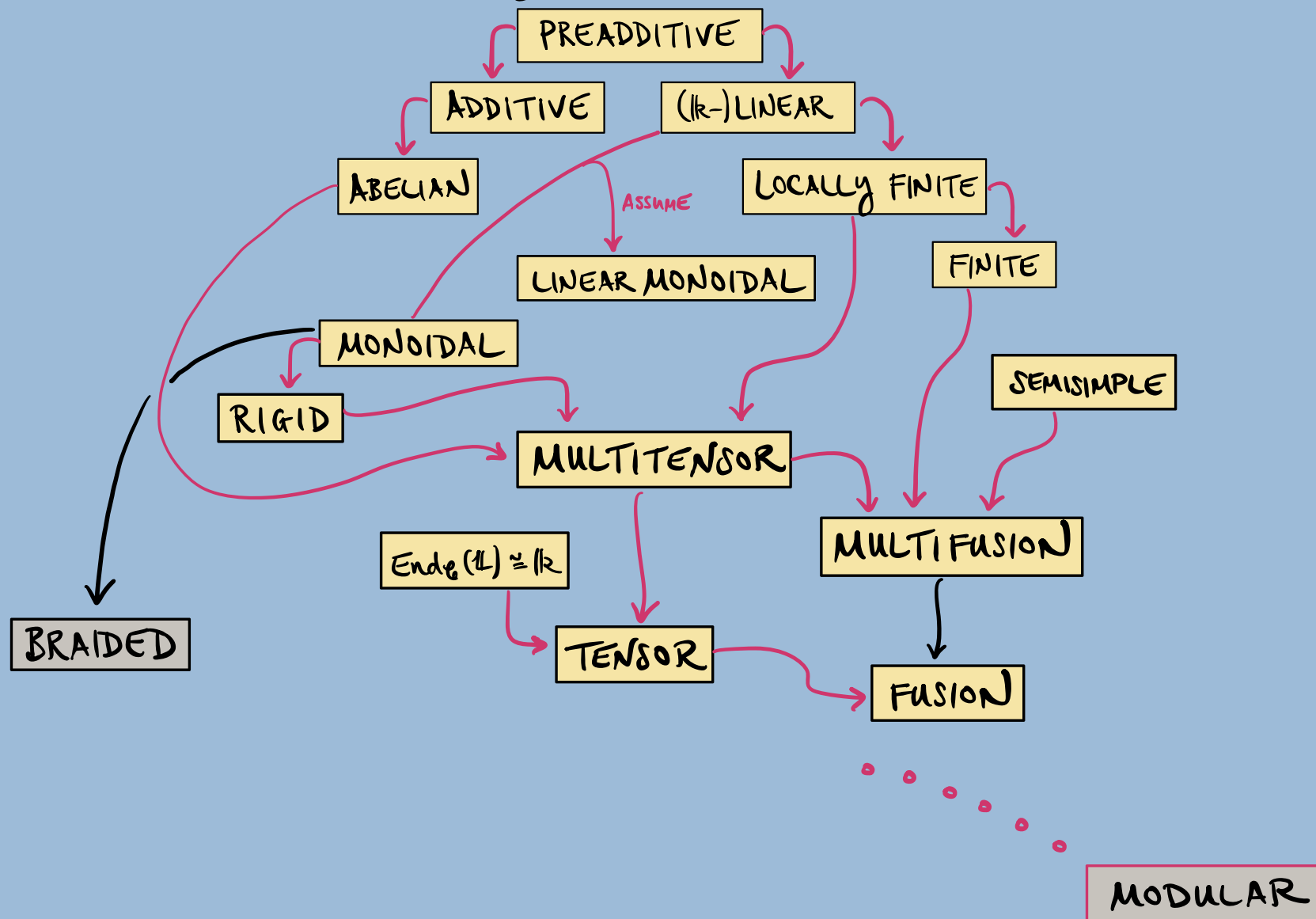
THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



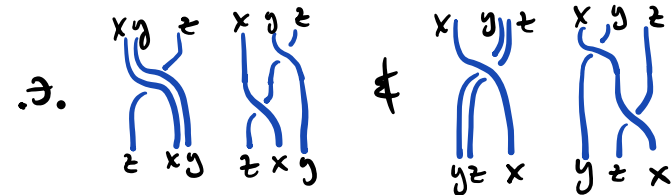
THE ROAD TO MODULARITY



NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{L})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

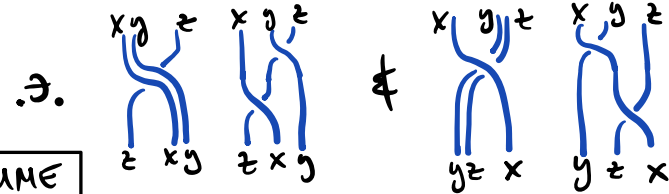
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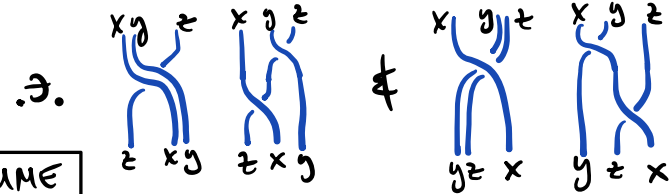
ASSUME
TENSOR

NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
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ASSUME
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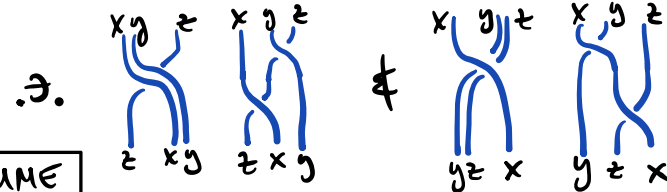
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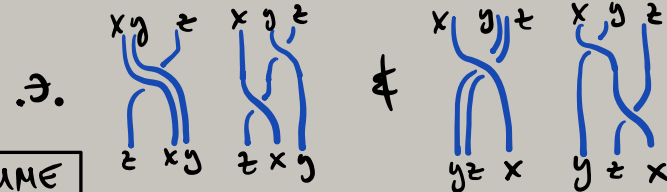
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ASSUME
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IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$$

NONDEGENERATE BRAIDINGS

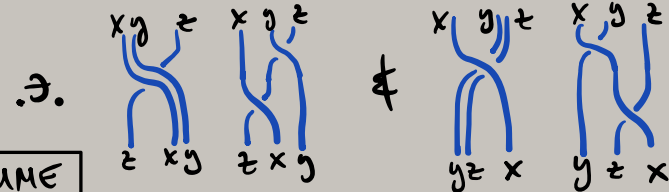
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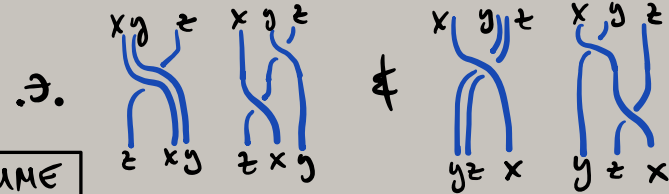
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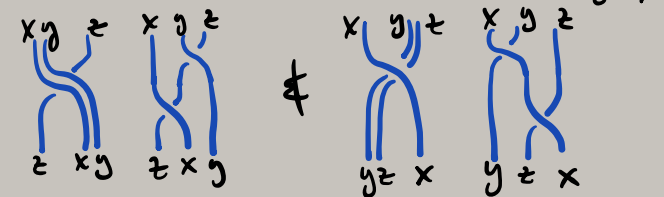
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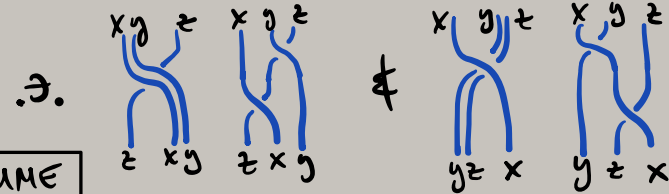
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[SHIMIZU: 1602.06534]

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SATISFYING BRAID AXIOMS

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ASSUME
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TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$

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TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

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ASSUME
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EXAMPLES:

H COCOMMUTATIVE,
 $R = 1 \otimes 1$

$\mathbb{K}G$
FOR $u \in \mathbb{Z}(G)$ w/ $u^2 = e$
 $R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$

SMALL Q.GROUP
 $u_q(\mathfrak{sl}_2)$

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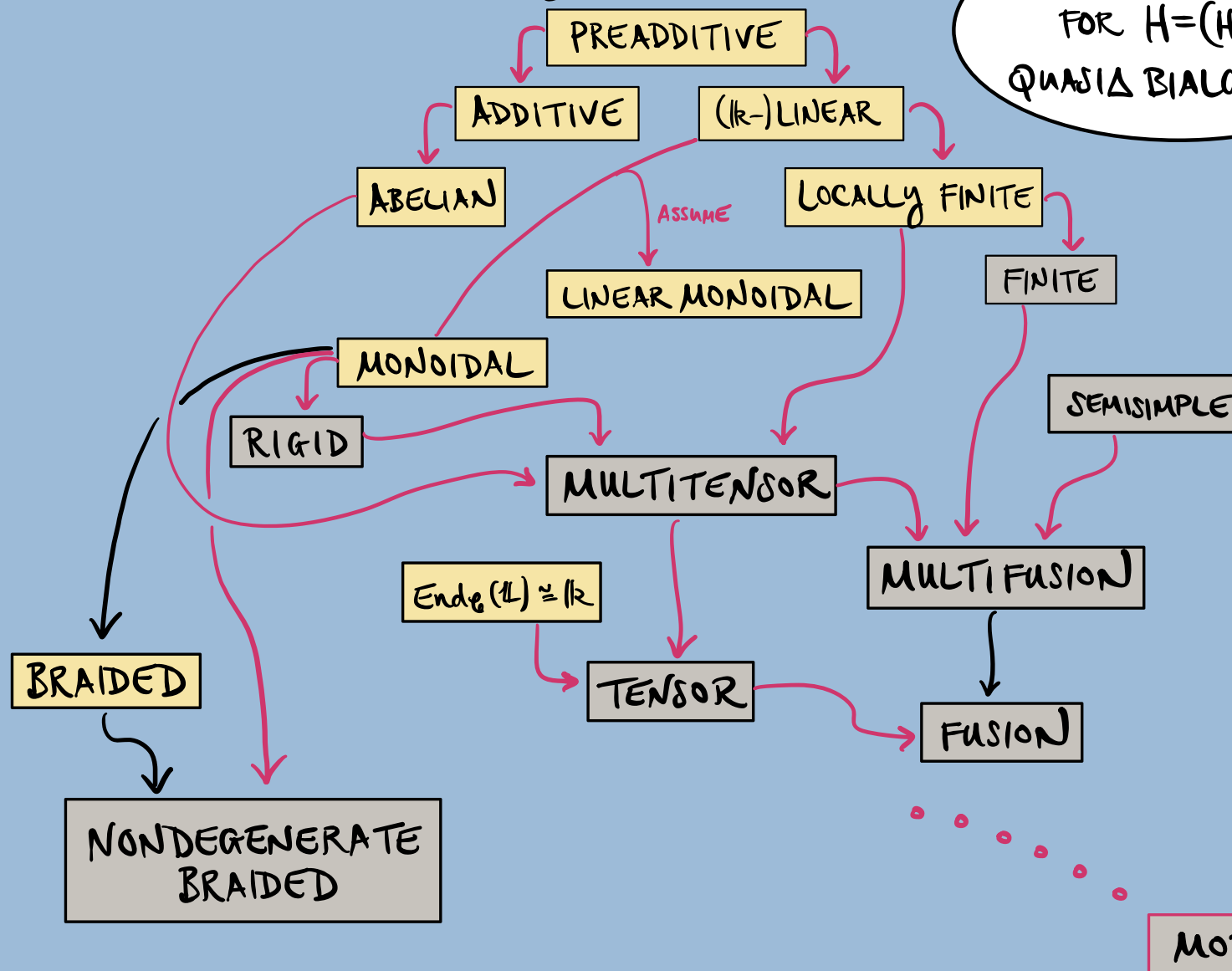
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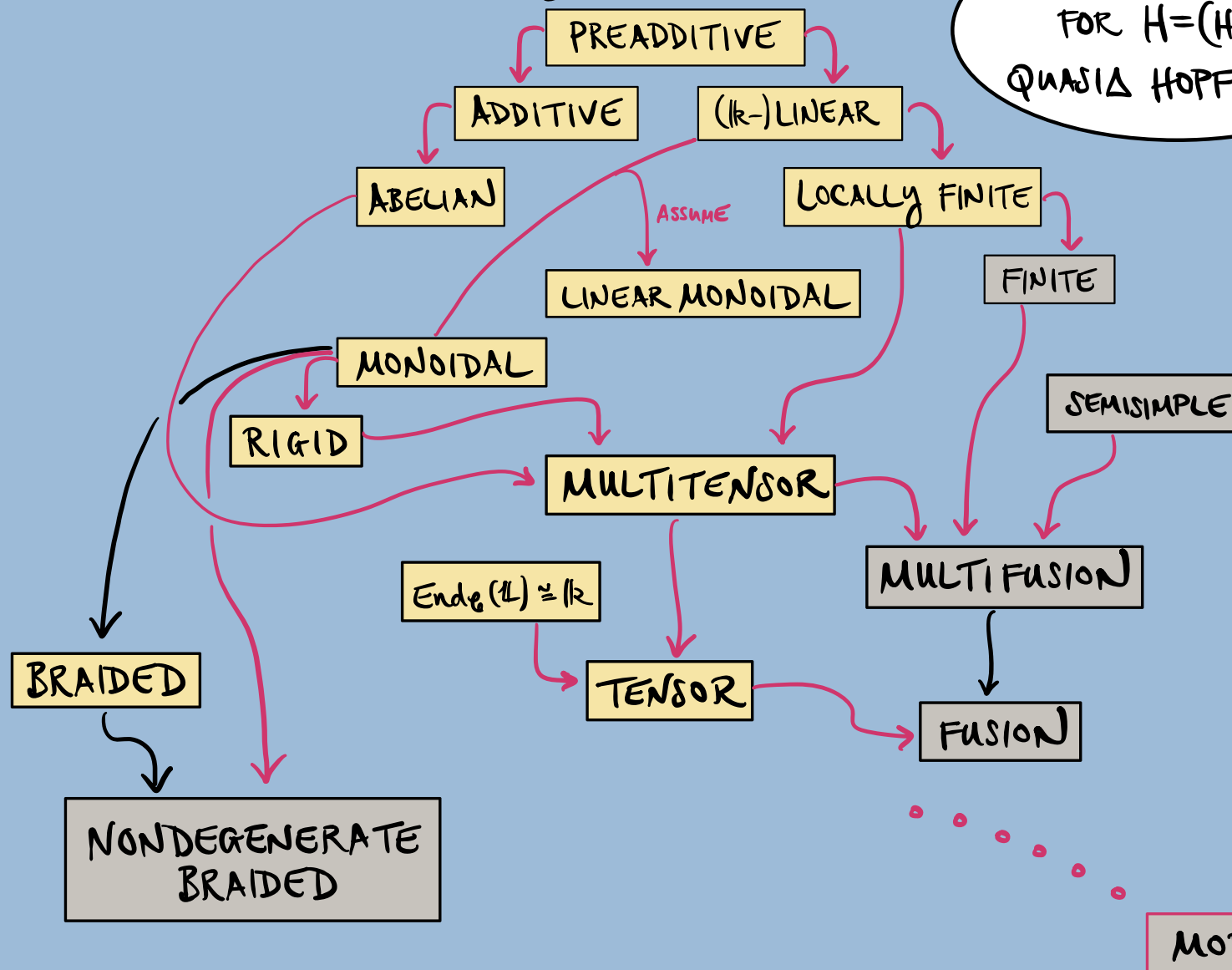
THE ROAD TO MODULARITY

EX. $H\text{-FdMod}$
FOR $H=(H,R)$
QUASIA Δ BIALGEBRA



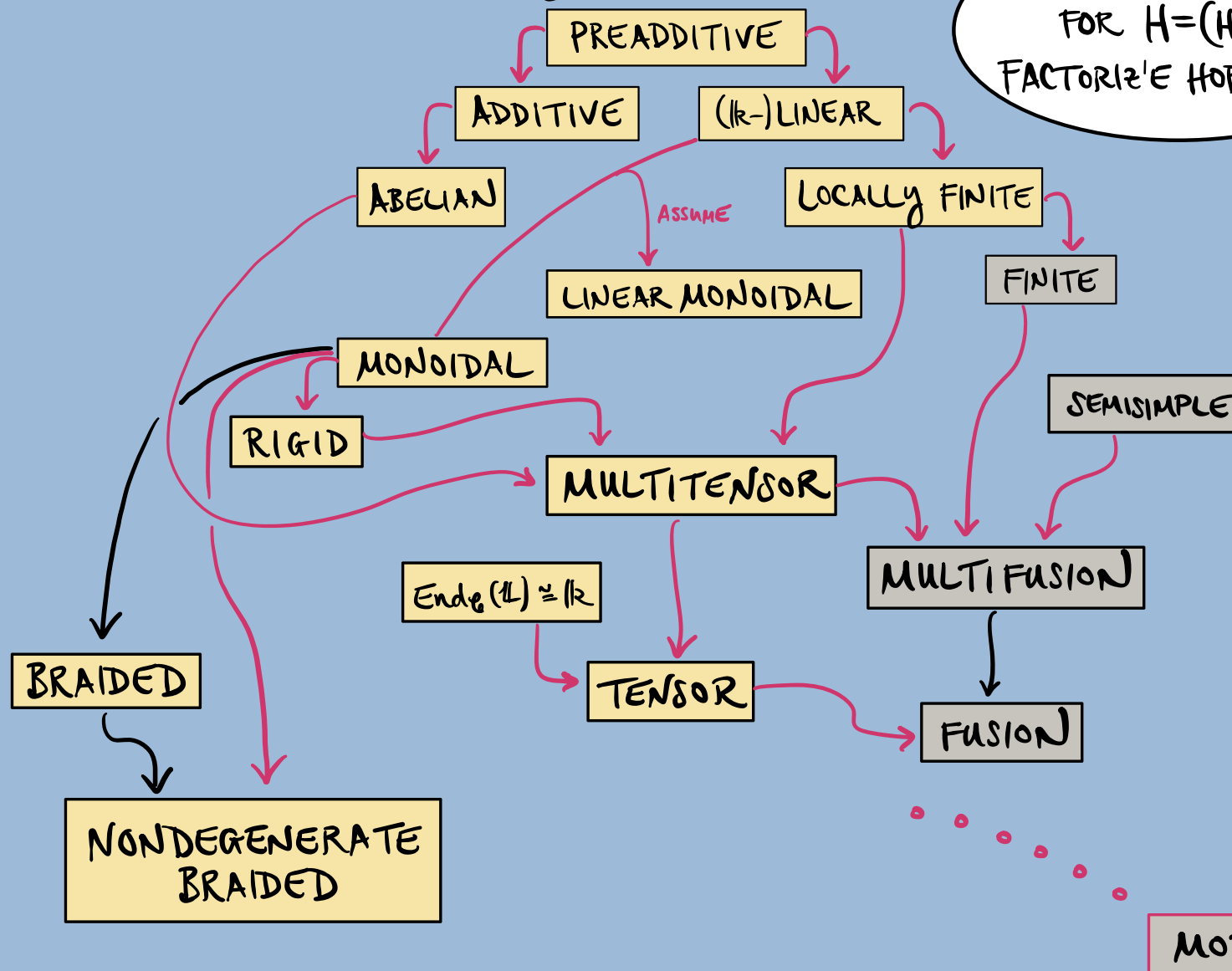
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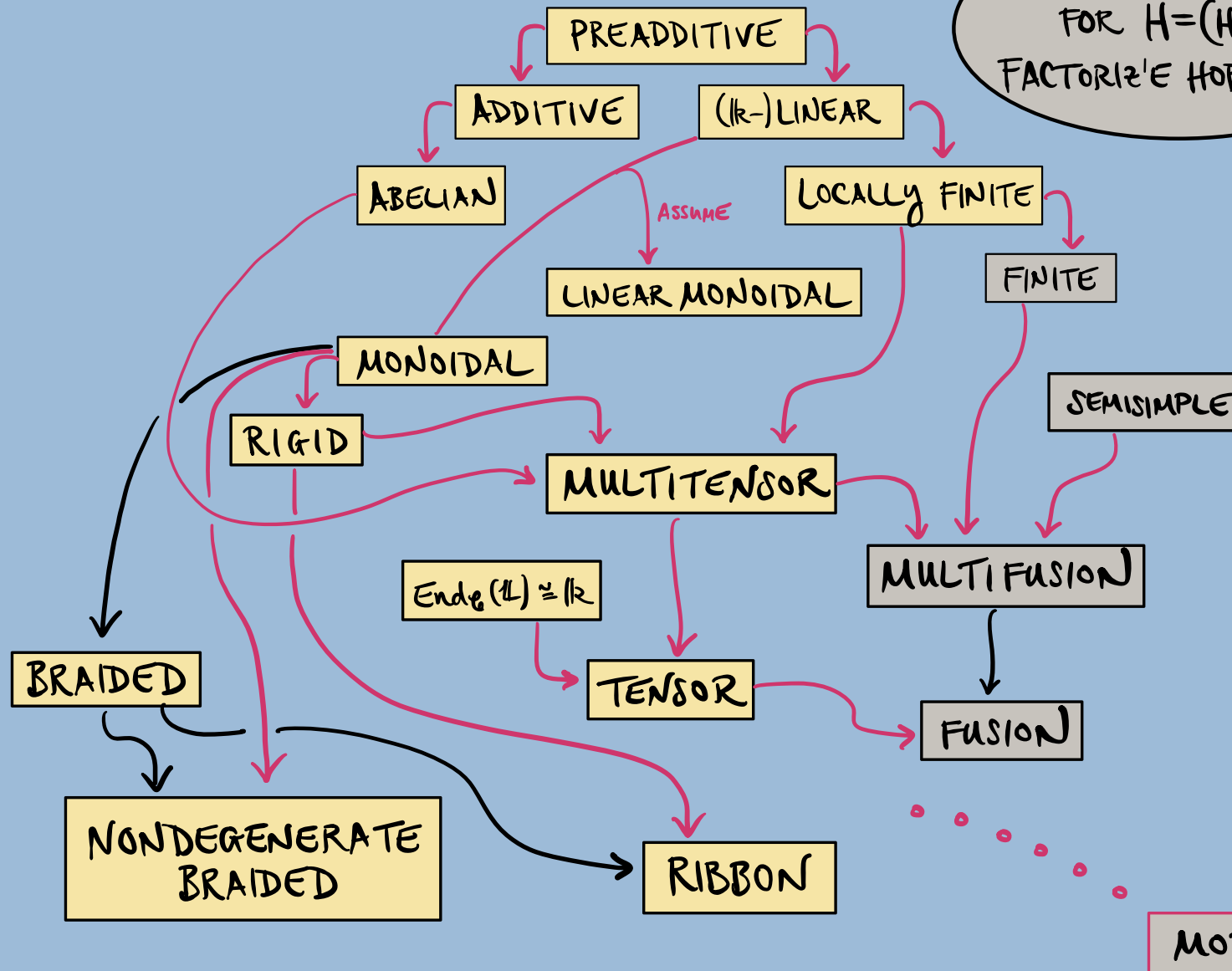
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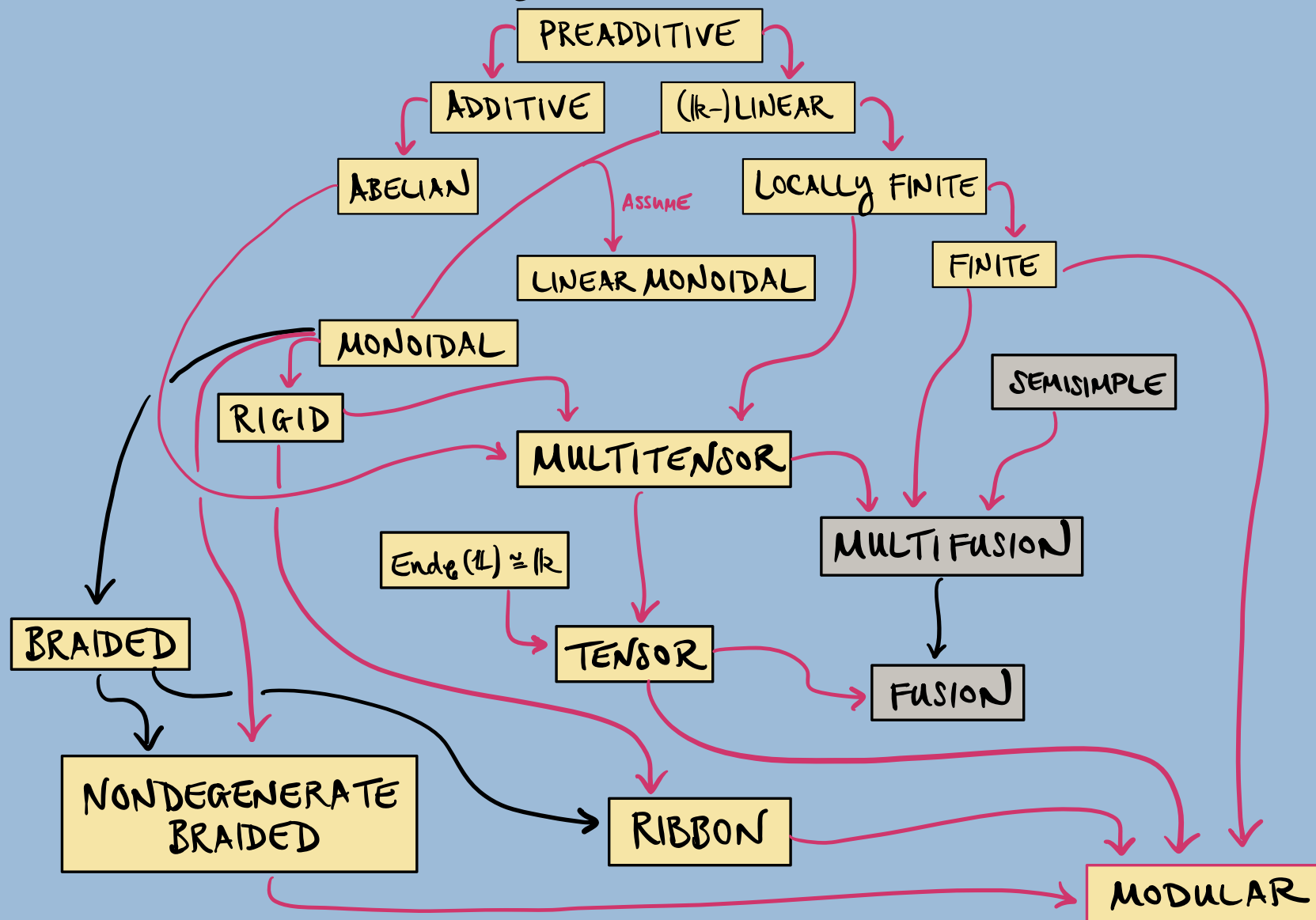


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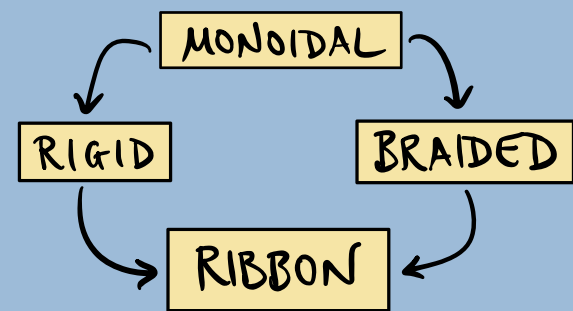
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THE ROAD TO MODULARITY



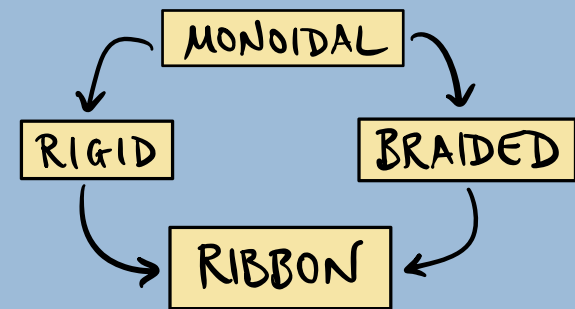
RIBBON CATEGORIES



RIBBON CATEGORIES

A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

IS "RIBBON"



RIBBON CATEGORIES

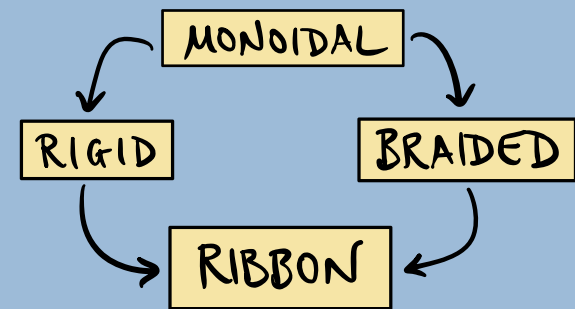
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IS "RIBBON" IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{ \theta_x : X \xrightarrow{\sim} X^* \}_{x \in \mathcal{C}} \quad \ni.$$

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y, x^*} \circ c_{x, y^*} \quad \& \quad (\theta_x)^* = \theta_{x^*}$$

$$\forall x, y \in \mathcal{C}$$



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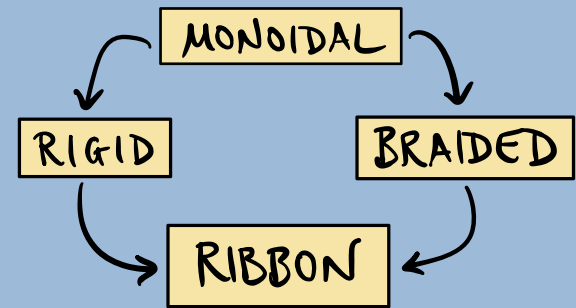
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$$\begin{array}{c} xy \\ \text{ } \\ \text{ } \\ \text{ } \\ xy \end{array} = \begin{array}{c} x \quad y \\ \text{ } \\ \text{ } \\ \text{ } \\ x \quad y \end{array}$$

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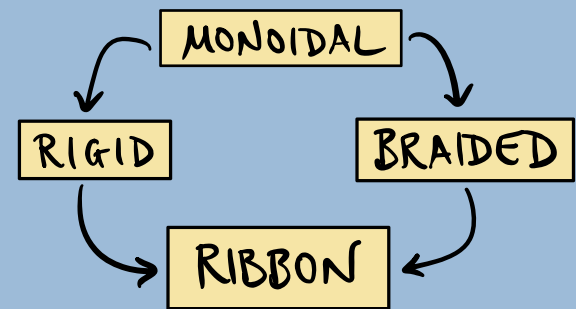
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$$\theta = \{\theta_x : X \xrightarrow{\sim} X^*\}_{x \in \mathcal{C}} \quad \text{with diagram } \theta_x \Rightarrow$$

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$$\text{Diagram 1} = \text{Diagram 2}$$

$$\text{Diagram 3} = \text{Diagram 4} \quad \forall x, y \in \mathcal{C}$$



[SHUM, 1994]

CATEGORY OF FRAMED,
 ORIENTED TANGLES

SI

FREE RIBBON CATEG.
 ON ONE GENERATOR

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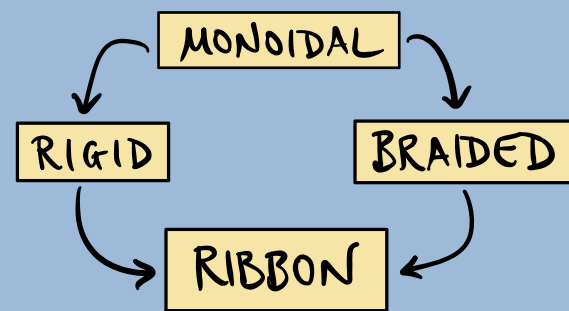
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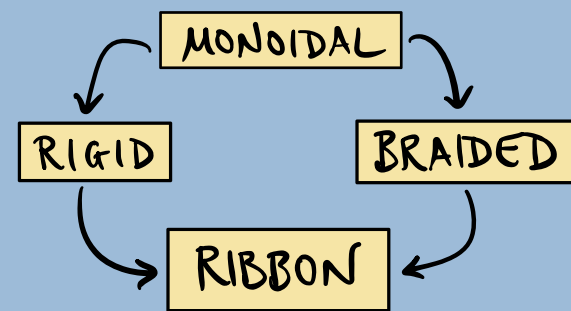
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$\leadsto$ H-Mod RIBBON

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MODULAR CATEGORIES



REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

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RIBBON HOPF ALGEBRA

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Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

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$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

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↑
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CONSTRUCTIONS:

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"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

$\uparrow$
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[J:1810.09055]

& REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

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RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

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"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872] [LW: 2202.08644] [S: 1707.09691] & REFS WITHIN

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[LW: 2010.11872]

$Z_\otimes(\mathcal{C})$

"RELATIVE
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[LW: 2202.08644] [S: 1707.09691]

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[LW: 2010.11872]

$Z_\delta(\mathcal{C})$

"RELATIVE
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[LW: 2202.08644]

$\text{LocMod}_{\mathcal{C}}(A)$

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$\mathcal{Z}_\delta(\mathcal{C})$

"RELATIVE
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[S: 1707.09691]

$A \mathcal{Y}_v^v = A \mathcal{Y}_v^{uv}$

$\text{LocMod}_\delta(A)$

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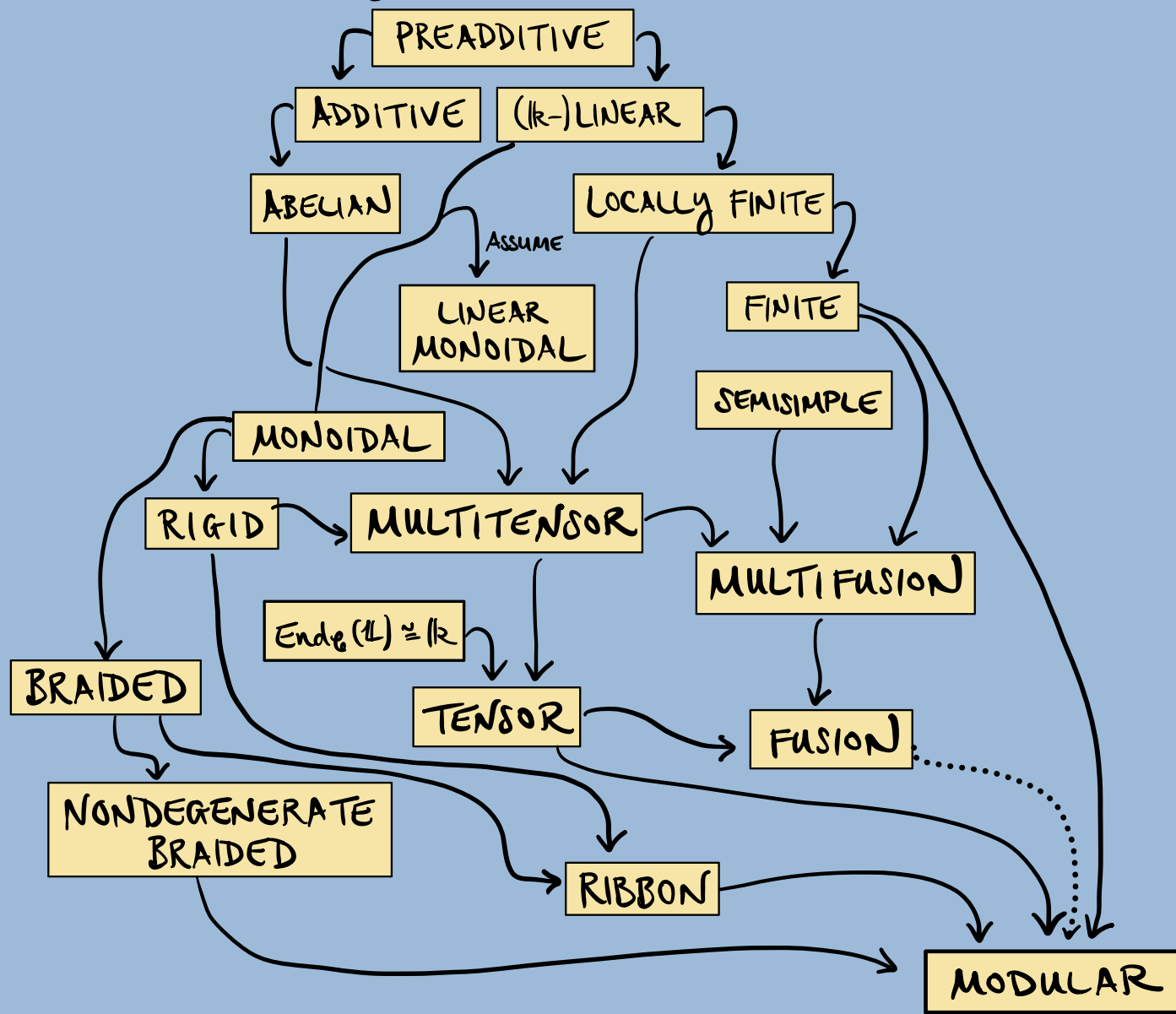
[LW: 2202.08644] [S: 1707.09691]

$\text{LocMod}_{\mathcal{C}}(A)$

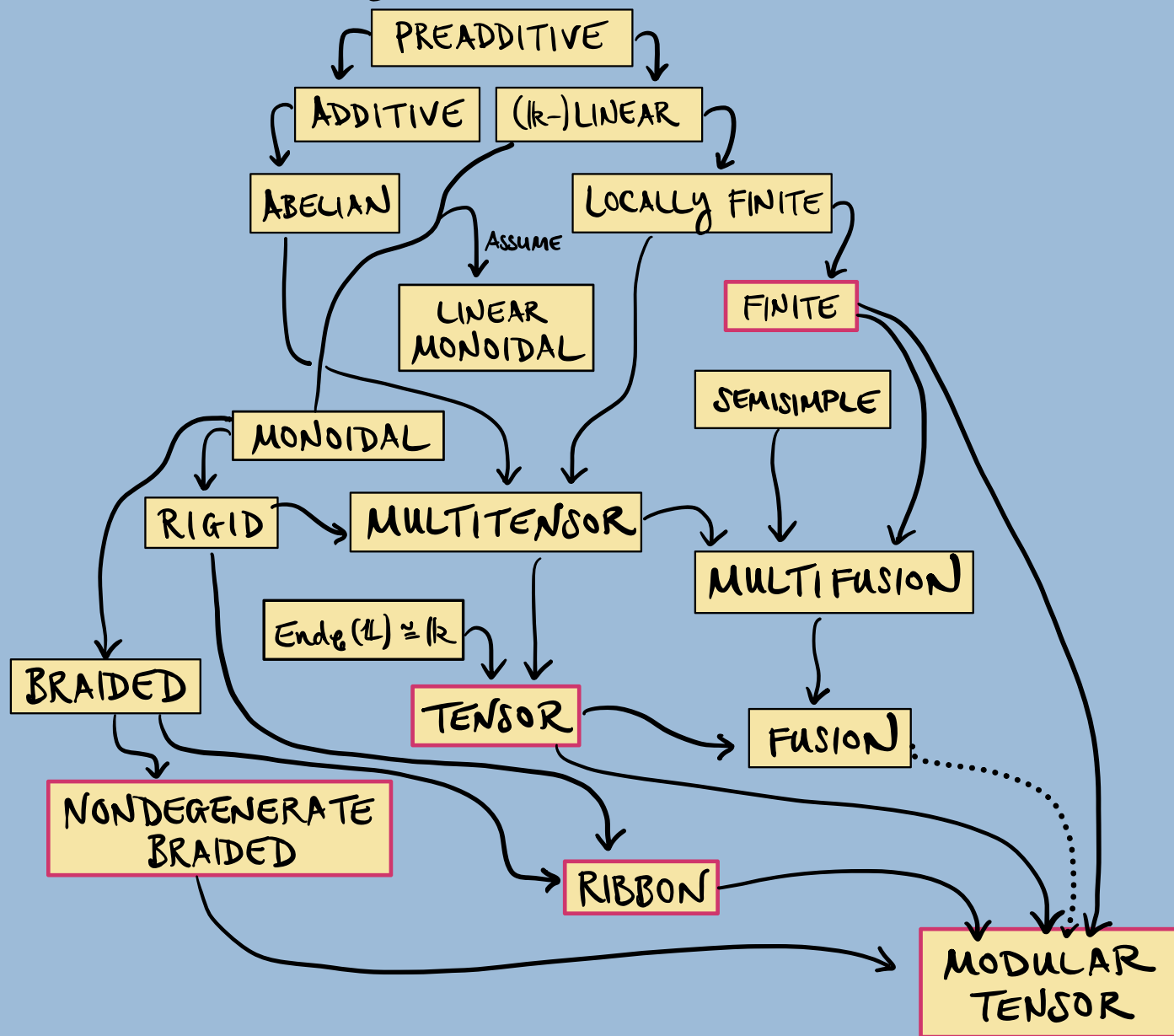
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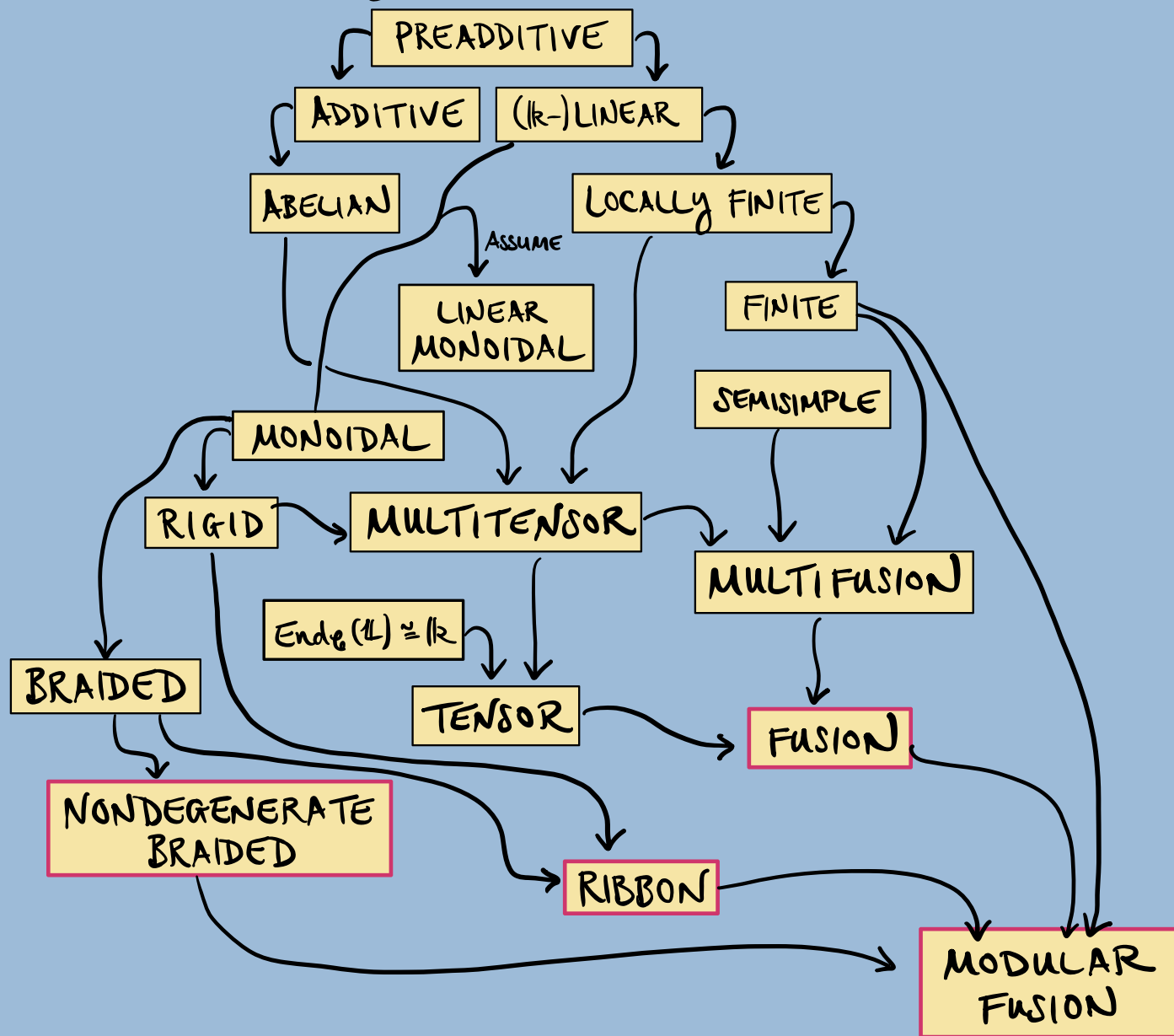
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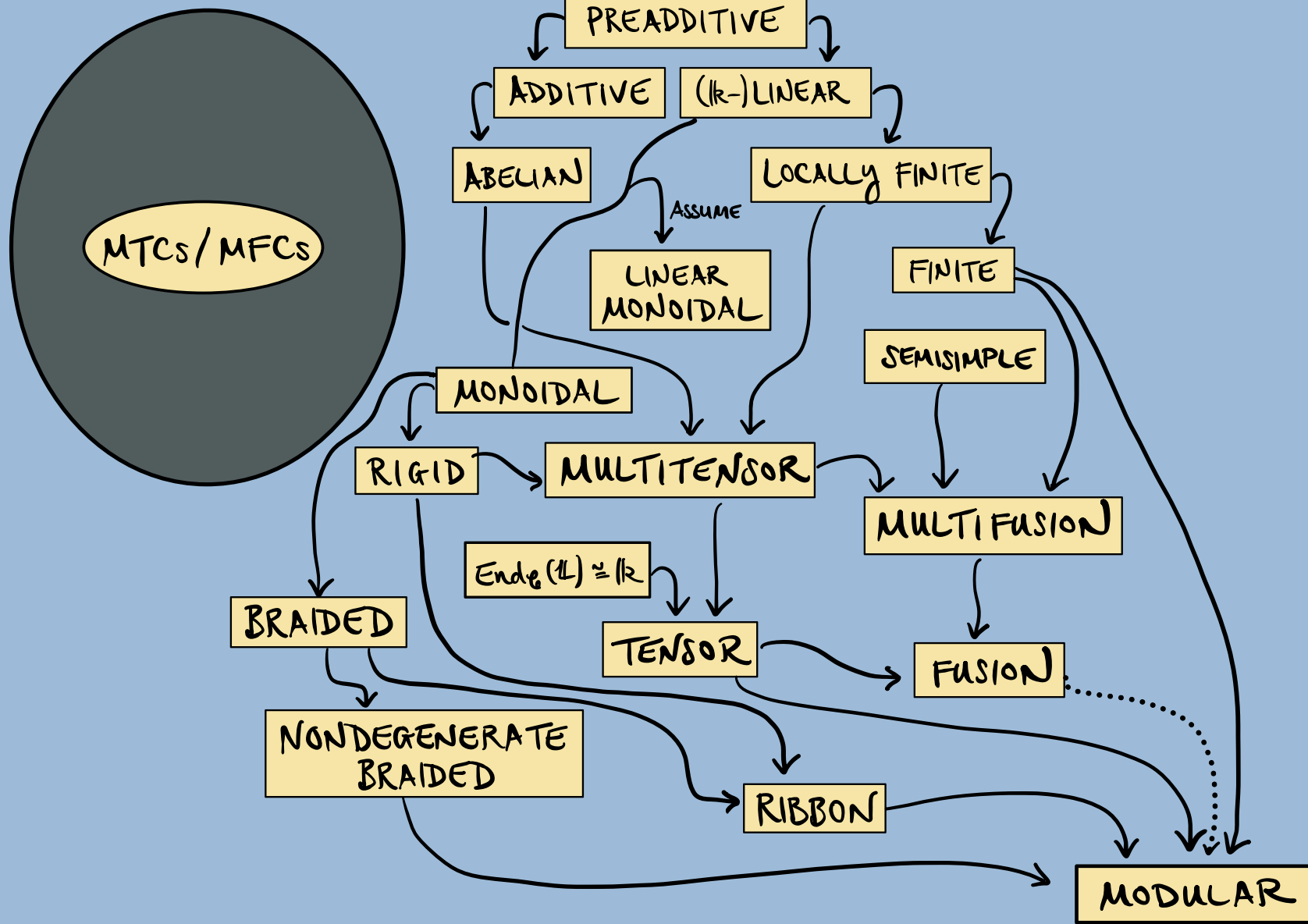
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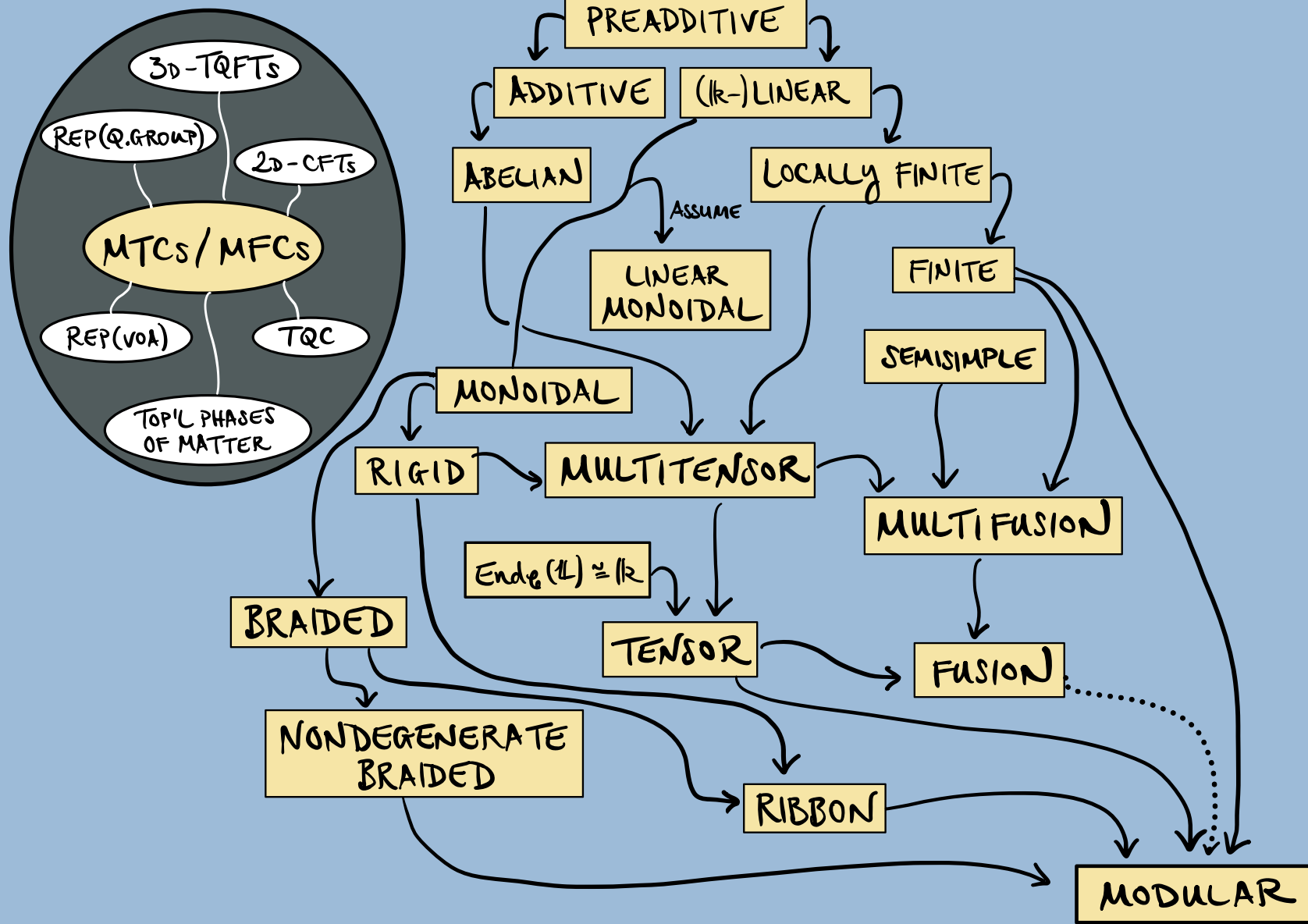
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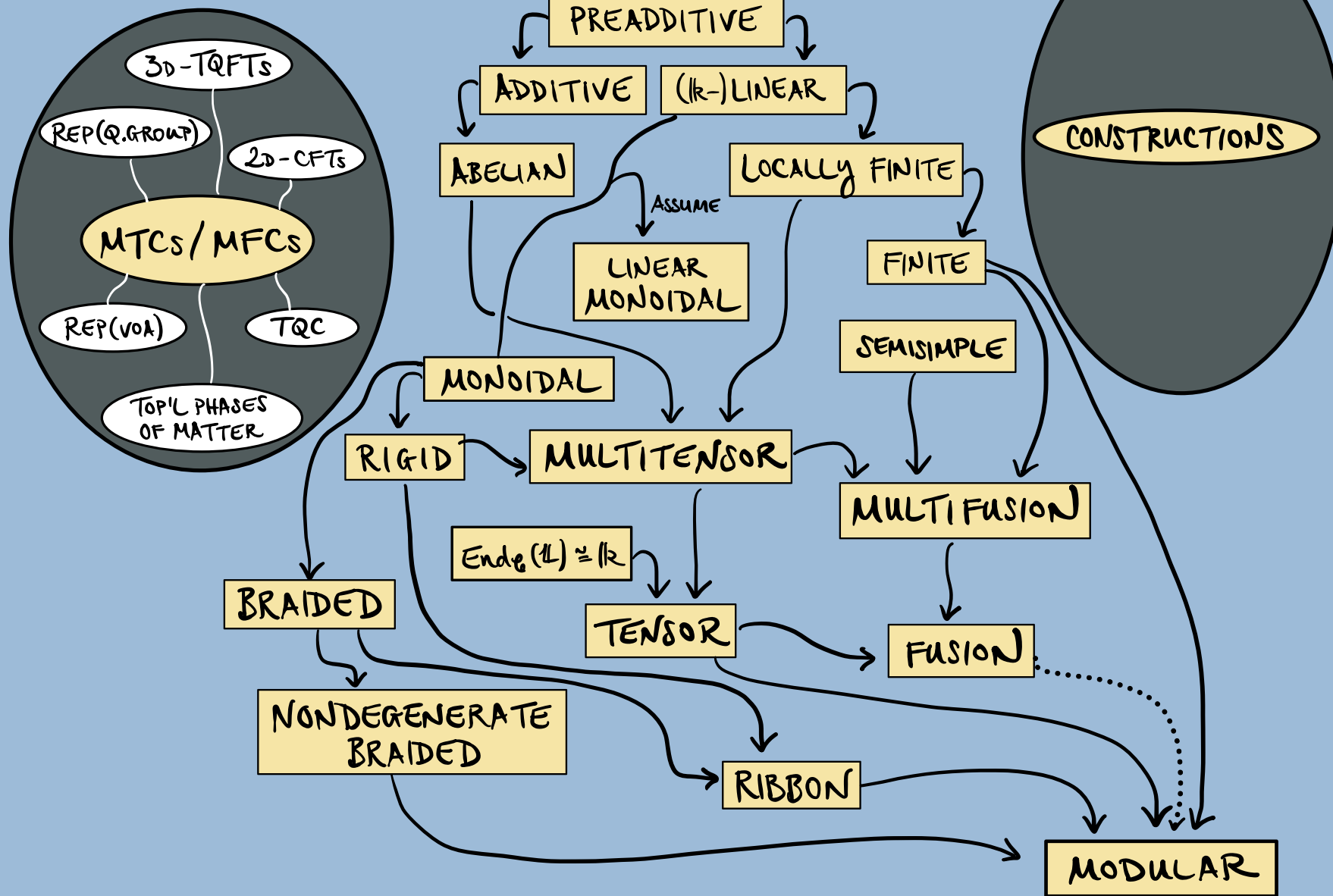
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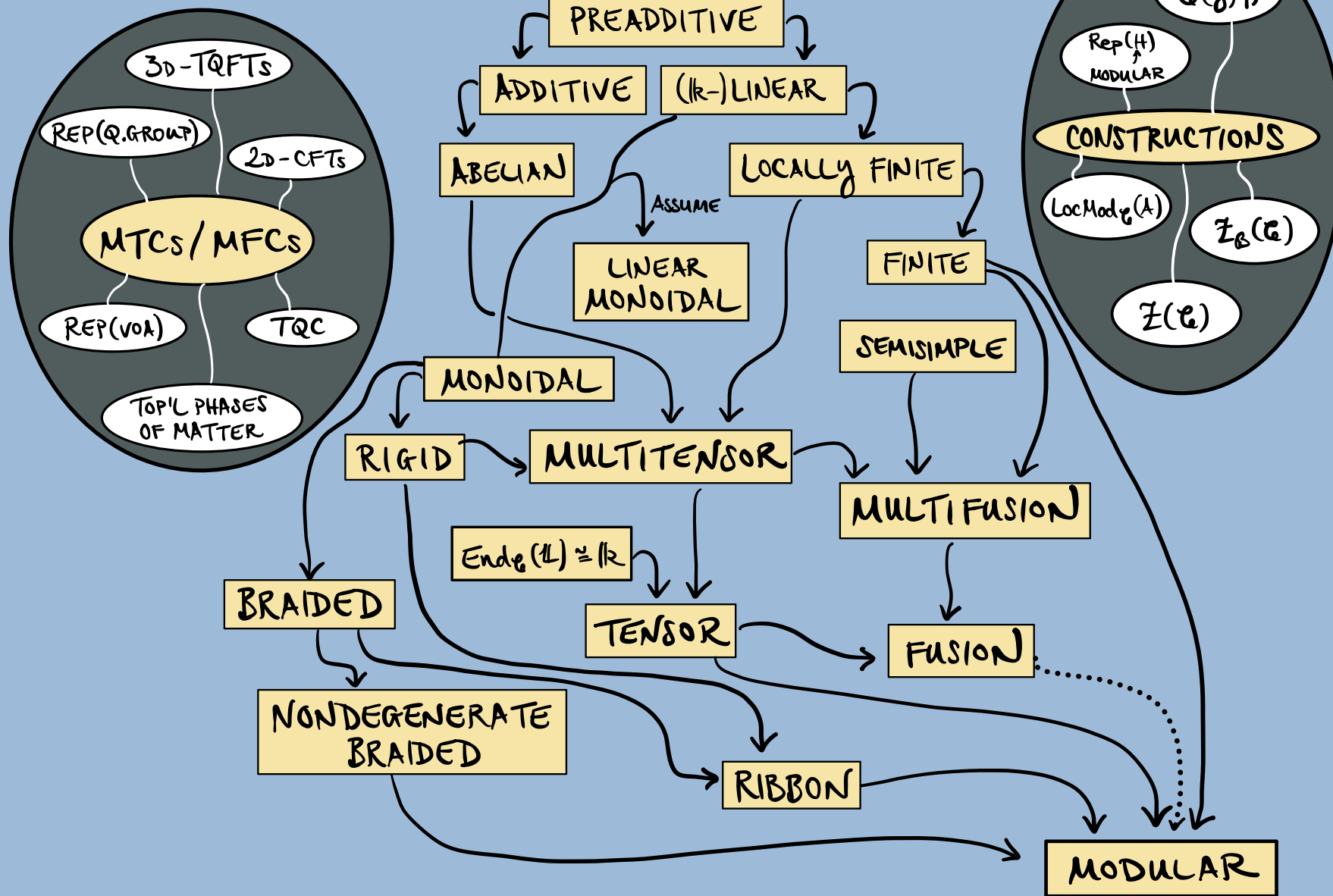
THE ROAD TO MODULARITY



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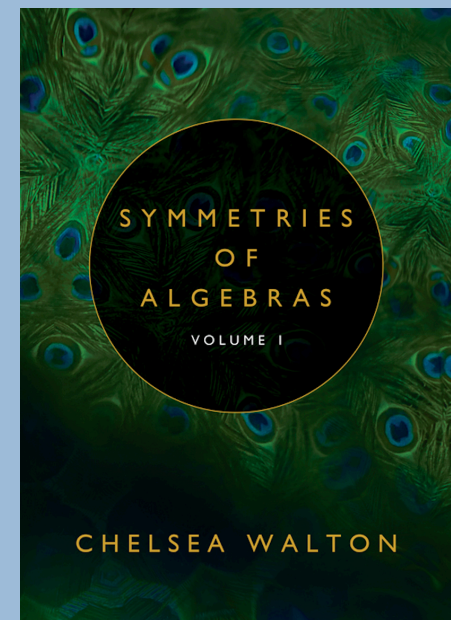
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Thanks for
listening!



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