November 12, 2018 On braided commutative algebras [Perimeter Institute

joint work with Robert Laugwitz, in preparation

Now available at https://arxiv.org/pdf/1901.08980.pdf

/ $\Bbbk$ field

One of my main interests is the notion of "quantum symmetry", specifically in the context of actions of Hopf algebras H on $\Bbbk$-algs A

* this framework for studying symmetry is quite rich for a few reasons

(1) behaves well under deformation

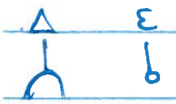
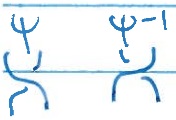
- Classical symmetries coming from actions of groups & Lie algs don't deform because such algebraic structures are "rigid", but the corresponding actions of group algebras & enveloping algs of Lie algs do deform by way via their Hopf algebra structure

(2) convenient categorical framework

- H acts on A $\overset{\text{def'n}}{\Longleftrightarrow}$ A is an algebra object in the monoidal category $\mathrm{Rep}(H)$.

The focus of this work is to construct & use algebras that are "commutative" in certain monoidal categorys of H-modules.

Let's make this more precise—

Graphical calculus	Algebraic structure $H / \Bbbk$	category $\mathcal{C} = \mathrm{Rep}(H)$	(categ of) Alg structures in $\mathcal{C}$
m u 	$\Bbbk$-algebra (H, m, u) — $\Bbbk$-vs, multip, unit	just H-mod	(nothing interesting)
Δ ε 	bialgebra $(\ \ , \Delta, \varepsilon)$ — comultip, counit	monoidal $(\mathcal{C}, \otimes)$ (For $V, W \in \mathcal{C}$, $V \otimes W \in \mathcal{C}$ via Δ)	$\mathrm{Alg}(\mathcal{C})$: algebras (A, m, u) in $\mathcal{C}$ ($\mathrm{Coalg}(\mathcal{C})$: coalgs in $\mathcal{C}$)
S 	Hopf algs $(\ \ , S)$ — antipode	rigid monoidal (have dual objects) (For $V, W \in \mathcal{C}$, $\mathrm{Hom}(V, W) \in \mathcal{C}$ via Δ, ε, S)	"duals of algs" Ex. $V \in \mathcal{C} \rightsquigarrow T(V^*) \in \mathcal{C}$ (really in a comp of $\mathcal{C}$) tensor alg; "Ext-algebra / cohom algs"
Ψ Ψ^{-1} 	quasi Δ bialg. (bialg, R); quasi Δ Hopf alg (Hopf alg, R) — R-matrix	braided, rigid monoidal $(\mathcal{C}, \otimes, \Psi)$; $\{\Psi_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V\}$ $\forall V, W \in \mathcal{C}$	$\mathrm{Bialg}(\mathcal{C})$, $\mathrm{HopfAlg}(\mathcal{C})$; $\mathrm{ComAlg}(\mathcal{C})$ $\Updownarrow$ $A \in \mathrm{Alg}(\mathcal{C})$ s.t. $m_A = m_A \Psi_{A,A}$ — main gadgets here — "braided com. algs"

Why care?

① Ex. $\Bbbk$ ↑ $\Bbbk$-Vec ⤴ & commutative $\Bbbk$-algs seem to be important.

② If you like the left column, braided com algebras yield natural examples of bialgebroids (= generalizations of bialgs w/ base larger than $\Bbbk$)

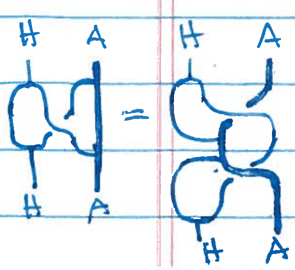
[Brzezinski-Militaru, Lu] H $\Bbbk$-bialgebra. $A \in H$-mod & H-comod

Then

$A \in \mathrm{ComAlg}({}^H_H\mathcal{YD}) \iff A \# H$ is a bialgebroid over base algebra A

left-left Yetter-Drinfeld modules ← with compatibility

braided monoidal catg



③ Important in rational 2-dim CFT: "extended chiral algebras" arise as commutative algebras in modular tensor algebras
[(Fröhlich-) Fuchs-Runkel-Schweigert, Kong-Runkel]

braided monoidal categ with much more structure
Ex. Rep(VOA)

④ Morita equivalence: Two $\Bbbk$-algs A, A' are Morita equiv $\overset{\text{def}}{\iff}$ A-mod $\sim$ A'-mod.
one Morita invariant - the center: $\Longrightarrow$ $Z(A) \cong Z(A')$ as $\Bbbk$-alg.

Likewise, $A, A' \in \mathrm{Alg}(\mathcal{C})$ are Morita equiv $\overset{\text{def}}{\iff}$ $\mathrm{Mod}_{\mathcal{C}}(A) \sim \mathrm{Mod}_{\mathcal{C}}(A')$ as $\mathcal{C}$-module categorys

one Morita invariant "full center" [FRS] $\Longrightarrow$ $Z^{\mathcal{C}}(A) \cong Z^{\mathcal{C}}(A')$ as commutative algebras in the center $\underbrace{\mathcal{Z}(\mathcal{C})}$ of $\mathcal{C}$
braided monoidal cat.

Application [FRS] When $\mathcal{C}$ is modular,
Morita equiv. classes of algebras* in $\mathcal{C}$ determine uniquely rational 2-dim'l CFTs w/ consistent boundary conditions
* nondegenerate, Frobenius, ..

Aim: Generalize [FFRS, KR] for MTCs & Davydov's full center construction in order to produce commutative algebras & Morita invariants more generally.

* Boils down to swapping out $\mathcal{Z}(\mathcal{C})$ monoidal center & replacing with $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ "relative monoidal center" ...

$\mathcal{Z}(\mathcal{C})$: objects $(V, c_{V,-})$
: for $V \in \mathcal{C}$, $c_{V,-}: V \otimes - \xrightarrow{\sim} - \otimes V \in \mathcal{C}$ "half braiding"

Ex. $\mathcal{C} = \mathrm{Rep}(H) \rightsquigarrow \mathcal{Z}(\mathcal{C}) \sim {}^H_H\mathcal{YD} \sim \mathrm{Rep}(D(H))$
H fin.dim'l Hopf alg
Drinfeld double of H

Ex. Rep categorys of q. groups $U_q(\mathfrak{g})$ ...

(Ignore red text until after page 6)

Theorem [Laugwitz-W]

Davydov (2010, 2012):

- Start with $\mathcal{B}$-augmented monoidal category $\mathcal{C}$ $\xleftarrow{\text{Forget}}$ $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ braided monoidal category

- If $\exists$ right adjoint to Forg, $R_{\mathcal{B}} : \mathcal{C} \longrightarrow \mathcal{Z}_{\mathcal{B}}(\mathcal{C})$, then

  - $R_{\mathcal{B}}$ is lax-monoidal: $\mathrm{Alg}(\mathcal{C}) \rightsquigarrow \mathrm{Alg}(\mathcal{Z}_{\mathcal{B}}(\mathcal{C}))$

  - Using "left center" construction for braided mon'l categories, get:
  $C^{\ell} : \mathrm{Alg}(\mathcal{Z}_{\mathcal{B}}(\mathcal{C})) \rightsquigarrow \mathrm{ComAlg}(\mathcal{Z}_{\mathcal{B}}(\mathcal{C}))$
  discussed in Ostrik 2003 & references w/in:
  $C^{\ell}(D)$ is the maximal subobject of D s.t. $m_D = m_D \Psi^{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})}_{C^{\ell}(D), D}$

- Also produced direct route $Z : \mathrm{Alg}(\mathcal{C}) \rightsquigarrow \mathrm{ComAlg}(\mathcal{Z}_{\mathcal{B}}(\mathcal{C}))$ "$\mathcal{B}$-center" "full center"
$A \longmapsto Z_{\mathcal{B}}(A)$

  object in $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ together with map $Z_{\mathcal{B}}(A) \to A$ in $\mathcal{C}$
  terminal among pairs (X, x) for $X \in \mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ & $x : X \to A$ s.t.

$$\begin{tikzcd} X \otimes A \arrow[r, "c_{X,A}"] \arrow[d, "x \otimes \mathrm{id}"'] & A \otimes X \arrow[d, "\mathrm{id} \otimes x"] \\ A \otimes A \arrow[dr, "m"'] & A \otimes A \arrow[dl, "m"] \\ & A \end{tikzcd}$$

- Established that $Z_{\mathcal{B}}(A) \cong C^{\ell}(R_{\mathcal{B}}(A))$ in $\mathrm{ComAlg}(\mathcal{Z}_{\mathcal{B}}(\mathcal{C}))$

- When $\mathcal{C} = \mathrm{Rep}(H)$ $\mathcal{C} = H\text{-mod}(\mathcal{B})$, $R_{\mathcal{B}}$ exists & $= \mathrm{Cent}^{\Psi}_{A \# H}(A)$ braided centralizer-alg in ${}^{H}_{H}\mathcal{YD}(\mathcal{B})$
(also due to Cohen-Fischman-Montgomery)

- Also showed that $Z_{\mathcal{B}}(A)$ is a Morita invariant: Take $A, A' \in \mathrm{Alg}(\mathcal{C})$
If $\mathrm{Mod}_{\mathcal{C}}(A) \sim \mathrm{Mod}_{\mathcal{C}}(A')$ as $\mathcal{C}$-mod categs, then $Z_{\mathcal{B}}(A) \cong Z_{\mathcal{B}}(A')$ in $\mathrm{ComAlg}(\mathcal{Z}_{\mathcal{B}}(\mathcal{C}))$

Our setting —

<u>Defn</u> [Laugwitz] Take $(\mathcal{B}, \otimes, \Psi^{\mathcal{B}})$ a braided monoidal category.
A monoidal category $\mathcal{C}$ is <u>$\mathcal{B}$-augmented</u> if it comes equipped with monoidal functors:

$$F: \mathcal{C} \longrightarrow \mathcal{B} \qquad \& \qquad T: \mathcal{B} \longrightarrow \mathcal{C}$$

and natural isoms: $\tau: FT \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{B}}$, $\sigma: \otimes_{\mathcal{C}}(id_{\mathcal{C}} \boxtimes T) \overset{\sim}{\Rightarrow} \otimes_{\mathcal{C}}^{op}(id_{\mathcal{C}} \boxtimes T)$,
so that σ descends to $\Psi^{\mathcal{B}}$ under F, and τ & σ are coherent w/ structure of $\mathcal{B}$.

<u>Main Example</u>: Take $H \in \mathrm{HopfAlg}(\mathcal{B})$. Build $\mathcal{C} = H\text{-mod}(\mathcal{B})$ category of left H-mods in $\mathcal{B}$.

- <u>monoidal</u>: $(V, a_V: H \otimes V \to V) \otimes (W, a_W: H \otimes W \to W)$

$$\|$$

$$\begin{tikzcd}
(V \otimes W, a_{V \otimes W}: H \otimes V \otimes W \arrow[rr] \arrow[d, "\Delta \otimes \mathrm{Id}"'] & & V \otimes W) \\
H \otimes H \otimes V \otimes W \arrow[rr, "id \otimes \Psi \otimes id"] & & H \otimes V \otimes H \otimes W \arrow[u, "a_V \otimes a_W"']
\end{tikzcd}$$

- <u>$\mathcal{B}$-augmented</u>: $F: \mathcal{C} \to \mathcal{B}$ forget, $T: \mathcal{B} \to \mathcal{C}$
  $X \longmapsto$ trivial H-mod structure on X

Ex. $\mathcal{B} = K\text{-mod}$, $K = \Bbbk \mathbb{Z}_n$ with $\mathcal{R}$-matrix $\frac{1}{n}\sum_{i,j \geq 0} q^{-2ij} g^i \otimes g^j$
for $n \geq 3$, order$(q^2) = n$, $\mathbb{Z}_n = \langle g \mid g^n = 1 \rangle \rightsquigarrow \Psi_{q^2}$ braiding on $\mathcal{B}$.
$H = \Bbbk[x]/(x^n) \in \mathrm{HopfAlg}(\mathcal{B})$ via $\Delta(x) = 1 \otimes x + x \otimes 1$
Then, $\mathcal{C} = H\text{-mod}(K\text{-mod}) \sim H \rtimes K\text{-mod}$

grouplike (g,1) skew prim

$$\underbrace{H \rtimes K}_{} =: u_q(\mathfrak{sl}_2^+) = T_n(q^2) = \frac{\Bbbk\langle g, x\rangle}{(g^n - 1, x^n, gx - q^2 xg,}$$

Sweedler Hopf alg.

↑ can generalize to get $\mathcal{C} \sim \mathrm{Rep}(u_q(\mathfrak{b}_+))$ for $K = u_q(\mathfrak{h})$, $H = u_q(\mathfrak{n}_+)$
for $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{b}_+$ the Cartan, pos. nilp, pos. borel part of a semisimple Lie alg $\mathfrak{g}$.

Def'n [Laugwitz] The relative monoidal center of a $\mathcal{B}$-augmented $\otimes$ categ $\mathcal{C}$ is a braided $\otimes$ category $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ consisting of objects $(V, c_{V,-})$ with $V \in \mathcal{C}$ & $c_{V,-} : V \otimes - \xrightarrow{\sim} - \otimes V$ in $\mathcal{C}$ so that

- $c_{V,-}$ is compatible with $\otimes$ of $\mathcal{C}$ &
- $c_{V,-}$ is compatible with $\mathcal{B}$-aug'n: $c_{V,T(B)} = \sigma_{V,B} \ \forall B \in \mathcal{B}$.

⋮

Prop [Laugwitz]

① $\mathcal{Z}_{Vec}(\mathcal{C}) \sim \mathcal{Z}(\mathcal{C})$

② In general, $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ is a full monoidal subcategory of $\mathcal{Z}(\mathcal{C})$.

③ If $\mathcal{C}$ is rigid (pivotal) then so is $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$.

Main Example [L] For $\mathcal{C} = H\text{-mod}(\mathcal{B})$, have

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \sim {}^H_H\mathcal{YD}(\mathcal{B}) \sim \operatorname{Rep}(\underbrace{D_K(H)})$$

(aka "double bosoniz'n" [Majid])

Category of left-left Yetter-Drinfeld modules w/ objs in $\mathcal{B}$

braided Drinfeld double of H w.r.t K & pairing $\langle , \rangle : H \overset{*}{\otimes} H \to \Bbbk$.

Ex. · $\mathcal{C} = u_q(\mathfrak{sl}_2^+)\text{-mod}$. Get that $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \sim u_q(\mathfrak{sl}_2)\text{-mod}$

· Likewise, $\mathcal{Z}_{\operatorname{Rep}(\Bbbk\mathbb{Z}_n)}(\operatorname{Rep}(u_q(\mathfrak{b}_+))) \sim \operatorname{Rep}(u_q(\mathfrak{g}))$
with Ψ_{q^2}

... same for big quantum groups: rep categ of highest weight modules

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What we [Laugwitz-W] achieve ... (add red text to page 4)

Main point of difficulty:
dealing with "internal braiding" $\Psi^{\mathcal{B}}$
& "external braiding" $\Psi^{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})}$ simultaneously.

Besides being able to get com. alg in rep categ of q.groups, One nice advantage – new supply of Morita invariants that are "easier" to compute....

Ex. Take $\mathcal{C} \sim H \rtimes K\text{-mod} \sim H\text{-mod}(\overbrace{K\text{-mod}}^{\mathcal{B}})$

for K quasi Δ, $H \in \text{HopfAlg}(K\text{-mod})$ finite dim'l

$\mathcal{Z}(\mathcal{C}) \sim \underbrace{D(H \rtimes K)}_{\|}\text{-mod}$, $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \sim \underbrace{D_K(H)}_{\|}\text{-mod}$

$H^* \otimes K^* \otimes K \otimes H$ as a k-vs subject to relations btw H^*, K^*, K, H

$H^* \otimes K \otimes H$ as a k-vs subject to relations btw H^*, K, H

smaller algebra

Extreme case: $H = k$, get $\mathcal{C} \sim \mathcal{B}$ & $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \sim \mathcal{B}$

It was stated (vaguely) in Ostrik (2003) that the left/right center is a mor. inv. of a braided (modular) tensor category, but we provide .. a proof of this statement.

In the braided case, we don't have to leave $\mathcal{C}$ to get Morita invariants of algebras in $\mathcal{C}$. In fact $Z_{\mathcal{B}}(A) = C^{\ell}(A)$ here

January 28, 2019:

This was later amended in paper so that \C is the negative Borel part of u_q(\g).

An extended example: $\mathcal{C} = u_q(b_+)\text{-mod}$ (not braided) b_+ positive Borel part of $\mathfrak{g} = \mathfrak{sl}_2$.

$A = k[u] \in \text{Alg}(\mathcal{C})$ via $g \cdot u = q^{-1}u$, $x \cdot u = \gamma$, $\gamma \in k$

$$\rightsquigarrow Z_{\mathcal{B}}(A) = \begin{cases} u_q(n_+) \otimes k[u^n] & \gamma = 0 \\ k[z] & \gamma \neq 0 \\ \quad \text{for } z = \sum_{i=0}^{n-1} \gamma^{-i} q^{-2\left(\binom{i+1}{2}+i\right)} (1-q^2)^i x^i u^{i+1} & \end{cases}$$

$\rightsquigarrow A_{\gamma=0}$ & $A_{\gamma\neq 0}$ are not Morita equiv as algs in $\mathcal{C}$.