

A CENTURY OF FROBENIUS ALGEBRAS

CHELSEA WALTON
RICE UNIVERSITY

ICM (WM)² MEETING, JULY 2026

STORY TIME!!

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FROBENIUS ALGEBRAS ARE
INTRIGUING ALGEBRAIC STRUCTURES

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... CHOCK-FULL OF SYMMETRY

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NUMEROUS CHARACTERIZATIONS HAVE
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LINEAR
-ALGEBRAIC

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LINEAR
-ALGEBRAIC

REPRESENTATION
-THEORETIC

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LINEAR ALGEBRAIC FROBENIUS ALGEBRAS

- DUE TO F.G. FROBENIUS (1849–1917).

A 1903-STYLE GAME...

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START WITH A FINITE DIM'L $\mathbb{C}$ -ALGEBRA A [F.D. $\mathbb{C}$ -VS + UNITAL RING]

GOAL: UNDERSTAND A VIA MATRICES

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- FOR $a \in A$, GET SCALARS $\beta_{i,j}^a \neq \gamma_{i,j}^a \in \mathbb{C}$ S.T. [UNITAL RING STRUCTURE]

$$v_i a = \sum_{j=1}^n \beta_{i,j}^a v_j \quad \neq \quad a v_i = \sum_{j=1}^n \gamma_{j,i}^a v_j$$

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⋮
[IN TERMS OF MATRICES]

• GET $\mathbb{C}$ -LINEAR MAPS $\beta: A \rightarrow \text{Mat}_n(\mathbb{C}) \neq \gamma: A \rightarrow \text{Mat}_n(\mathbb{C})$

$$a \mapsto (\beta_{i,j}^a)_{i,j=1}^n$$

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QUESTION:

WHEN DOES $\exists P \in \text{GL}_n(\mathbb{C})$ S.T.

$$P \beta(a) = \gamma(a) P \quad \forall a \in A?$$

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REP'N OF A "

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... BUT ~~X~~ REPTHY IN 1903,
JUST HAD LINEAR ALG.

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TOWARDS AN ANSWER...

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FOR $\underline{\lambda} := (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, DEFINE
THE PARASTROPHIC MATRIX
OF A AT $\underline{\lambda}$ AS:

$$P_{\underline{\lambda}}^A := \left(\sum_{k=1}^n \lambda_k m_{i,j}^k \right)_{i,j=1}^n$$

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A : [FROBENIUS, 1903] $\exists \underline{\lambda} \in \mathbb{C}^n$ S.T.

$$P := P_{\underline{\lambda}}^A \text{ IS INVERTIBLE.}$$

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A FINITE DIM'L $\mathbb{C}$ -ALG A
IS "FROBENIUS"



$\exists \underline{\lambda} \in \mathbb{C}^n$ s.t. $\det(P_{\underline{\lambda}}^A) \neq 0$

NOT SELF-NAMED!

GET $\mathbb{C}$ -LINEAR MAPS:

$$\beta: A \rightarrow \text{Mat}_n(\mathbb{C}) \quad \& \quad \gamma: A \rightarrow \text{Mat}_n(\mathbb{C})$$
$$a \mapsto (\beta_{ij}^a)_{i,j=1}^n \quad \quad a \mapsto (\gamma_{ij}^a)_{i,j=1}^n$$

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$\exists \underline{\lambda} \in \mathbb{C}^n$ s.t. $\det(P_{\underline{\lambda}}^A) \neq 0$

Ex. $A = \mathbb{C}[x]/(x^2) \dots$

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$A = \mathbb{C}1_A \oplus \mathbb{C}x$ AS A $\mathbb{C}$ -VS.

MULT. TABLE:

	1_A	x
1_A	1_A	x
x	x	0

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$$m_{1,1}^1 = m_{1,x}^x = m_{x,1}^x = 1 \quad \text{ELSE } m_{*,*}^* = 0$$

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 $\equiv$ FROBENIUS $\equiv$

Ex. $A = \mathbb{C}[x, y]/(x^2, xy, y^2)$ ^{you do!}
 $\equiv$ NOT FROBENIUS $\equiv$

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$\left[P_{\underline{\lambda}}^A := \left(\sum_{k=1}^n \lambda_k m_{i,j}^k \right)_{i,j=1}^n \right]$

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FOR $\lambda_x \neq 0$

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$$[v_i v_j = \sum_{k=1}^n m_{i,j}^k v_k]$$

$$1 \cdot x = M_{x,1}^x = 1 \quad \text{ELSE } M_{*,*}^* = 0$$

Ex. $A = \mathbb{C}[x]/$

$\equiv$ FROB

Ex. $A = \mathbb{C}[x],$

$\equiv$ NOT FROBENIUS $\equiv$

AND THEN THIS GAME
WAS OVER FOR A WHILE

... UNTIL 35 YEARS LATER

INVERTIBLE
FOR $\lambda x \neq 0$

REP'N-THEORETIC FROBENIUS ALGEBRAS

- DUE TO BRAUER, NAKAYAMA, & NESBITT.

FAST FWD TO MID 1930's...

A FINITE DIM'L $\mathbb{C}$ -ALG A
IS "FROBENIUS"



$\exists \underline{\lambda} \in \mathbb{C}^n$ s.t. $\det(P_{\underline{\lambda}}^A) \neq 0$

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TOOLS OF ABSTRACT ALGEBRA
(RINGS, ALGEBRAS, MODULES $\equiv$ REP'NS)

NOW AVAILABLE!

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FROM T.Y. LAM'S BOOK, "LECTURES ON MODULES AND RINGS":

C. Nesbitt wrote: "The writer, in collaborating with T. Nakayama, adopted the term Frobeniusean algebra, but now, quailing before our critics, we return to simply Frobenius algebra." So apparently, people were not attracted by a six-syllable word.

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$p_L: A \rightarrow \text{End}_{\mathbb{C}}(A)$ LEFT REG. REP'N

$\equiv$ IS EQUIV. TO $\equiv$

$p_R: A \rightarrow \text{End}_{\mathbb{C}}(A^*)$ RIGHT REG. REP'N

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- COHOMOLOGY ALGS $H^*(X)$,
 $X = \text{FIN. DIM CPT ORIENTED MFLD}$
(VIA 4. WITH POINCARÉ DUALITY)

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ARTIN-WEDDERBURN THM



FIN. DIM SEMISIMPLE $\mathbb{C}$ -ALGS
($\equiv$ SEPARABLE)
ARE FROBENIUS!

THEOREM [BNN, 1937-1942]

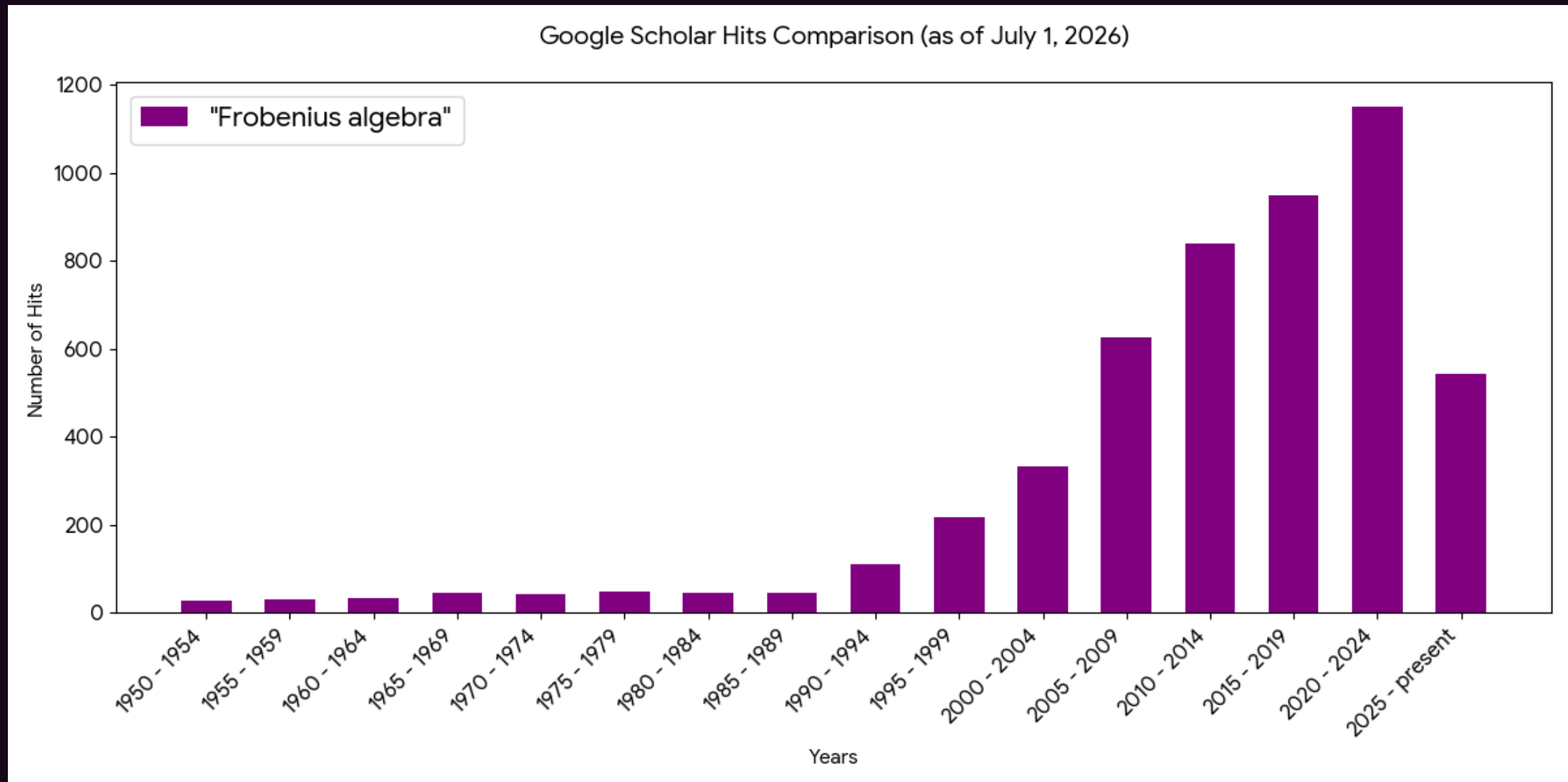
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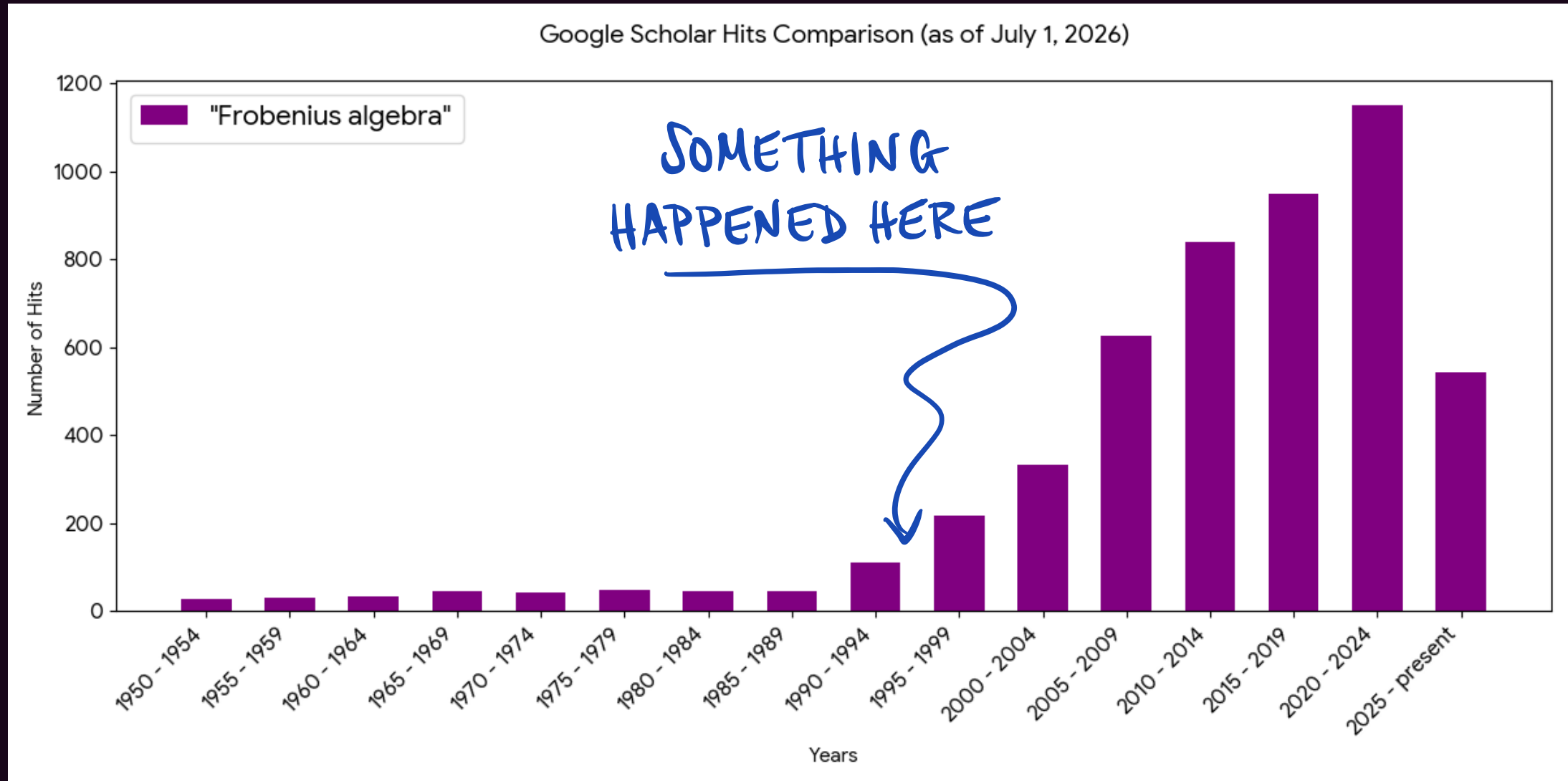
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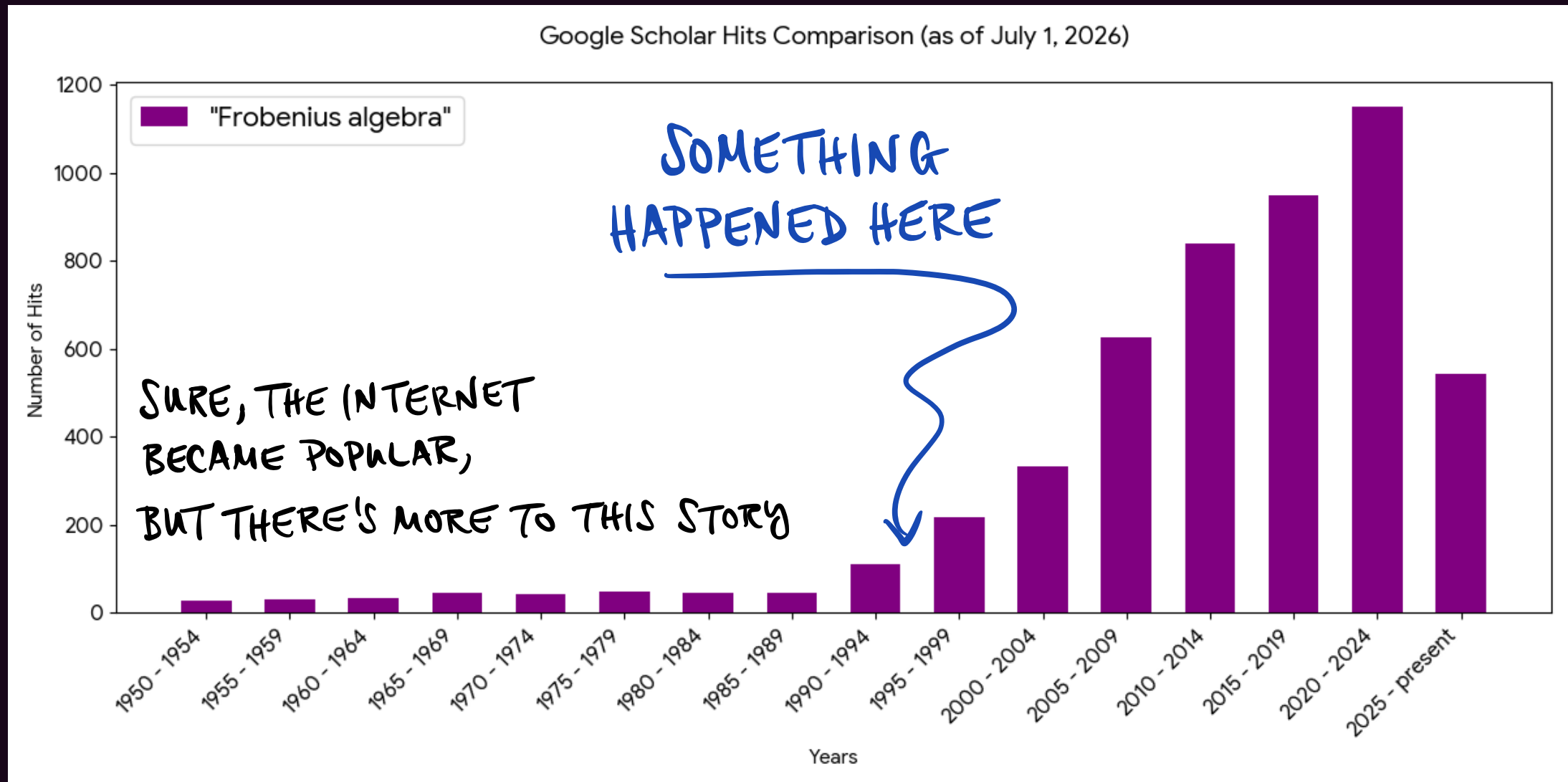
≡ POPULARITY THROUGHOUT THE YEARS ≡



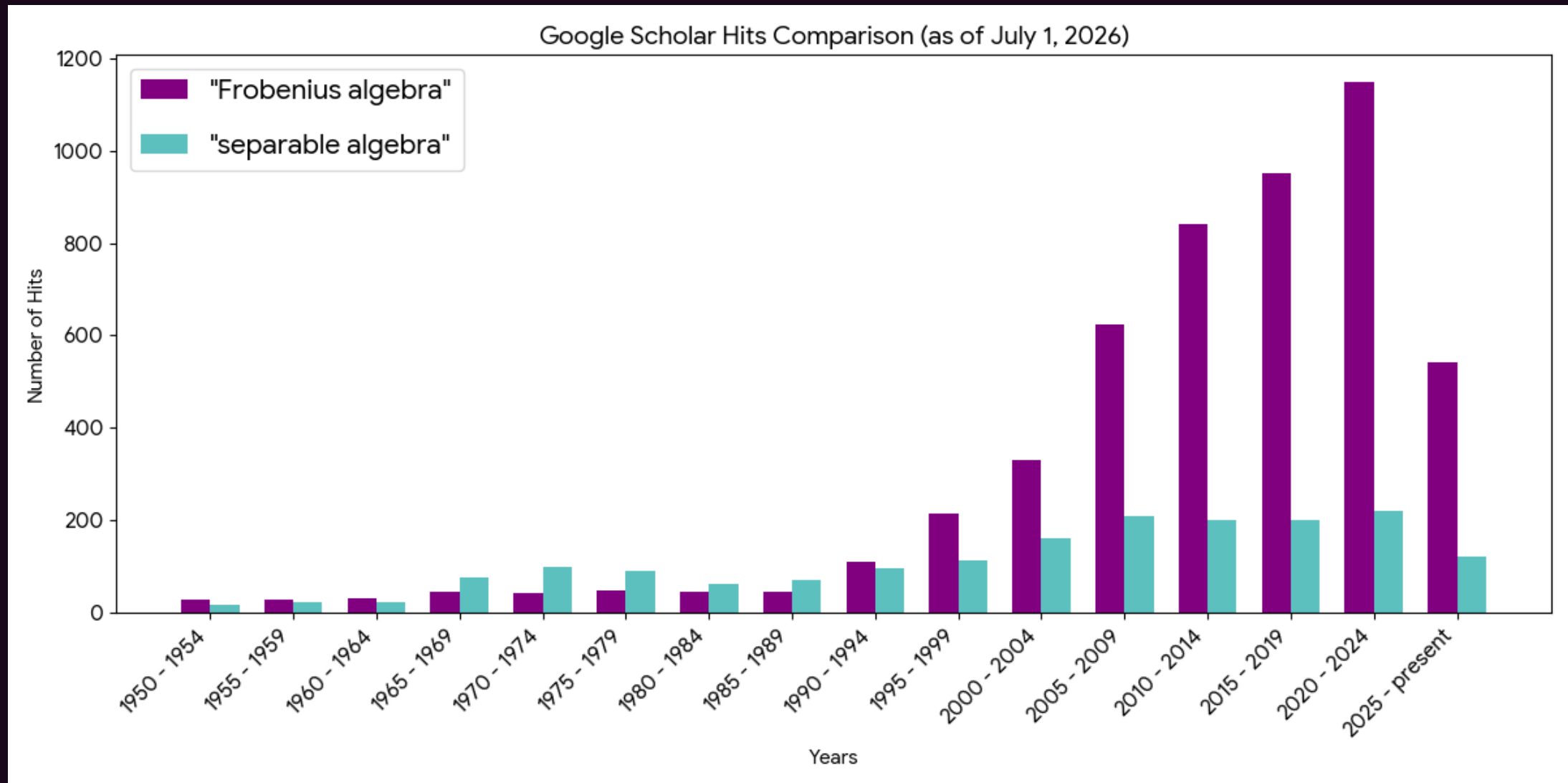
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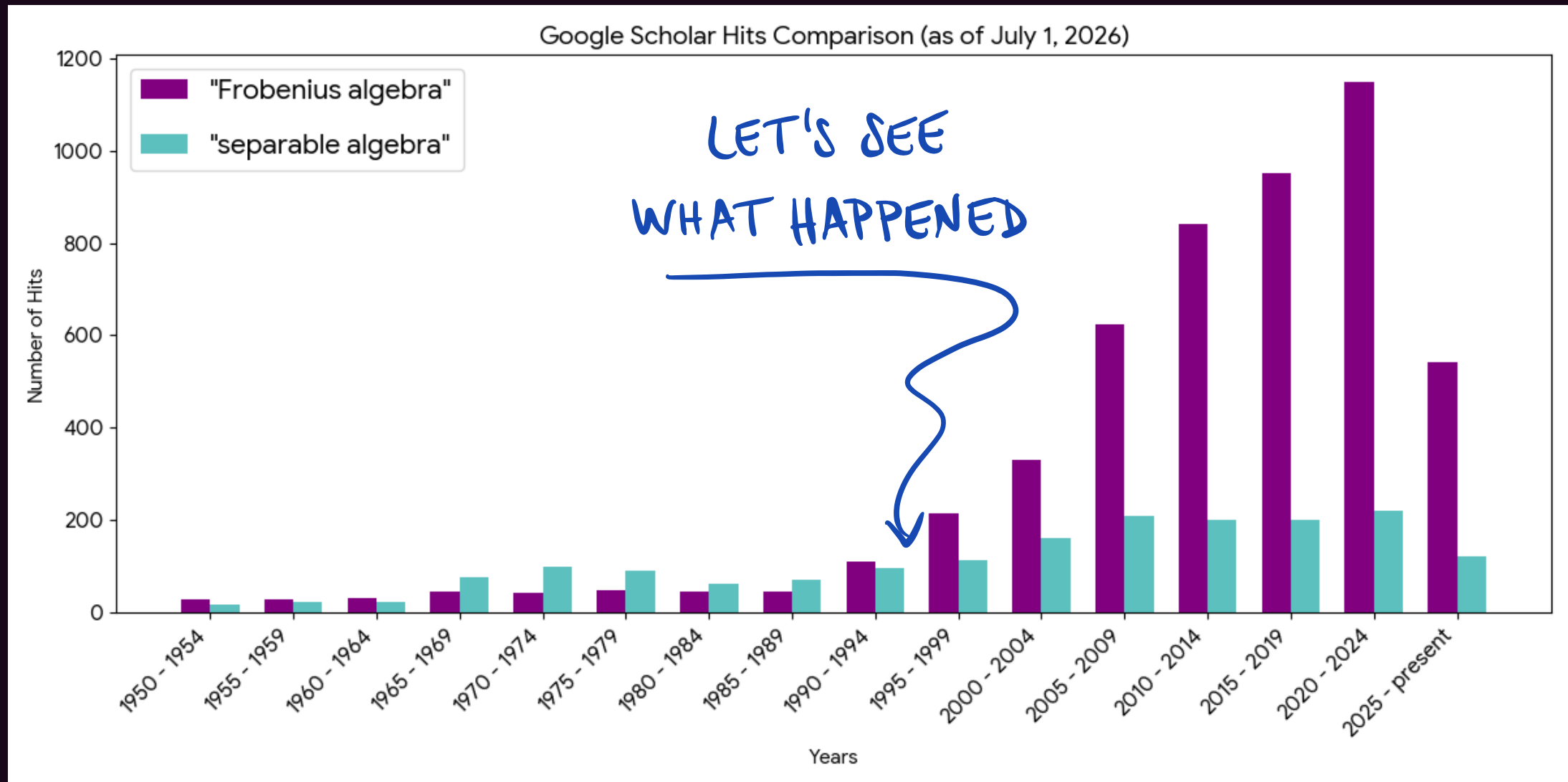
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TOPOLOGICAL QUANTUM FIELD THEORIES

- DUE TO ATIYAH, SEGAL, WITTEN •

FAST FWD TO LATE 1980's ...

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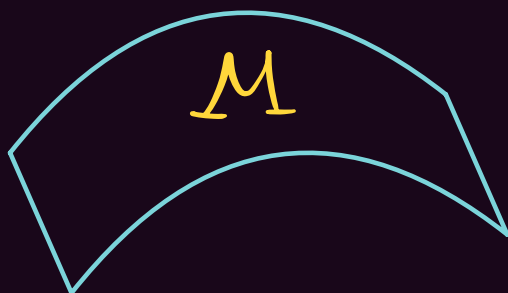
FAST FWD TO LATE 1980's...

MANIFOLD

|||

SPACE OF A

PHYSICAL SYSTEM



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FAST FWD TO LATE 1980's ...

V

VECTOR SPACE OF
"QUANTUM FIELD STATES" ON M

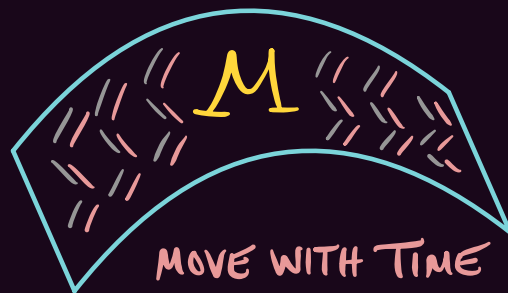
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IMPORTANT TO KNOW HOW THE
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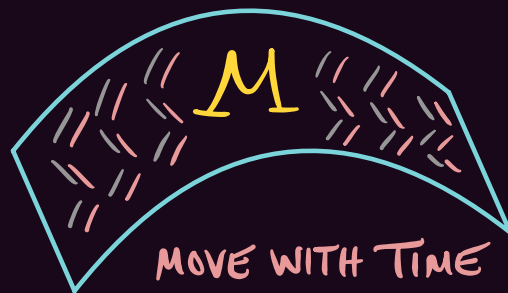
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$$V \xrightarrow{f}$$

VECTOR SPACE OF
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⋮

CAPTURED BY A LINEAR MAP

$$V \xrightarrow{f} V$$

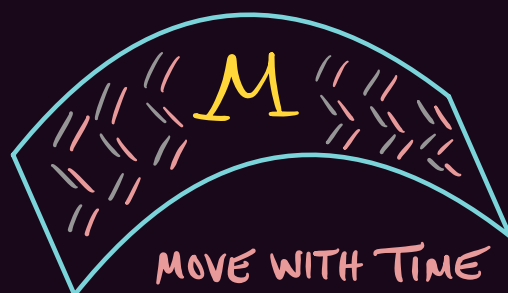
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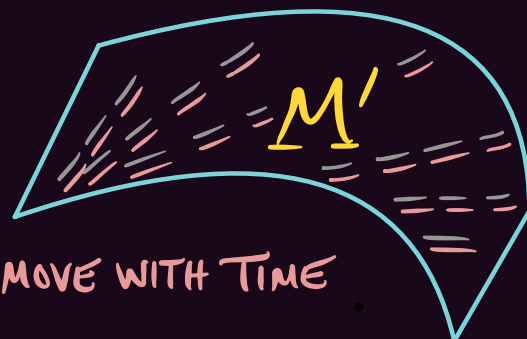
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TAKE ANOTHER M'



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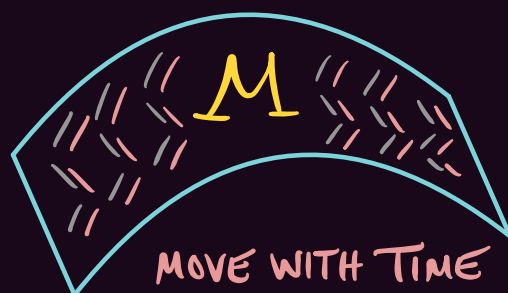
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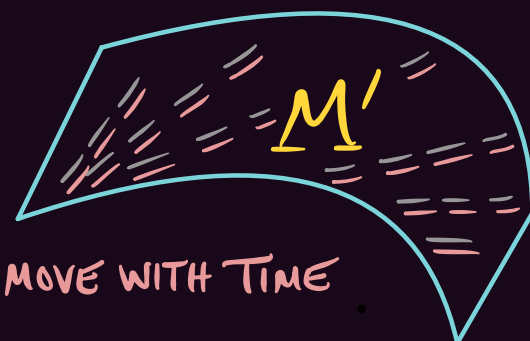
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$V \xrightarrow{f} V$

AND GET

⋮

$V' \xrightarrow{f'} V'$

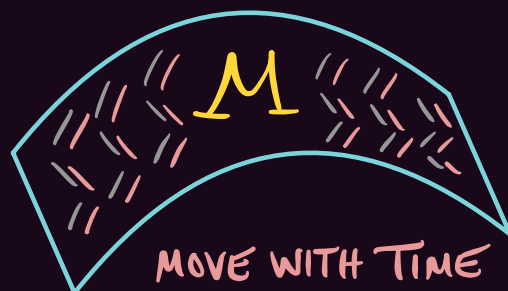
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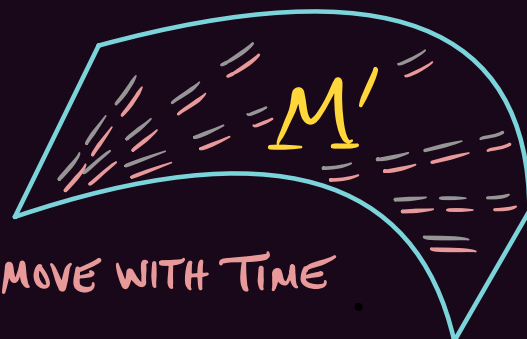
MANIFOLD

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SPACE OF A
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TAKE ANOTHER M'



• CHELSEA WALTON (RICE UNIV) •

$$\begin{array}{ccc} \text{DISJOINT UNION} & \xrightarrow{\quad} & \text{TENSOR PRODUCT} \\ M \sqcup M' & \leadsto & V \otimes V' \end{array}$$

• A CENTURY OF FROBENIUS ALGEBRAS •

FAST FWD TO LATE 1980's ...

$$V \xrightarrow{f}$$

VECTOR SPACE OF
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• ICM - (WM)², JULY 2024 •

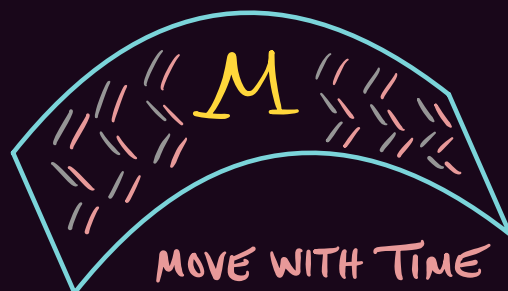
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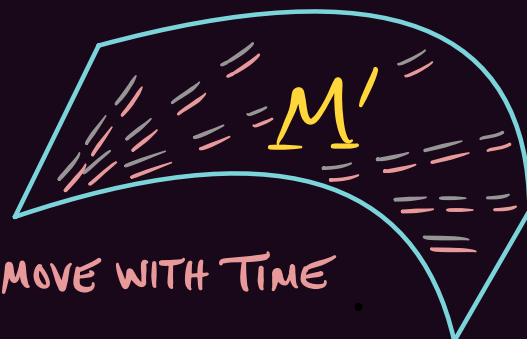
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⋮

CAPTURED BY A LINEAR MAP

$V \xrightarrow{f} V$

AND GET

$V' \xrightarrow{f'} V'$

DISJOINT
UNION



TENSOR
PRODUCT

$M \sqcup M'$

$\leadsto$

$V \otimes V'$

SWAP



FLIP

$M \sqcup M' \rightarrow M' \sqcup M$

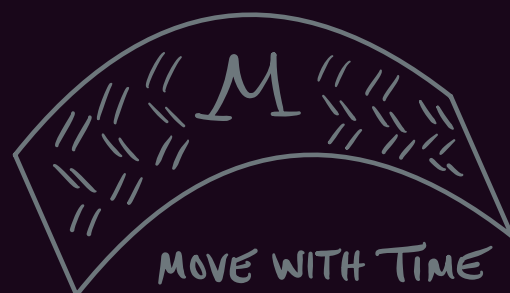
$V \otimes V' \rightarrow V' \otimes V$

TOPOLOGICAL QUANTUM FIELD THEORIES

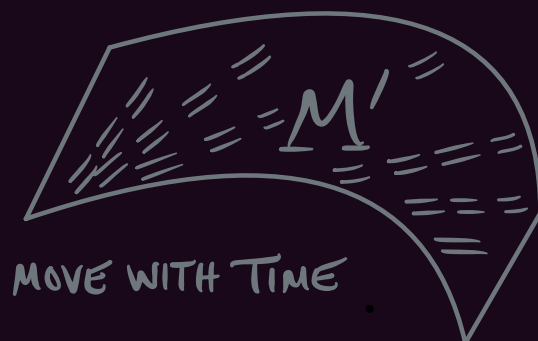
FAST FWD TO LATE 1980's ...

- DUE TO ATIYAH, SEGAL, WITTEN • ↘

MANIFOLD
" "
SPACE OF A
PHYSICAL SYSTEM



TAKE ANOTHER M'



CAN REPRESENT THE
"SYSTEM" OF SUCH
TOPOL STATES
BY THE "SYSTEM"
OF VECTOR SPACES

$$\begin{array}{ccc} \text{DISJOINT UNION} & \xrightarrow{\quad} & \text{TENSOR PRODUCT} \\ M \sqcup M' & \leadsto & V \otimes V' \\ \text{SWAP} & \xrightarrow{\quad} & \text{FLIP} \\ M \sqcup M' \rightarrow M' \sqcup M & & V \otimes V' \rightarrow V' \otimes V \end{array}$$

$V \otimes f$
VECTOR SPACE OF
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IMPORTANT TO KNOW HOW THE
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⋮

CAPTURED BY A LINEAR MAP

$$V \xrightarrow{f} V$$

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TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO LATE 1980's ...

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"SYSTEM": SYMMETRIC MONOIDAL CATEGORY

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- COLLECTION OF OBJECTS
‡ MAPS BTW OBJECTS ... + AXIOMS] "CATEGORY"

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"CATEGORY"

- OPERATION $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
COMBINING OBJS & COMBINING MAPS

$(\mathcal{C}, \otimes, \mathbb{1})$

... BEHAVES

LIKE A MONOID

"MONOIDAL"

- OBJECT LABELED AS $\mathbb{1}$

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- OBJECT LABELED AS $\mathbb{1}$

- NICE SET OF MAPS $c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$ WITH $c^2 = \text{id}$] "SYMMETRIC"

TOPOLOGICAL QUANTUM FIELD THEORIES

- DUE TO ATIYAH, SEGAL, WITTEN •

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SYMMETRIC MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

- OBJECTS & MAPS $\mathcal{C}$

- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

- $\mathbb{1} \in \mathcal{C}$



... $(\mathcal{C}, \otimes, \mathbb{1})$ LIKE A MONOID

- $c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$

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EXAMPLE (ALGEBRAIC)

$$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$$



CATEG. OF

$\mathbb{C}$ -VECTOR
SPACES

& $\mathbb{C}$ -LINEAR
MAPS

TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO LATE 1980's...

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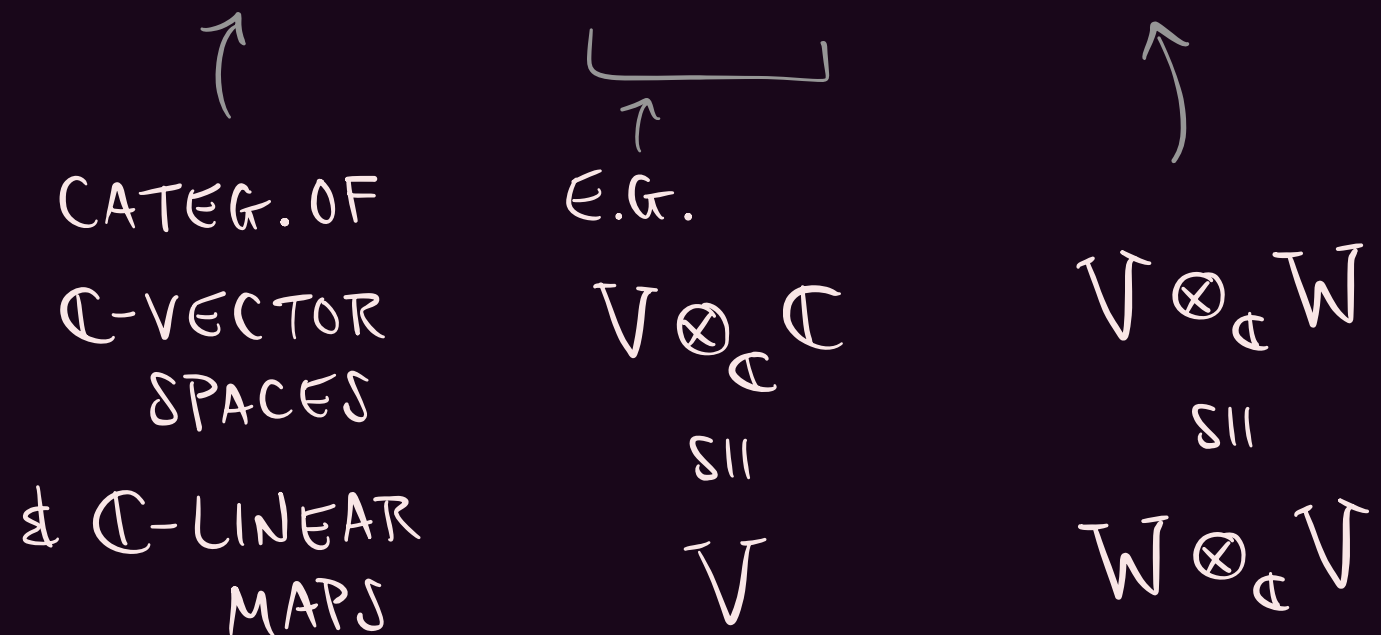
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$$(2\text{Cob}^{\text{OR}}, \sqcup, \emptyset, \text{SWAP})$$

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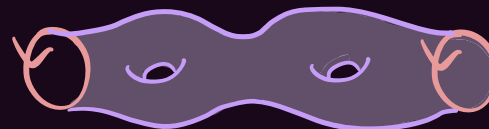
$$\left(2\text{Cob}^{\text{OR}}, \sqcup, \emptyset, \text{SWAP} \right)$$

OBJ: ORIENTED CLOSED
SMOOTH 1-MFLDS

(DISJOINT UNION OF $\bigcirc$)

MAPS: Bdy & ORIENTIN PRES.

DIFF. CLASSES OF CPT 2-MFLDS



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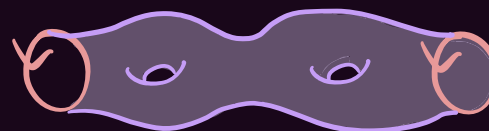
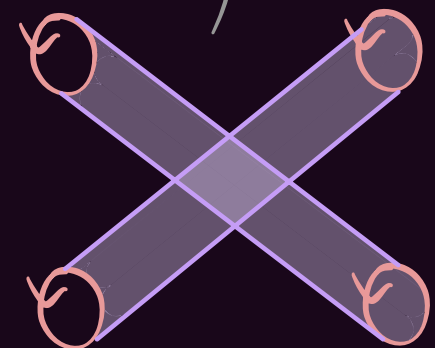
DISJ. UNION EMPTY MFLD

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TOPOLOGICAL QUANTUM FIELD THEORIES

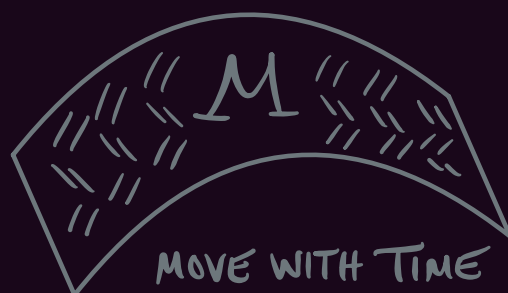
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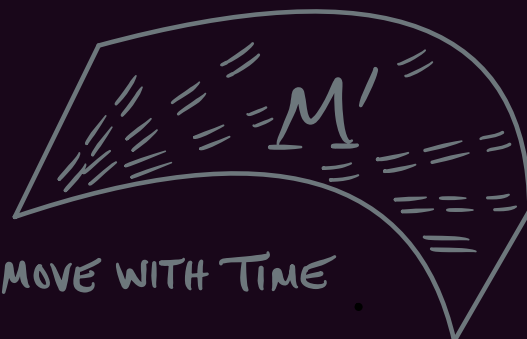
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DISJOINT
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TENSOR
PRODUCT

$M \sqcup M'$



$V \otimes V'$

SWAP



FLIP

$M \sqcup M' \rightarrow M' \sqcup M$

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$V \xrightarrow{f}$

VECTOR SPACE OF
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IMPORTANT TO KNOW HOW THE
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TOPOLOGICAL QUANTUM FIELD THEORIES

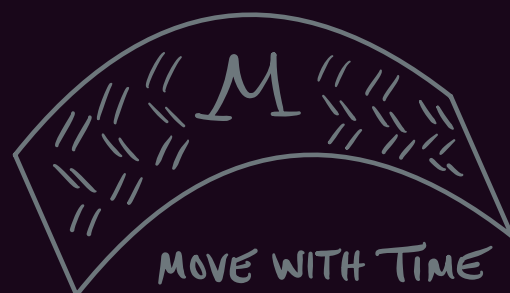
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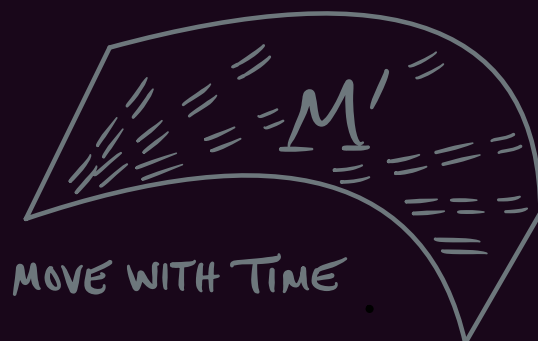
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- DUE TO ATIYAH, SEGAL, WITTEN • 2

FAST FWD TO LATE 1980's ...

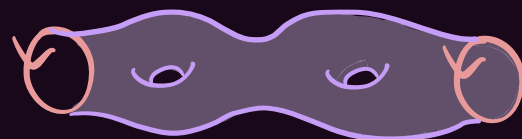
$(2\text{Cob}^{\text{OR}}, \perp, \otimes, \text{SWAP})$

OBJ: ORIENTED CLOSED
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DIFF. CLASSES OF CPT 2-MFLDS



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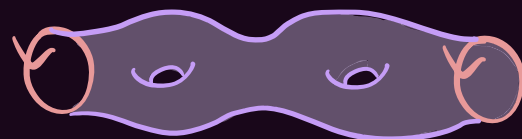
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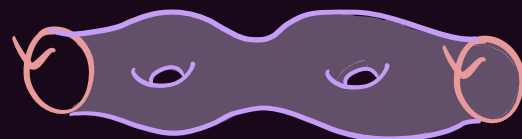
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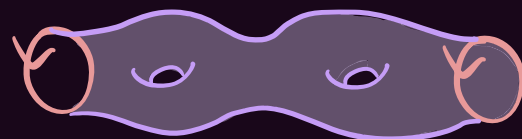
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GET SYMMETRIC
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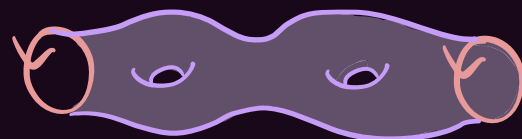
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2-MFLD $\mathbb{C}$ -LIN. MAP

PRESERVING $\otimes, \mathbb{1}, c$

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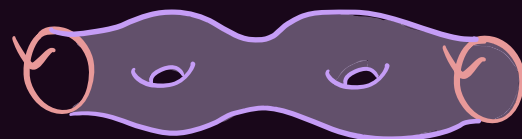
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OMITTING ORIENTATION
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⊗ GEN. OBJECTS

⊗ GEN. MORPHISMS

MORPHISM RELATIONS

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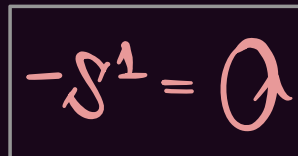
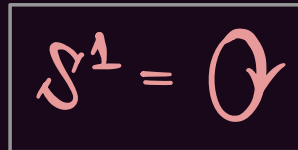
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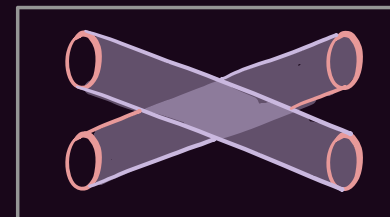
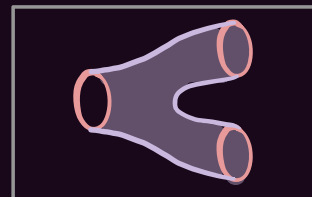
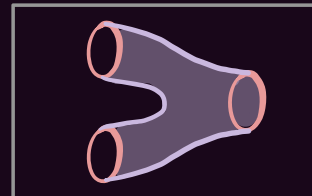
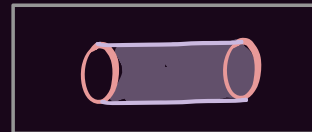
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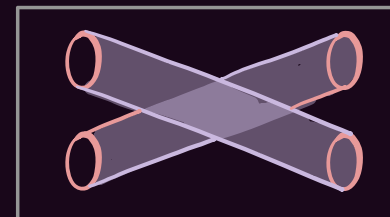
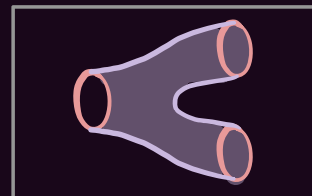
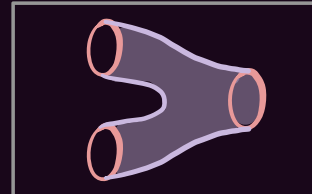
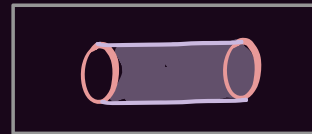
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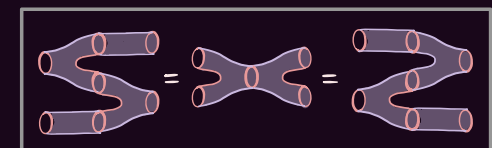
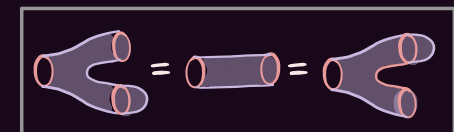
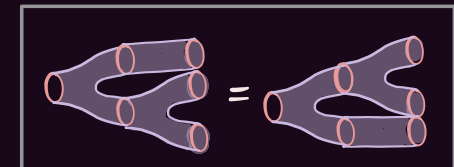
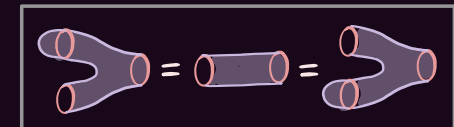
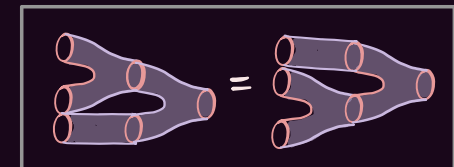
$$S^1 = \bigcirc \curvearrowright$$

$$-S^1 = \bigcirc \curvearrowleft$$

⊗ GEN. MORPHISMS



MORPHISM RELATIONS



TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO MID 1990's...

• DUE TO ABRAMS, QUINN •

$(2\text{Cob}^{\text{OR}}, \sqcup, \emptyset, \text{SWAP})$

2D-TQFT $\downarrow$ $\mathbb{Z}$

$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$

OBJ: $\mathbb{C}$ -VECTOR SPACES

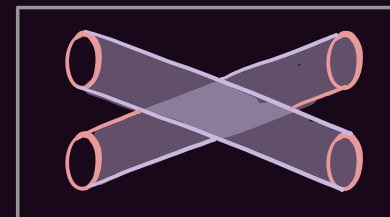
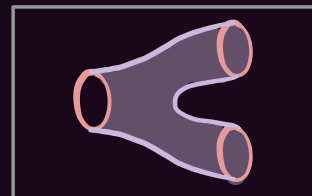
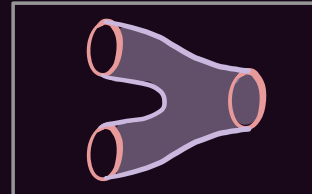
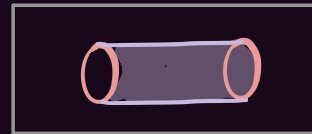
MAPS: $\mathbb{C}$ -LINEAR MAPS

⊗ GEN. OBJECTS

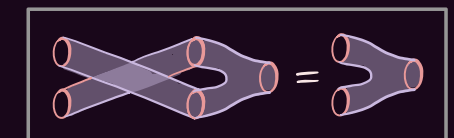
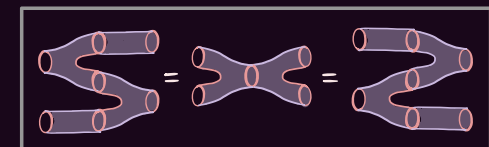
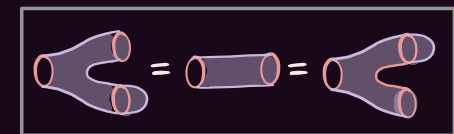
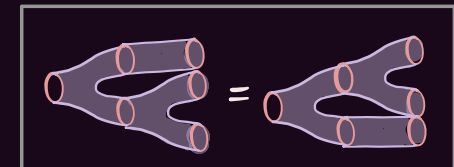
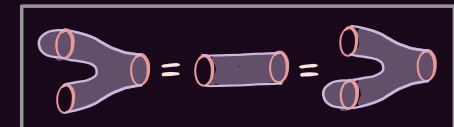
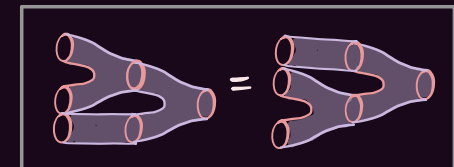
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⊗ GEN. MORPHISMS



MORPHISM RELATIONS



TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO MID 1990's...

• DUE TO ABRAMS, QUINN •

$(2\text{Cob}^{\text{OR}}, \sqcup, \emptyset, \text{SWAP})$

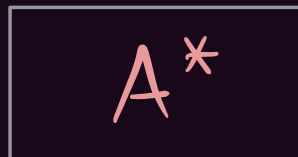
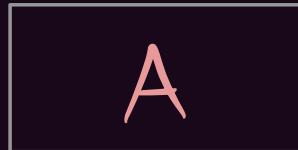
2D-TQFT $\downarrow$ $\mathbb{Z}$

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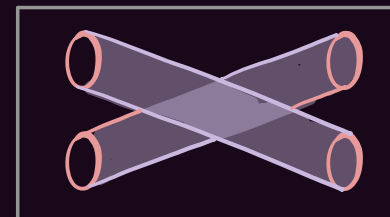
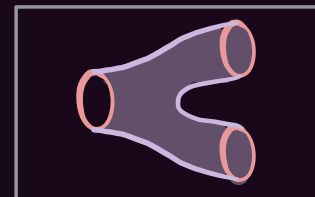
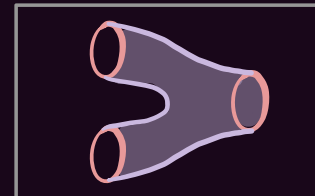
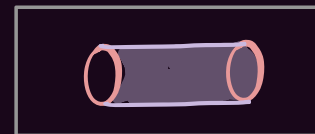
⊗ GEN. OBJECTS



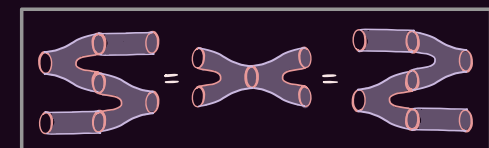
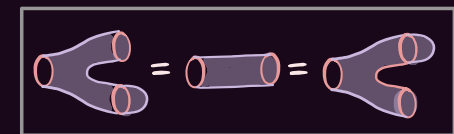
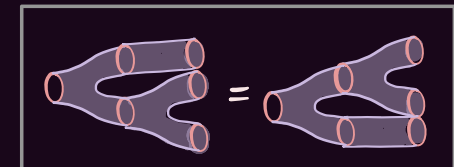
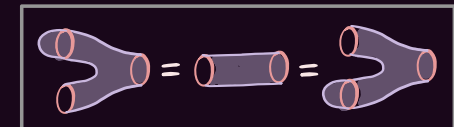
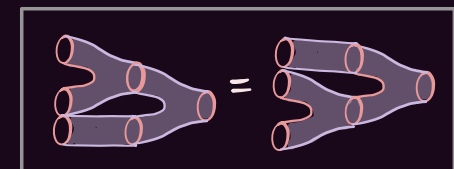
A FIN. DIM.

$\Uparrow$
NONDEGENERACY
IN MORPHISM
RELATIONS
LATER...

⊗ GEN. MORPHISMS



MORPHISM RELATIONS



TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO MID 1990's...

• DUE TO ABRAMS, QUINN •

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2D-TQFT $\downarrow$ $\mathbb{Z}$

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OBJ: $\mathbb{C}$ -VECTOR SPACES

MAPS: $\mathbb{C}$ -LINEAR MAPS

⊗ GEN. OBJECTS

A

A^*

A FIN. DIM.

$\Uparrow$
NONDEGENERACY
IN MORPHISM
RELATIONS
LATER...

⊗ GEN. MORPHISMS

$A \xrightarrow{\text{id}} A$

$A^{\otimes 2} \xrightarrow{\mu} A$

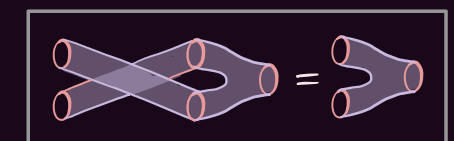
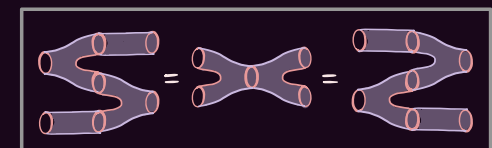
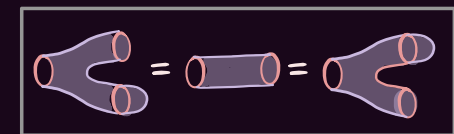
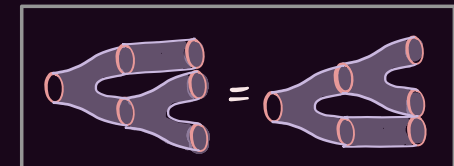
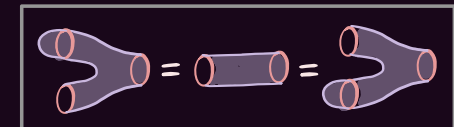
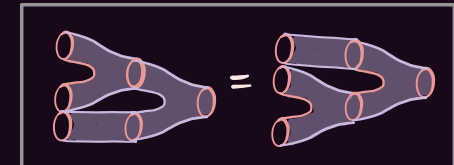
$\mathbb{C} \xrightarrow{\eta} A$

$A \xrightarrow{\Delta} A^{\otimes 2}$

$A \xrightarrow{\epsilon} \mathbb{C}$

$A^{\otimes 2} \xrightarrow{\text{flip}} A^{\otimes 2}$

MORPHISM RELATIONS



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2D-TQFT $\downarrow$ $\mathbb{Z}$

$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$

$(A, m, u) :$

FIN DIM. $\mathbb{C}$ -ALGEBRA

⊗ GEN. OBJECTS

A

A^*

⊗ GEN. MORPHISMS

$A \xrightarrow{\text{id}} A$

$A^{\otimes 2} \xrightarrow{m} A$

$\mathbb{C} \xrightarrow{u} A$

$A \xrightarrow{\Delta} A^{\otimes 2}$

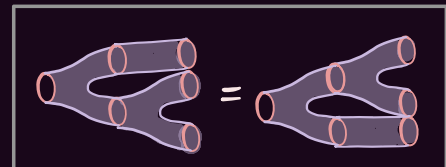
$A \xrightarrow{\varepsilon} \mathbb{C}$

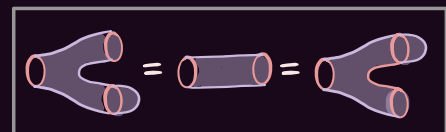
$A^{\otimes 2} \xrightarrow{\text{flip}} A^{\otimes 2}$

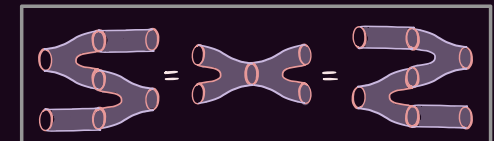
MORPHISM RELATIONS

ASSOCIATIVITY

UNITALITY









TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO MID 1990's...

• DUE TO ABRAMS, QUINN •

$(2\text{Cob}^{\text{OR}}, \sqcup, \emptyset, \text{SWAP})$

2D-TQFT $\downarrow$ $\mathbb{Z}$

$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$

$(A, m, u) :$

FIN DIM. COMMUTATIVE $\mathbb{C}$ -ALGEBRA

⊗ GEN. OBJECTS

A

A^*

⊗ GEN. MORPHISMS

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$A^{\otimes 2} \xrightarrow{m} A$

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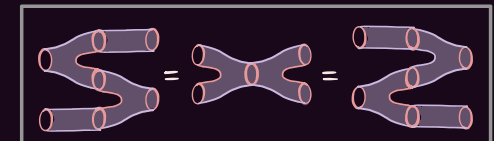
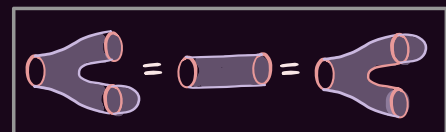
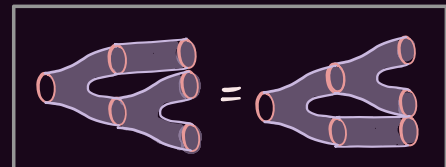
$A \xrightarrow{\varepsilon} \mathbb{C}$

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MORPHISM RELATIONS

ASSOCIATIVITY

UNITALITY



COMMUTATIVITY

TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO MID 1990's...

• DUE TO ABRAMS, QUINN •

$(2\text{Cob}^{\text{OR}}, \sqcup, \emptyset, \text{SWAP})$

$2\text{D-TQFT} \downarrow \mathbb{Z}$

$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$

$(A, m, u, \Delta, \varepsilon) :$

FIN DIM. COMMUTATIVE $\mathbb{C}$ -ALGEBRA
+ $\mathbb{C}$ -COALGEBRA

⊗ GEN. OBJECTS

A

A^*

⊗ GEN. MORPHISMS

$A \xrightarrow{\text{id}} A$

$A^{\otimes 2} \xrightarrow{m} A$

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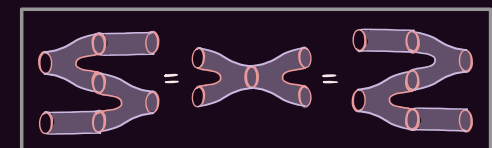
MORPHISM RELATIONS

ASSOCIATIVITY

UNITALITY

COASSOCIATIVITY

COUNITALITY



COMMUTATIVITY

TOPOLOGICAL QUANTUM FIELD THEORIES FAST FWD TO MID 1990's...

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2D-TQFT $\downarrow$ $\mathbb{Z}$

$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$

$(A, m, u, \Delta, \varepsilon) :$

COMMUTATIVE
FROBENIUS ALGEBRA!
OVER $\mathbb{C}$

⊗ GEN. OBJECTS

A

A^*

⊗ GEN. MORPHISMS

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MORPHISM RELATIONS

ASSOCIATIVITY

UNITALITY

COASSOCIATIVITY

COUNITALITY

"FROB. LAW"

COMMUTATIVITY

CATEGORICAL FROBENIUS ALGEBRAS

• DUE TO ABRAMS, QUINN •

FAST FWD TO MID 1990's...

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$(A, m, u, \Delta, \varepsilon) :$

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FROBENIUS ALGEBRA!

OVER $\mathbb{C}$

⊗ GEN. OBJECTS

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⊗ GEN. MORPHISMS

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MORPHISM RELATIONS

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COMMUTATIVITY

CATEGORICAL FROBENIUS ALGEBRAS

• DUE TO ABRAMS, QUINN •

FAST FWD TO MID 1990's...

THEOREM [BNN, 1937-1942]

LET A BE A FIN. DIM. $\mathbb{C}$ -ALGEBRA.

THEN, THE FOLLOWING ARE EQUIVALENT.

1. A IS FROBENIUS $[\exists \lambda$ s.t. P_λ^A IS INV.]

2. $\exists$ ISOMORPHISM OF RIGHT A -MODULES:

$$\phi: A \xrightarrow{\sim} A^*$$

3. $\exists$ $\mathbb{C}$ -LINEAR MAP $\psi: A \rightarrow \mathbb{C}$ s.t. $\ker(\psi)$
DOESN'T CONTAIN $\neq 0$ RIGHT IDEAL OF A

4. $\exists$ NONDEG., ASSOC., $\mathbb{C}$ -LINEAR MAP

$$p: A \otimes_{\mathbb{C}} A \rightarrow \mathbb{C}$$

$(A, m, u, \Delta, \varepsilon) :$

FROBENIUS ALGEBRA!
OVER $\mathbb{C}$

CATEGORICAL FROBENIUS ALGEBRAS

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$$p: A \otimes_{\mathbb{C}} A \rightarrow \mathbb{C} \quad + \quad q: \mathbb{C} \rightarrow A \otimes_{\mathbb{C}} A \dots$$

$(A, m, u, \Delta, \varepsilon) :$

FROBENIUS ALGEBRA!
OVER $\mathbb{C}$

CATEGORICAL FROBENIUS ALGEBRAS

• DUE TO ABRAMS, QUINN •

FAST FWD TO MID 1990's...

By TAKING:

$$p: A \otimes_{\mathbb{C}} A \xrightarrow{\mu} A \xrightarrow{\varepsilon} \mathbb{C}$$

$$q: \mathbb{C} \xrightarrow{\eta} A \xrightarrow{\Delta} A \otimes_{\mathbb{C}} A$$

$$(A, \mu, \eta, \Delta, \varepsilon):$$

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4. $\exists$ NONDEG., ASSOC., $\mathbb{C}$ -LINEAR MAP

$$p: A \otimes_{\mathbb{C}} A \rightarrow \mathbb{C} + q: \mathbb{C} \rightarrow A \otimes_{\mathbb{C}} A \dots$$

CATEGORICAL FROBENIUS ALGEBRAS

• DUE TO ABRAMS, QUINN •

FAST FWD TO MID 1990's...

By TAKING:

$$\Delta: A \cong \mathbb{C} \otimes_{\mathbb{C}} A \xrightarrow{q \otimes \text{id}} A^{\otimes 3} \xrightarrow{\text{id} \otimes m} A^{\otimes 2}$$

$$\varepsilon: A \cong \mathbb{C} \otimes_{\mathbb{C}} A \xrightarrow{u \otimes \text{id}} A^{\otimes 2} \xrightarrow{p} \mathbb{C}$$

$(A, m, u, \Delta, \varepsilon):$ 

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CATEGORICAL FROBENIUS ALGEBRAS

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$(2\text{Cob}^{\text{or}}, \sqcup, \emptyset, \text{SWAP})$



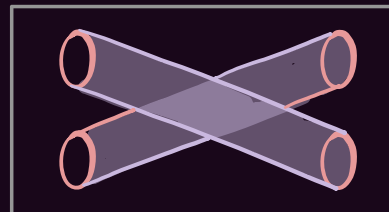
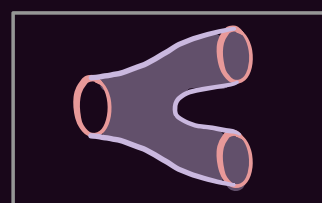
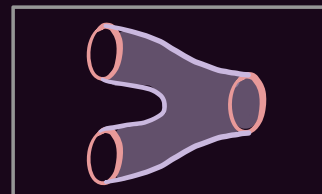
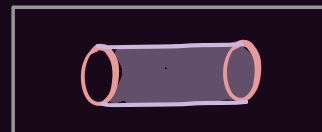
$(\text{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip})$

⊗ GEN. OBJECTS

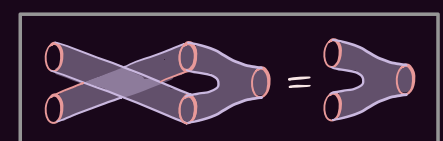
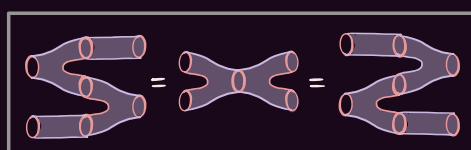
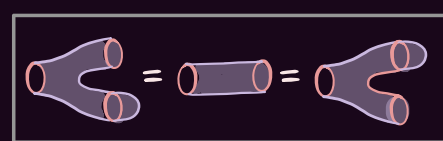
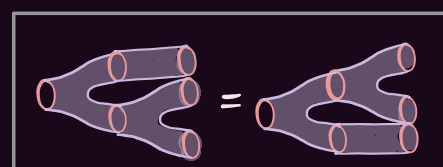
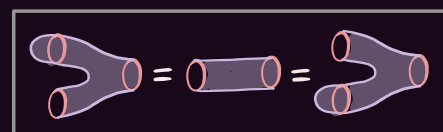
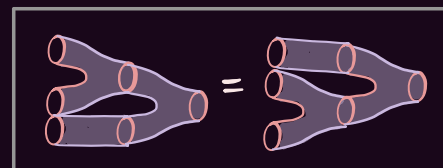
$S^1 = \bigcirc \curvearrowright$

$-S^1 = \bigcirc \curvearrowleft$

⊗ GEN. MORPHISMS



MORPHISM RELATIONS



CATEGORICAL FROBENIUS ALGEBRAS

• DUE TO ABRAMS, QUINN •

FAST FWD TO MID 1990's...

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$(A, m, u, \Delta, \varepsilon) :$

COMMUTATIVE
FROBENIUS ALGEBRA!
OVER $\mathbb{C}$

⊗ GEN. OBJECTS

A

A^*

⊗ GEN. MORPHISMS

$A \xrightarrow{\text{id}} A$

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MORPHISM RELATIONS

ASSOCIATIVITY

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COASSOCIATIVITY

COUNITALITY

"FROB. LAW"

COMMUTATIVITY

CATEGORICAL FROBENIUS ALGEBRAS

↓ • DUE TO ABRAMS, QUINN •
1996 1995

FAST FWD TO MID 1990's...

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{c} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \xrightarrow{\text{2-D TQFT}} \text{Vec}_{\mathbb{C}} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{COMMUTATIVE} \\ \text{FROB. ALGS / } \mathbb{C} \end{array} \right\}$$

$$\mathcal{Z} \longmapsto \mathcal{Z}(S^1)$$

CATEGORICAL FROBENIUS ALGEBRAS

↓ • DUE TO ABRAMS, QUINN •
1996 1995

FAST FWD TO MID 1990's...

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{c} 2\text{Cob}^{\text{OR SYM. MON. FUN.}} \\ \xrightarrow{\quad\quad\quad} \text{Vec}_{\mathbb{C}} \\ \text{2-D TQFT} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{COMMUTATIVE} \\ \text{FROB. ALGS } / \mathbb{C} \end{array} \right\}$$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) =: A_{\text{vs}}$$

↑
EQUIPPED W/ MAPS IN $\text{Vec}_{\mathbb{C}}$

$$\mu: A^{\otimes 2} \rightarrow A \quad \Delta: A \rightarrow A^{\otimes 2}$$

$$\eta: \mathbb{C} \rightarrow A \quad \varepsilon: A \rightarrow \mathbb{C}$$

CATEGORICAL FROBENIUS ALGEBRAS

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1996 1995

FAST FWD TO MID 1990's...

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$$\mathcal{Z} \longmapsto \mathcal{Z}(S^1) =: \mathbf{A}_{\text{vs}}$$

(COM.) FROBENIUS ALGS / $\mathbb{C}$
→ CATEGORICAL MACHINES
THAT CHURN OUT
TOP. INVARIANTS IN $\text{Vec}_{\mathbb{C}}$

EQUIPPED W/ MAPS IN $\text{Vec}_{\mathbb{C}}$

$$\begin{array}{ll} \mu: A^{\otimes 2} \rightarrow A & \Delta: A \rightarrow A^{\otimes 2} \\ \eta: \mathbb{C} \rightarrow A & \varepsilon: A \rightarrow \mathbb{C} \end{array}$$

CATEGORICAL FROBENIUS ALGEBRAS

↓ • DUE TO ABRAMS, QUINN •
1996 1995

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob} \xrightarrow[\text{2-D TQFT}]{\text{OR SYM. MON. FUN.}} \text{Vec}_{\mathbb{C}} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGS } / \mathbb{C} \end{array} \right\}$$

FAST FWD TO MID 1990's...

THERE'S NOTHING
SPECIAL ABOUT $\text{Vec}_{\mathbb{C}}$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) =: A \text{ vs}$$

(COM.) FROBENIUS ALGS $/ \mathbb{C}$
→ CATEGORICAL MACHINES
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↓ • DUE TO ABRAMS, QUINN •
1996 1995

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FAST FWD TO MID 1990's...

WORKS FOR ANY
SYM. MONOIDAL CATEG.
 $(\mathbb{C}, \otimes, \mathbb{1}, c)$

$$\mathcal{Z} \longmapsto \mathcal{Z}(S^1) =: A \text{ vs}$$

(COM.) FROBENIUS ALGS $/ \mathbb{C}$
→ CATEGORICAL MACHINES
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TOP. INVARIANTS IN $\text{Vec}_{\mathbb{C}}$

EQUIPPED W/ MAPS IN $\text{Vec}_{\mathbb{C}}$

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CATEGORICAL FROBENIUS ALGEBRAS

↓ • DUE TO ABRAMS, QUINN •
1996 1995

THEOREM: $\exists$ BIJECTION:

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FAST FWD TO MID 1990's...

WORKS FOR ANY
SYM. MONOIDAL CATEG.
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) =: A \in \mathcal{C}$$

(COM.) FROBENIUS ALGS IN $\mathcal{C}$
→ CATEGORICAL MACHINES
THAT CHURN OUT
TOP. INVARIANTS IN $\mathcal{C}$

EQUIPPED W/ MAPS IN $\mathcal{C}$

$$\begin{array}{ll} \mu: A^{\otimes 2} \rightarrow A & \Delta: A \rightarrow A^{\otimes 2} \\ \eta: \mathbb{1} \rightarrow A & \varepsilon: A \rightarrow \mathbb{1} \end{array}$$

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1996 1995

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGS IN } \mathcal{C} \end{array} \right\}$$

FAST FWD TO MID 1990's...

WORKS FOR ANY
SYM. MONOIDAL CATEG.
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$

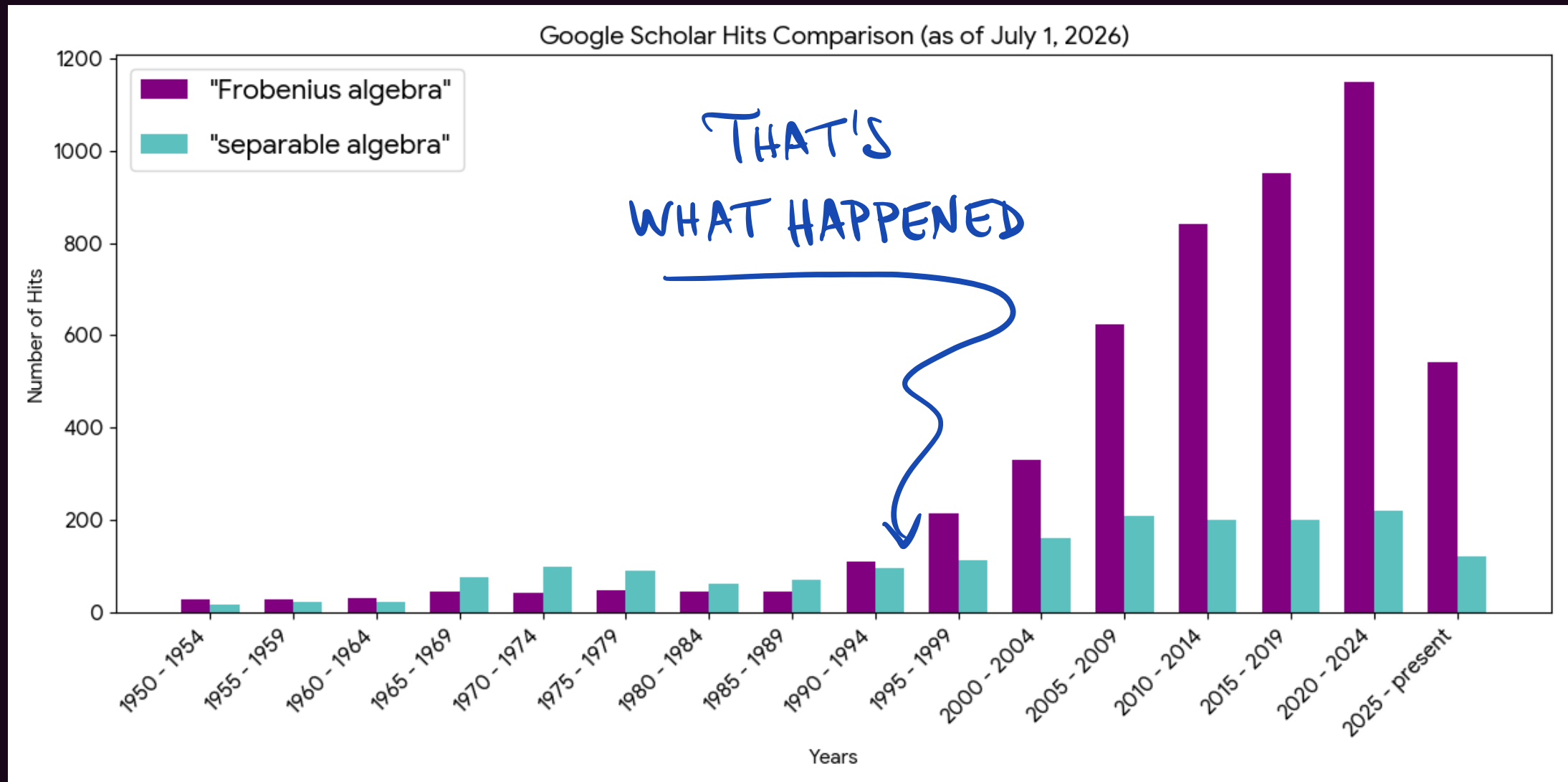
$$\mathcal{I} \longmapsto \mathcal{I}(S^1) =: A \in \mathcal{C}$$

(COM.) FROBENIUS ALGS IN $\mathcal{C}$
→ CATEGORICAL MACHINES
THAT CHURN OUT
TOP. INVARIANTS IN $\mathcal{C}$

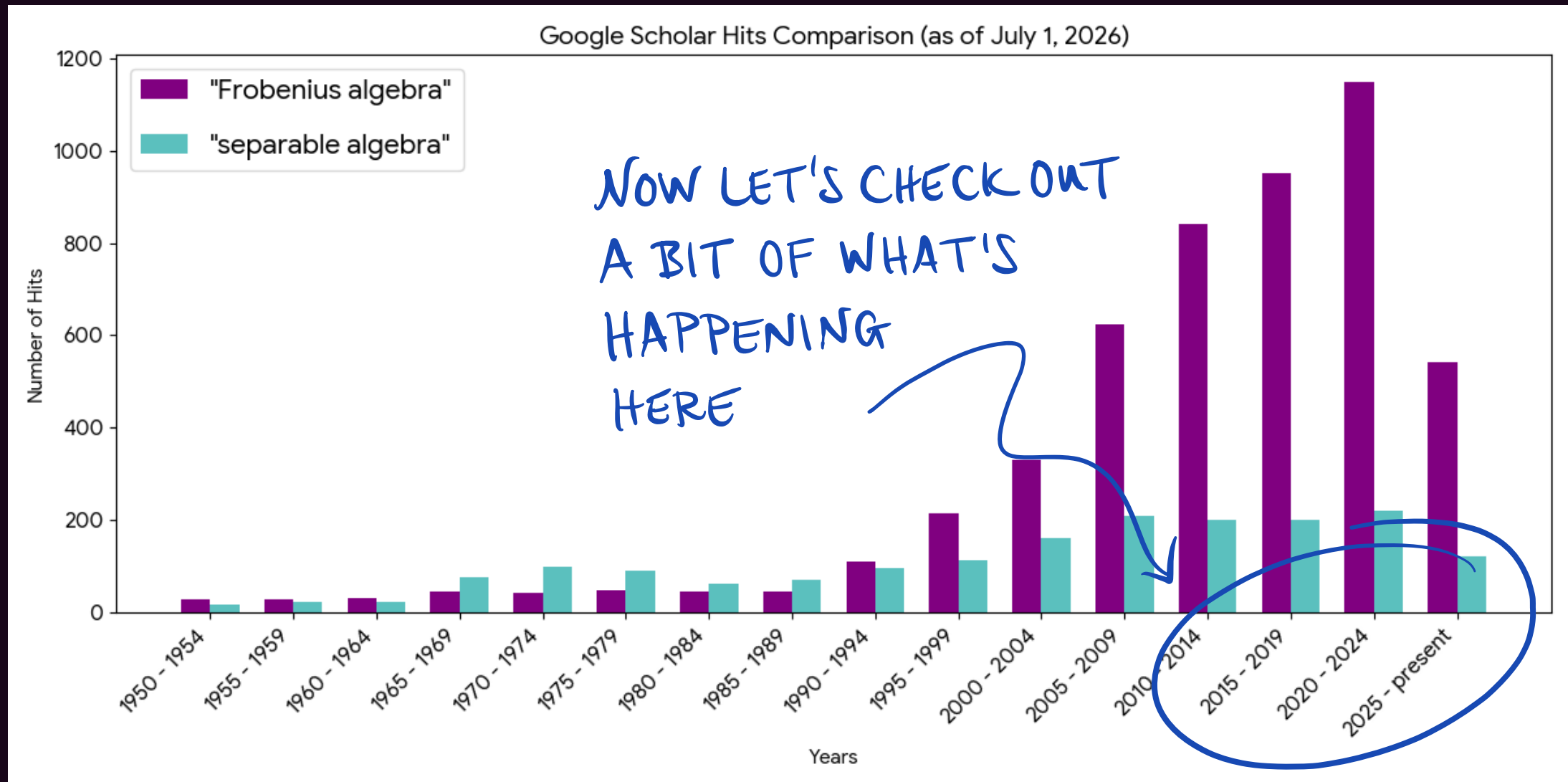
EQUIPPED W/ MAPS IN $\mathcal{C}$

$$\begin{array}{ll} \mu: A^{\otimes 2} \rightarrow A & \Delta: A \rightarrow A^{\otimes 2} \\ \eta: \mathbb{1} \rightarrow A & \varepsilon: A \rightarrow \mathbb{1} \end{array}$$

≡ POPULARITY THROUGHOUT THE YEARS ≡



≡ POPULARITY THROUGHOUT THE YEARS ≡



FROBENIUS ALGEBRAS THESE DAYS

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \xleftrightarrow{1-1} \mathcal{C} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGAS IN } \mathcal{C} \end{array} \right\}$$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) = (A, m, u, \Delta, \varepsilon)$$

FROBENIUS ALGEBRAS THESE DAYS

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \xleftrightarrow{\quad} \mathcal{C} \xleftrightarrow{\quad} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGAS IN } \mathcal{C} \end{array} \right\}$$

$$\mathcal{Z} \longmapsto \mathcal{Z}(S^1) = (A, m, u, \Delta, \varepsilon)$$

THERE'S A SUPER
ACTIVE PROGRAM
IN STUDYING TQFTs...

FROBENIUS ALGEBRAS THESE DAYS

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \mathcal{C} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGAS IN } \mathcal{C} \end{array} \right\}$$

$$\mathcal{Z} \longmapsto \mathcal{Z}(S^1) = (A, m, u, \Delta, \varepsilon)$$

COULD INCREASE
TOPOLOGICAL
DIMENSION

FROBENIUS ALGEBRAS THESE DAYS

HAVE TO UP THE
CATEGORICAL LEVEL HERE

THEOREM: $\exists$ BIJECTION:

3-D TQFTs $\Leftrightarrow$ MTCs

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \xrightarrow{\quad} \mathcal{C} \xleftrightarrow{\quad} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGS IN } \mathcal{C} \end{array} \right\}$$

$$\mathbb{Z} \longmapsto \mathbb{Z}(S^1) = (A, m, u, \Delta, \varepsilon)$$

COULD INCREASE
TOPOLOGICAL
DIMENSION

FROBENIUS ALGEBRAS THESE DAYS.

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \xrightarrow{\quad} \mathcal{C} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGAS IN } \mathcal{C} \end{array} \right\}$$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) = (A, m, u, \Delta, \varepsilon)$$

COULD ALTER THE
TOPOLOGICAL
FLAVOR

FROBENIUS ALGEBRAS THESE DAYS

HAVE TO ALTER THE
FLAVOR OF FROB. ALG. HERE

THEOREM: $\exists$ BIJECTION:

$$\left\{ \begin{array}{l} \text{2Cob}^{\text{OR SYM. MON. FUN.}} \\ \text{2-D TQFT} \end{array} \right\} \xrightarrow{\quad} \mathcal{C} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{FROB. ALGS IN } \mathcal{C} \end{array} \right\}$$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) = (A, m, u, \Delta, \varepsilon)$$

COULD ALTER THE
TOPOLOGICAL
FLAVOR

FROBENIUS ALGEBRAS THESE DAYS

↓ • DUE TO TURAEV-TURNER, 2006 •

THEOREM: $\exists$ BIJECTION:

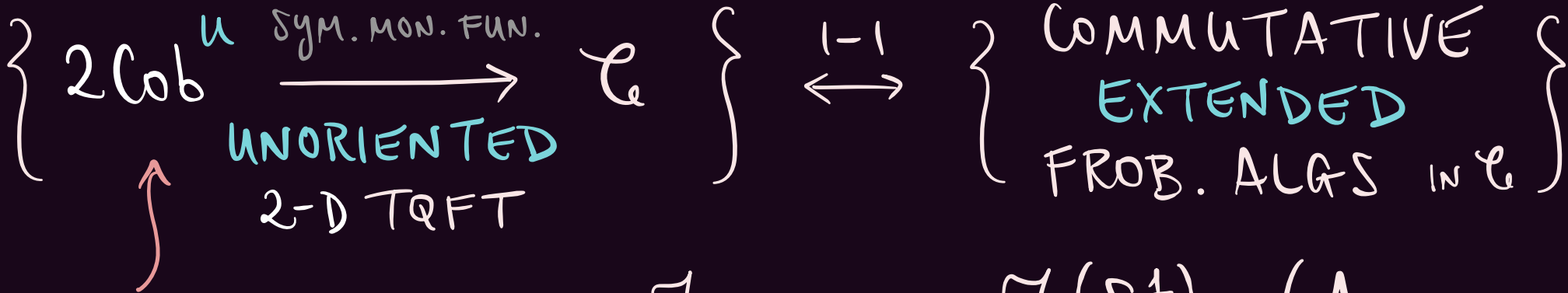
$$\left\{ \begin{array}{l} 2\text{Cob} \\ \vdots \\ \text{UNORIENTED} \\ \text{2-D TQFT} \end{array} \right\} \xrightarrow[\text{UNORIENTED}]{\text{u sym. mon. fun.}} \mathcal{C} \left\{ \begin{array}{l} \text{COMMUTATIVE} \\ \text{EXTENDED} \\ \text{FROB. ALGAS in } \mathcal{C} \end{array} \right\}$$

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) = (A, m, u, \Delta, \varepsilon) \\ \vdots$$

FROBENIUS ALGEBRAS THESE DAYS

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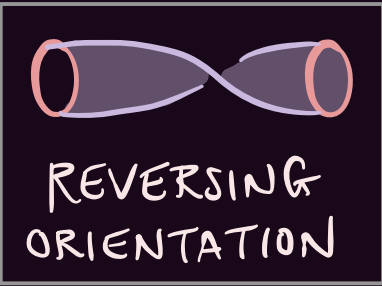
THEOREM: $\exists$ BIJECTION:



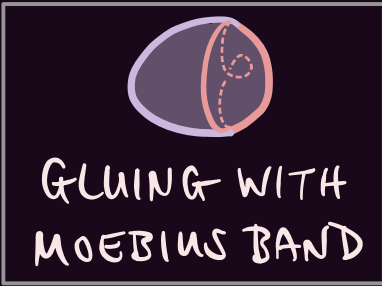
ADD'L $\otimes$ GEN. MORPHISMS

$$\mathcal{I} \longmapsto \mathcal{I}(S^1) = (A, m, u, \Delta, \varepsilon)$$

⋮



REVERSING ORIENTATION

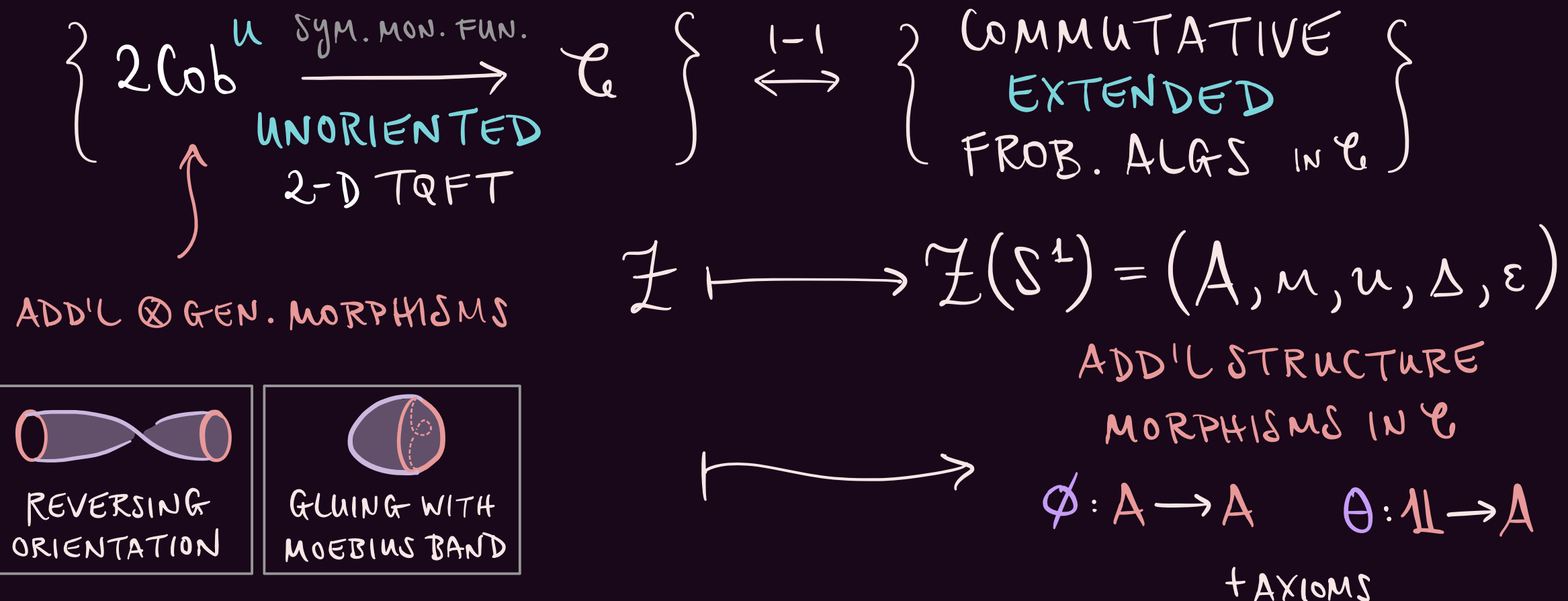


GLUING WITH MOEBIUS BAND

FROBENIUS ALGEBRAS THESE DAYS

↓ • DUE TO TURAEV-TURNER, 2006 •

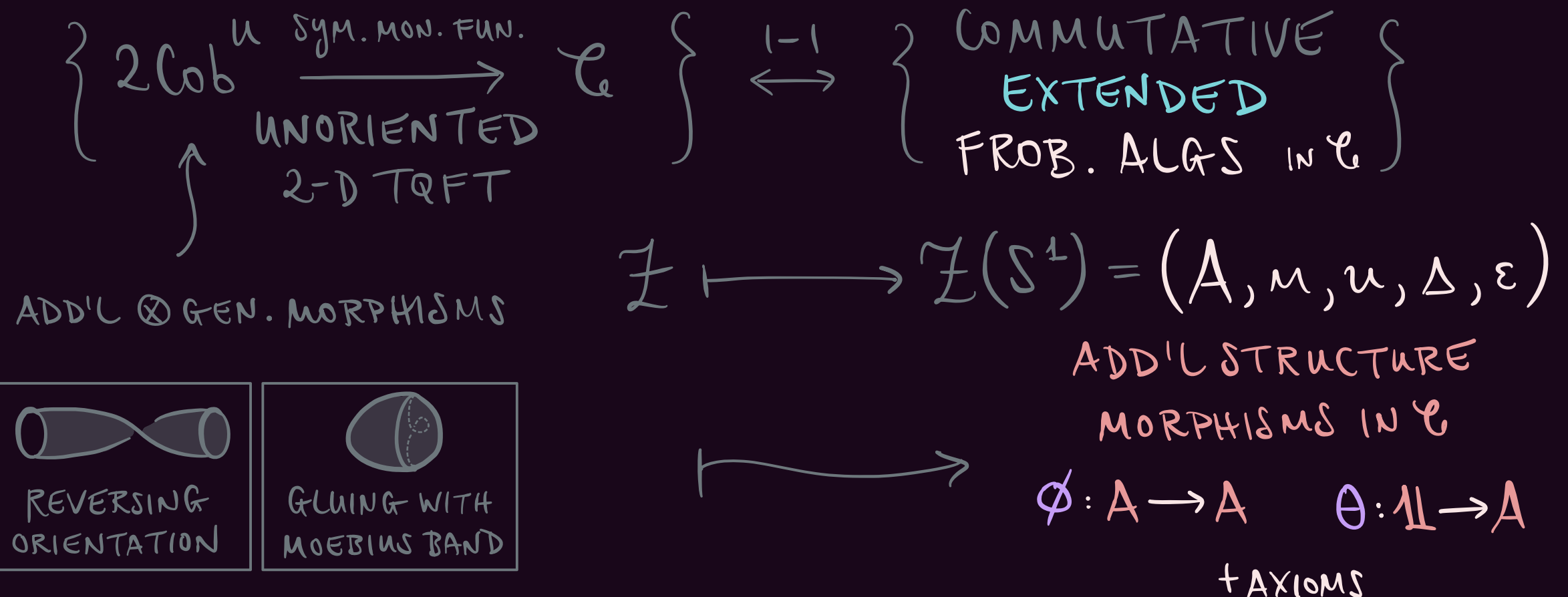
THEOREM: $\exists$ BIJECTION:



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FROBENIUS ALGEBRAS THESE DAYS

Theory and Applications of Categories, Vol. 44, No. 36, 2025, pp. 1218–1252.

ON EXTENDED FROBENIUS STRUCTURES

A. CZENKY, J. KESTEN, A. QUINONEZ, AND C. WALTON

SYSTEMATIC
STUDY OF
EXTENDED
FROB. ALGS IN $\mathcal{C}$

$(A, \mu, \nu, \Delta, \varepsilon)$

+

ADD'L STRUCTURE
MORPHISMS IN $\mathcal{C}$

$\phi: A \rightarrow A$

$\theta: 1 \rightarrow A$

+ AXIOMS

FROBENIUS ALGEBRAS THESE DAYS

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- CLASSIFIES (ϕ, θ) FOR MANY FROB ALGS / $\mathcal{C}$
E.G. $\mathbb{C}, \mathbb{C}C_2, \mathbb{C}C_3, \mathbb{C}C_4, \mathbb{C}[x]/(x^n)$

SYSTEMATIC
STUDY OF
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$$(A, \mu, \nu, \Delta, \varepsilon) \\ +$$

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E.G. $\mathbb{C}, \mathbb{C}C_2, \mathbb{C}C_3, \mathbb{C}C_4, \mathbb{C}[x]/(x^n)$
... BUT YOU COULD DO MORE!

SYSTEMATIC
STUDY OF
EXTENDED
FROB. ALGS IN $\mathcal{C}$

$(A, \mu, \nu, \Delta, \varepsilon)$
+

ADD'L STRUCTURE
MORPHISMS IN $\mathcal{C}$

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... BUT YOU COULD DO MORE!
- EXAMINES RELATED HOPF
& FUNCTORIAL CONSTRUCTIONS

SYSTEMATIC
STUDY OF
EXTENDED
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$(A, \mu, \nu, \Delta, \varepsilon)$
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MORPHISMS IN $\mathcal{C}$

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+ AXIOMS

FROBENIUS ALGEBRAS THESE DAYS

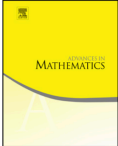
Advances in Mathematics 307 (2017) 598–639



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Consistent systems of correlators in non-semisimple conformal field theory

Jürgen Fuchs^a, Christoph Schweigert^{b,*}

^a Teoretisk Fysik, Karlstads Universitet, Universitetsgatan 21, S-65188 Karlstad, Sweden
^b Fachbereich Mathematik, Universität Hamburg, Bereich Algebra und Zahlentheorie, Bundesstraße 55, D-20146 Hamburg, Germany



TIES TO
CONFORMAL
FIELD
THEORY
IN
MATH
PHYSICS



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Quantum games and synchronicity

Adina Goldberg

University of Waterloo
Institute for Quantum Computing

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Citation: Quantum 10, 1964 (2026).

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"Q-SYSTEMS"
|||
UNITARY
FROB. ALGS
IN
FUNCTIONAL
ANALYSIS

Journal of Functional Analysis 283 (2022) 109524



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Full Length Article

Q-system completion for C^* 2-categories

Quan Chen, Roberto Hernández Palomares*, Corey Jones, David Penneys



Ms. Adina Goldberg | Synchronous quantum games

YouTube : INI Seminar Room 1 : Dec 18, 2024



COOL TALK: FROB. ALGS HIDING @ 11:20

• CHELSEA WALTON (RICE UNIV) •

• A CENTURY OF FROBENIUS ALGEBRAS •

• $(CM - (WM)^2)$, JULY 2026 •

FROBENIUS ALGEBRAS THESE DAYS

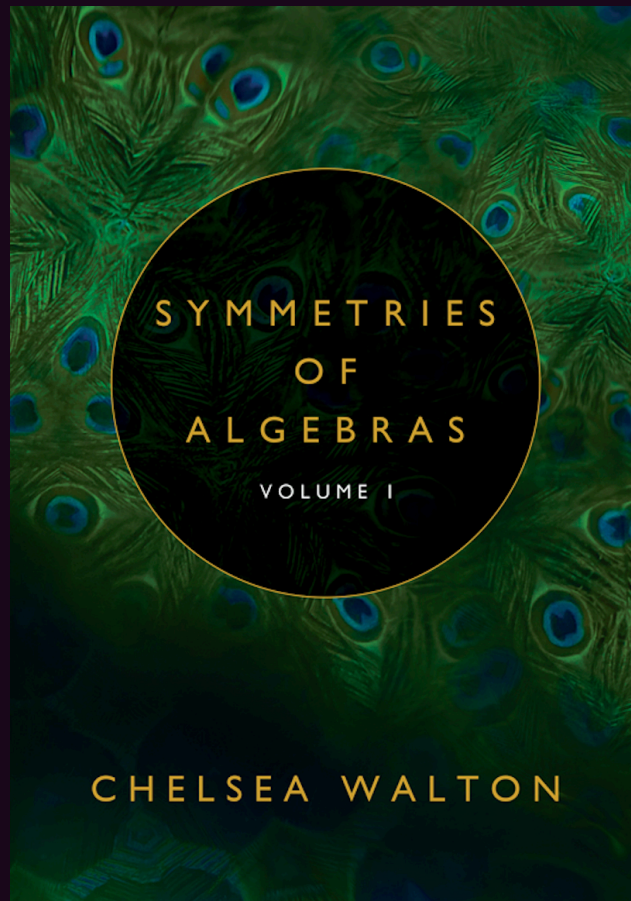
≡ TOPICS ≡

ALGEBRAS
OVER A FIELD

CATEGORIES

MONOIDAL
CATEGORIES

ALGEBRAS IN
MON. CATS.

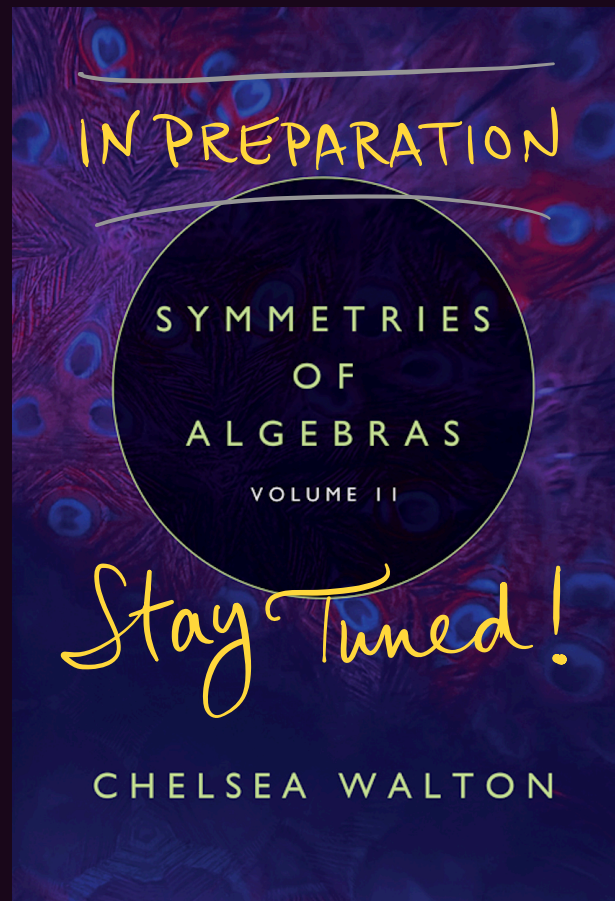


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ARE GREATLY APPRECIATED ☺

• CHELSEA WALTON (RICE UNIV) •

• A CENTURY OF FROBENIUS ALGEBRAS •



≡ TOPICS ≡

COALGEBRAS
OVER A FIELD

COALGEBRAS IN
MON. CATS.

FROBENIUS
ALGEBRAS
OVER A FIELD

FROBENIUS
ALGEBRAS IN
MON. CATS.

• ICM - (WM)², July 2024 •

Thanks for listening!!

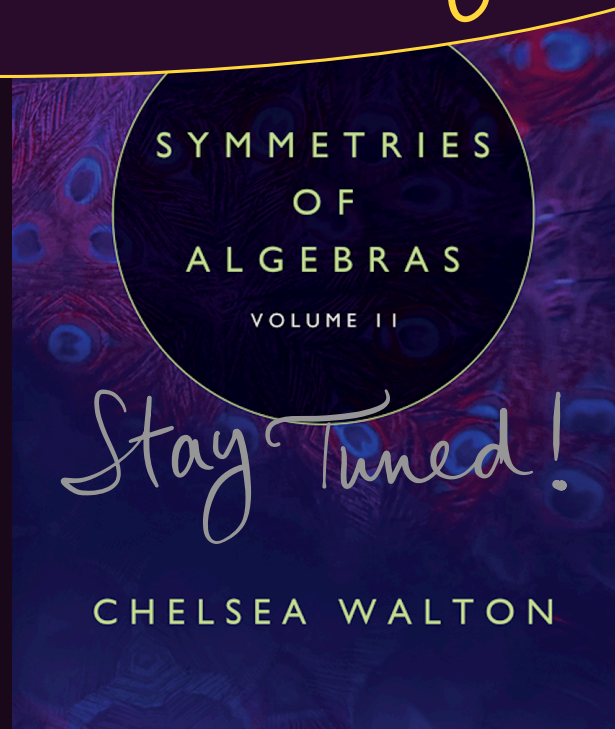
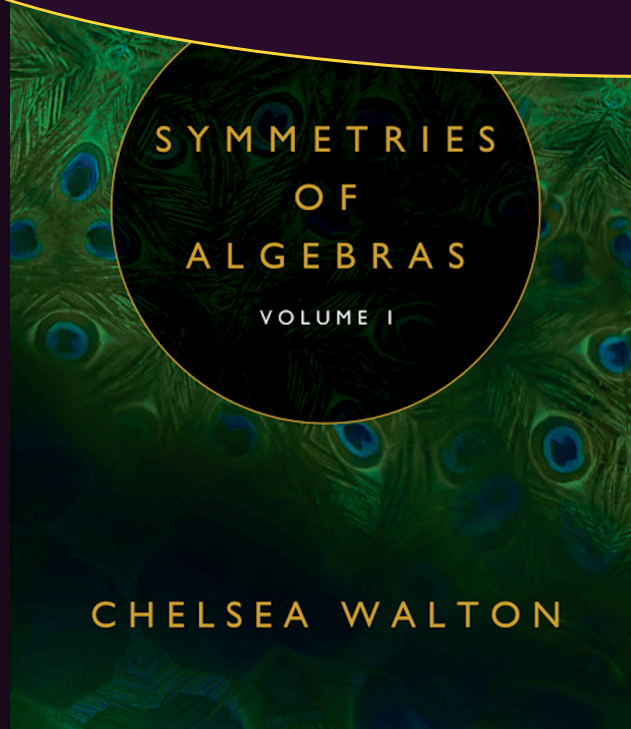
≡ TOPICS ≡

ALGEBRAS
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CATEGORIES

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ALGEBRAS IN
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≡ TOPICS ≡

COALGEBRAS
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OUT NOW!

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