# QUANTUM SYMMETRY

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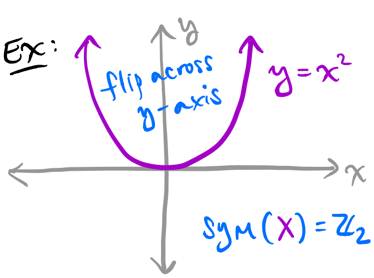
WHAT IS SYMMETRY?

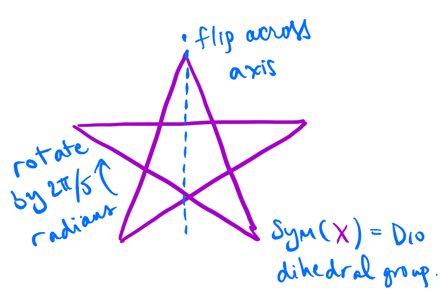
↪ ONE OF THE OLDEST NOTIONS IN MATHEMATICS.

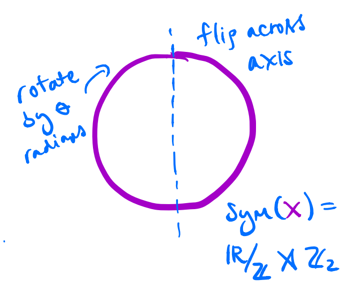
DEF'N: GIVEN AN OBJECT $X$,
A SYMMETRY OF $X$ IS
AN (INVERTIBLE) STRUCTURE
-PRESERVING TRANSFORMATION
FROM $X$ TO $X$.

THE COLLECTION
FORMS A MONOID
(OR A GROUP)
$\mathrm{Sym}(X)$
UNDER COMPOSITION

EX:



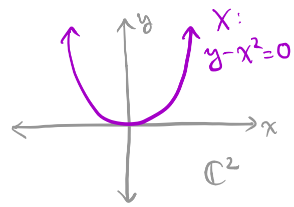



SYMMETRIES OF GEOMETRIC OBJECTS $X$ $\longleftrightarrow$ SYMMETRIES OF FUNCTION ALGEBRAS $\mathcal{O}(X)$

BEST ILLUSTRATED FOR AFFINE VARIETIES $X$

SHAPES CUT OUT BY SETTING POLYNOMIALS EQUAL TO 0




$\mathrm{Sym}(X) = \mathbb{Z}_2$
$= \langle g := \text{flip across y-axis} \rangle$

$\longleftrightarrow$

$\mathcal{O}(X) = \dfrac{\mathbb{C}[x,y]}{(y-x^2)}$

$\mathbb{Z}_2 \curvearrowright \mathcal{O}(X)$

$g \longmapsto \begin{bmatrix} x \mapsto -x \\ y \mapsto y \end{bmatrix}$

ACTIONS ON AFFINE VARIETIES $X$ $\longleftrightarrow$ ACTIONS ON FUNCTION/COORDINATE ALGS. $\mathcal{O}(X)$

LET'S PACKAGE THIS IN THE LANGUAGE OF CATEGORY THEORY

FRAMEWORK FOR RELATING ONE COLLECTION OF OBJECTS & MAPS BETWEEN OBJECTS WITH ANOTHER COLLECTION OF OBJECTS & MAPS BETWEEN OBJECTS

SHIFTING THE STUDY OF SYMMETRY FROM A GEOMETRIC POINT OF VIEW TO AN ALGEBRAIC POINT OF VIEW

$\mathrm{AFF}/\mathbb{C}$ CATEGORY OF AFFINE VARIETIES/$\mathbb{C}$ $\xrightarrow[\text{(CONTRA-VARIANT)}]{X \mapsto \mathcal{O}(X)}$ $\mathrm{COMALG}/\mathbb{C}$ CATEGORY OF COMMUTATIVE ALGS./$\mathbb{C}$

$G \curvearrowright X : \quad G \times X \longrightarrow X$ $\longrightarrow$ $\mathcal{O}(X) \longrightarrow \underbrace{\mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{O}(X)}_{\simeq\, \mathcal{O}(G \times X)}$

GROUP ACTION

"COACTION"

$G$ IS AN OBJECT OF AFF/$\mathbb{C}$

& $G$ IS A GROUP (ALG. GROUP)

$\exists$ MAPS IN AFF/$\mathbb{C}$

$m : G \times G \longrightarrow G$ (MULTIP.)

$e : \{\cdot\} \longrightarrow G$ (IDENTITY/UNIT)
ONE PT VAR.

$\iota : G \longrightarrow G$ (INVERSION)

SATISFYING GROUP AXIOMS

$\longrightarrow$

$\mathcal{O}(G)$ IS AN OBJECT OF COMALG/$\mathbb{C}$

& $\mathcal{O}(G)$ HAS ADDITIONAL STRUCTURE

$\exists$ MAPS IN COMALG/$\mathbb{C}$

$\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ (COMULTIP.)

$\varepsilon : \mathcal{O}(G) \longrightarrow \mathbb{C} = \mathcal{O}(\{\cdot\})$ (COUNIT)

$S : \mathcal{O}(G) \longrightarrow \mathcal{O}(G)$ (ANTIPODE)

SATISFYING HOPF ALGEBRA AXIOMS

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# CLASSICAL SYMMETRY

## GEOMETRICALLY

SYMMETRIES ARE CAPTURED BY GROUP ACTIONS

## ALGEBRAICALLY

SYMMETRIES ARE CAPTURED BY COACTIONS OF COMMUTATIVE HOPF ALGEBRAS

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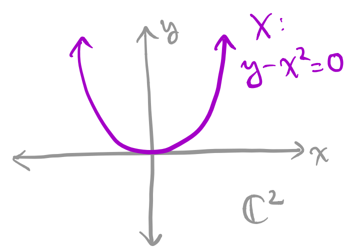
EX.




$\mathrm{Sym}(X) = \mathbb{Z}_2$

$= \langle g := \text{flip across y-axis} \rangle$

WHICH IS $g^2 - e = 0$ AS A VARIETY

$\mathcal{O}(X) = \dfrac{\mathbb{C}[x,y]}{(y-x^2)}$ & $\mathcal{O}(\mathbb{Z}_2) = \dfrac{\mathbb{C}[g]}{(g^2-e)}$

($e$ = unit of $\mathcal{O}(\mathbb{Z}_2)$)

$$\begin{aligned} \mathcal{O}(X) &\longrightarrow \mathcal{O}(\mathbb{Z}_2) \otimes \mathcal{O}(X) \\ x &\longmapsto -g \otimes x \\ y &\longmapsto e \otimes y \end{aligned}$$

REST OF HOPF ALG. STRUCTURE OF $\mathcal{O}(\mathbb{Z}_2)$

$\Delta(g) = g \otimes g \quad \varepsilon(g) = 1_{\mathbb{C}} \quad S(g) = g$

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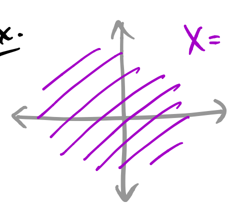
EX.



$X = \mathbb{C}^2$

TAKE SYMMETRIES
- ORIGIN-PRESERVING
- LINEAR
- INVERTIBLE

$$\underbrace{\mathcal{O}(\mathbb{C}^2)}_{= \mathbb{C}[x,y]} \longrightarrow \underbrace{\mathcal{O}(GL_2(\mathbb{C}))}_{= \mathbb{C}[a,b,c,d,t]/(\det \cdot t - 1)} \otimes \mathcal{O}(\mathbb{C}^2)$$

$$\begin{aligned} x &\longmapsto a \otimes x + b \otimes y \\ y &\longmapsto c \otimes x + d \otimes y \end{aligned}$$

$= GL_2(\mathbb{C})$

WHICH IS THE VARIETY

$\underbrace{(ad-bc)}_{=:\det}t - 1 = 0$ in $\mathbb{C}^5_{a,b,c,d,t}$

FOR $a =: a_{11}$ $b =: a_{12}$ $c =: a_{21}$ $d =: a_{22}$

$\Delta(a_{ij}) = \sum_{k=1}^{2} a_{ik} \otimes a_{kj}$, $\varepsilon(a_{ij}) = \delta_{ij}$, $S$ involves det

## CLASSICAL SYMMETRY

**GEOMETRICALLY**

INVERTIBLE SYMMETRIES ARE CAPTURED BY GROUP ACTIONS

**ALGEBRAICALLY**

INVERTIBLE SYMMETRIES ARE CAPTURED BY COACTIONS OF COMMUTATIVE HOPF ALGEBRAS

## WHAT IS QUANTUM SYMMETRY?

↳ THIS IS LESS WELL-UNDERSTOOD & IS AN ACTIVE AREA OF RESEARCH

## PHILOSOPHY

THE MORE SOPHISTICATED THE OBJECT

THE MORE SOPHISTICATED THE FRAMEWORK NEEDED TO ANALYZE ITS SYMMETRIES

TO ILLUSTRATE THIS, LET'S CONSIDER

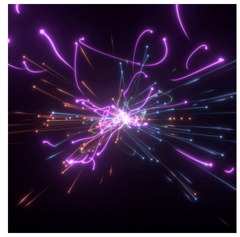
SYMMETRIES OF QUANTUM OBJECTS

EX.

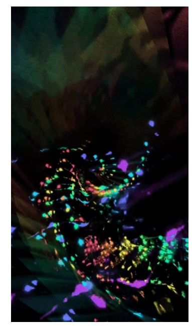
$\mathbb{C}^2$ COMPLEX PLANE

$\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$

$\mathbb{C}^2_q$ QUANTUM PLANE ....



WHICH IS IMPOSSIBLE TO VISUALIZE ....



THROUGH THE LENS OF QUANTUM PHYSICS,

WE CANNOT "OBSERVE" QUANTUM OBJECTS

YET WE CAN STILL MODEL THEIR BEHAVIOR

ALGEBRAICALLY (VIA COORDINATE/FUNCTION ALGEBRAS)

$\mathbb{C}^2$ COMPLEX PLANE

$\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$

$\mathbb{C}^2_q$ QUANTUM PLANE

$\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y] = \dfrac{\mathbb{C}\langle x,y\rangle}{(yx-xy)}$ POLYNOMIAL ALGEBRA

$\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$ ← SAME $\mathbb{C}$-VS, ALTERED MULTIP.

$\mathcal{O}_q(\mathbb{C}^2) = \dfrac{\mathbb{C}\langle x,y\rangle}{(yx-qxy)}$ q-POLYNOMIAL ALGEBRA

ONE WIDELY ACCEPT MODEL FOR UNDERSTANDING QUANTIZED 2-SPACE

BACK TO SYMMETRY – IN A NICE FRAMEWORK:

**SYMMETRY SHOULD COMMUTE WITH DEFORMATION**

TRY THE FRAMEWORK OF GROUP ACTIONS

*LINEAR, INVERTIBLE, ORIGIN-PRESERVING

! JUMP IN SYMMETRIES, EVEN THOUGH q VARIES SMOOTHLY

$\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y]$ POLYNOMIAL ALGEBRA

$\big\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$

$\mathcal{O}_q(\mathbb{C}^2) = \dfrac{\mathbb{C}\langle x,y\rangle}{(yx - qxy)}$ q-POLYNOMIAL ALGEBRA

$\mathrm{Sym}^*(\mathcal{O}(\mathbb{C}^2)) = GL_2(\mathbb{C})$

$\big\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$

$$\mathrm{Sym}^*(\mathcal{O}(\mathbb{C}_q^2)) = \begin{cases} GL_2(\mathbb{C}) & q = 1 \\ \text{ANTI/DIAG. MATRICES} & q = -1 \\ \text{DIAG. MATRICES} & q \neq \pm 1 \end{cases}$$

TRY THE FRAMEWORK OF HOPF ALGEBRA CO/ACTIONS
(EXPANSION OF GROUP ACTION FRAMEWORK)

$\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y]$ POLYNOMIAL ALGEBRA

COMMUTATIVE HOPF ALG

$$\underbrace{\mathcal{O}(\mathbb{C}^2)}_{=\mathbb{C}[x,y]} \longrightarrow \underbrace{\mathcal{O}(GL_2(\mathbb{C}))}_{=\mathbb{C}[a,b,c,d,t]/(\det\cdot t - 1)} \otimes \mathcal{O}(\mathbb{C}^2)$$

$$x \longmapsto a \otimes x + b \otimes y$$

$$y \longmapsto c \otimes x + d \otimes y$$

WITH $\Delta, \varepsilon, S$ AS STATED ABOVE

$\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$

✓ GET SMOOTH TRANSITION OF SYMMETRIES

$\downarrow$ q-DEFORMATION $q \in \mathbb{C}^\times$

$$\mathcal{O}_q(\mathbb{C}^2) = \frac{\mathbb{C}\langle x, y\rangle}{(yx - qxy)}$$ q-POLYNOMIAL ALGEBRA

NONCOMMUTATIVE HOPF ALG

$$\underset{=\,\mathbb{C}_q[x,y]}{\mathcal{O}(\mathbb{C}^2_q)} \longrightarrow \underset{=\,\mathbb{C}_q[a,b,c,d,t]/(\det_q \cdot t - 1)}{\mathcal{O}_q(GL_2(\mathbb{C}))} \otimes \mathcal{O}(\mathbb{C}^2_q)$$

$$x \longmapsto a \otimes x + b \otimes y$$

$$y \longmapsto c \otimes x + d \otimes y$$

WITH $\Delta, \varepsilon, S_q$

## TAKEAWAY POINT:

JUST AS GROUP ACTIONS PROVIDE A NICE FRAMEWORK FOR **CLASSIC SYMMETRY**

(E.G., OF FUNCTION ALGS OF OBSERVABLE OBJECTS)

CO/ACTIONS OF HOPF ALGEBRAS $\longrightarrow$ "QUANTUM GROUPS"

PROVIDE A NICE FRAMEWORK FOR **QUANTUM SYMMETRY**

(E.G., OF FUNCTION ALGS OF QUANTIZED OBJECTS)

NICE REF: DRINFELD'S 1986 ICM MANUSCRIPT

## ON HOPF ALGEBRAS / $\mathbb{C}$

DEF'N: A HOPF ALGEBRA / $\mathbb{C}$ IS A 6-TUPLE:

$$\Big(\underset{\substack{\uparrow\\ \mathbb{C}\text{-VS}}}{H},\ \underbrace{m: H\otimes H \to H,\ u: \mathbb{C}\to H,\ \Delta: H\to H\otimes H,\ \varepsilon: H\to \mathbb{C},\ S: H\to H}_{\mathbb{C}\text{-LINEAR MAPS}}\Big)$$

SO THAT

$(H, m, u)$ IS AN ASSOCIATIVE UNITAL ALGEBRA & $(H, \Delta, \varepsilon)$ IS AN COASSOCIATIVE COUNITAL COALGEBRA

WITH COMPATIBILITY CONDITIONS:

$$\begin{tikzcd} H\otimes H \arrow[rr, "m"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & H \arrow[dl, "\varepsilon"] \\ & \mathbb{C} & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C} \arrow[dl, "u"'] \arrow[dr, "u\otimes u"] & \\ H \arrow[rr, "\Delta"] & & H\otimes H \end{tikzcd}$$

$$\begin{tikzcd} & H\otimes H \arrow[r, "m"] \arrow[dl, "\Delta\otimes\Delta"'] & H \arrow[dr, "\Delta"] & \\ H^{\otimes 4} \arrow[r, "id\otimes flip\otimes id"'] & H^{\otimes 4} \arrow[rr, "m\otimes m"'] & & H\otimes H \end{tikzcd}$$

$(H, m, u, \Delta, \varepsilon)$ IS A BIALGEBRA

& SO THAT $S$ IS AN ANTI-MULTIP. MAP SATISFYING ADD'L COMP. CONDNS.

# HOPF ALGEBRAS

$(\underbrace{H, m, u}_{\text{ALGEBRA}}, \Delta, \varepsilon, \underset{\substack{\uparrow\\ \text{ANTIPODE}}}{S})$

$\underbrace{H, m, u, \Delta, \varepsilon}_{\text{BIALGEBRA}}$; $\underbrace{H, \Delta, \varepsilon}_{\text{COALGEBRA}}$

ARE QUITE NICE...

(1) THEY GENERALIZE GROUPS

- TAKE $G$ A FINITE GROUP

  $\rightsquigarrow$ $\mathbb{C}G$ GROUP ALGEBRA IS A HOPF ALG.

  WITH $\Delta(g) = g\otimes g$

  $\varepsilon(g) = 1_{\mathbb{C}}$

  $S(g) = g^{-1} \quad \forall g\in G$

- TAKE $G$ AN ALGEBRAIC GROUP (GROUP + VARIETY)

  $\rightsquigarrow$ $\mathcal{O}(G)$ IS A HOPF ALGEBRA

WITH $\left\{\begin{array}{l}\Delta \\ \varepsilon \\ S\end{array}\right.$ BEING THE PULLBACK OF AFFINE VARIETY MAPS: $\left\{\begin{array}{ll}m_G & \text{(MULT)} \\ u_G & \text{(UNIT)} \\ i_G & \text{(INVERSION)}\end{array}\right.$

## HOPF ALGEBRAS

$(H, m, u, \Delta, \varepsilon, S)$

ALGEBRA: $(H, m, u)$

COALGEBRA: $(H, \Delta, \varepsilon)$

BIALGEBRA: $(H, m, u, \Delta, \varepsilon)$

ANTIPODE: $S$

ARE QUITE NICE...

## (2) ACTIVE CLASSIFICATION PROGRAM

(DESPITE CONTAINING CLASSIFICATION OF GROUPS)

TWO TRACKS

**SEMISIMPLE**

H-MODULES ARE $\oplus$s OF SIMPLE H-MODULES

(USING TECHNIQUES FROM GROUP THEORY)

**POINTED**

SIMPLE H-COMODULES ARE 1-DIMENSIONAL

(USING TECHNIQUES FROM LIE THEORY)

## HOPF ALGEBRAS

$(H, m, u, \Delta, \varepsilon, S)$

ALGEBRA: $(H, m, u)$

COALGEBRA: $(H, \Delta, \varepsilon)$

BIALGEBRA: $(H, m, u, \Delta, \varepsilon)$

ANTIPODE: $S$

ARE QUITE NICE...

## (3) GET MONOIDAL CATEGORY OF CO/REPRESENTATIONS (≡ CO/MODULES)

A CATEGORY EQUIPPED WITH $\otimes$ AND $\mathbb{1}$

- IF $M, N$ ARE H-MODULES, THEN SO IS $M \otimes N$:

$$\begin{tikzcd}
H \otimes M \otimes N \arrow[rr, dashed, "H \curvearrowright (M \otimes N)"] \arrow[d, "\Delta \otimes id \otimes id"'] & {?_{\mathrm{DEF}}} & M \otimes N \\
H \otimes H \otimes M \otimes N \arrow[rr, "id \otimes flip \otimes id"'] & & H \otimes M \otimes H \otimes N \arrow[u, "H \curvearrowright M \,\otimes\, H \curvearrowright N"']
\end{tikzcd}$$

• ALSO, $\mathbb{C}$ $(= \mathbb{1})$ IS AN H-MODULE

$$\begin{tikzcd} H \otimes \mathbb{C} \arrow[rr, dashed, "H \curvearrowright \mathbb{C}"] \arrow[dr, "\varepsilon \otimes id"'] & \arrow[d, phantom, "\circlearrowright_{\text{DEF}}"] & \mathbb{C} \\ & \mathbb{C} \otimes \mathbb{C} \arrow[ur, "\sim"'] & \end{tikzcd}$$

## SYMMETRIES OF ALGEBRAS VIA HOPF ALG. CO/ACTIONS

DEF'N: TAKE A $\mathbb{C}$-ALGEBRA $A = (A, \, m: A \otimes A \to A, \, u: \mathbb{C} \to A)$

A HOPF ALGEBRA $H$ CO/ACTS ON $A$ IF:

$A$ IS AN "ALGEBRA OBJECT" IN THE MONOIDAL CATEGORY OF $H$-CO/MODULES $\iff$ THE $\mathbb{C}$-VS $A$ IS AN $H$-CO/MOD. & $m, u$ ARE $H$-CO/MOD MAPS.

EX. $\mathcal{O}(GL_2(\mathbb{C}))$ COACTS ON $\mathbb{C}[x,y]$

$\mathcal{O}_q(GL_2(\mathbb{C}))$ COACTS ON $\mathbb{C}_q[x,y]$

### RESULTS I : NO QUANTUM SYMMETRY

QUESTION: GIVEN AN ALGEBRA $A$, DO WE NEED A HOPF ALG. ($\equiv$ QUANTUM GROUP) TO CAPTURE ITS SYMMETRIES? OR WILL GROUPS SUFFICE?

DEF'N: WE SAY THAT AN ALGEBRA $A$ ADMITS NO QUANTUM SYMMETRY

IF EVERY ACTION $\underset{\text{HOPF ALG}}{H} \curvearrowright A$ MUST FACTOR THROUGH $\underset{\text{COCOMMUTATIVE HOPF ALG.}}{L} \curvearrowright A$

$\Delta_L = \text{flip} \circ \Delta_L \longrightarrow$ COCOMMUTATIVE HOPF ALG.

(FACT: $L$ FINITE DIM'L & COCOM $\Rightarrow L \cong \mathbb{C}G$ GROUP ALG.)

THEOREM: NO QUANTUM SYMMETRY OCCURS FOR

| ACTIONS OF H | ON | ALGEBRAS A | REFERENCES |
| --- | --- | --- | --- |
| SEMISIMPLE | | COMMUTATIVE DOMAINS | ETINGOF–W, 2014 |
| FINITE DIMENSIONAL | | SEVERAL DEFORMATIONS OF COMMUTATIVE DOMAINS | CUADRA–ETINGOF–W<br>ETINGOF–W, 2015–2017 |
| | | • $\mathbb{C}_q[x_1, \dots, x_n]$<br>$q \in \mathbb{C}^\times$ NON ROOT OF 1. | SKRYABIN, 2017<br>(IN POSITIVE CHAR.) |
| NOT AN EXHAUSTIVE LIST | | • WEYL ALGEBRAS<br>E.g. $\mathbb{C}\langle x,y\rangle / (yx - xy - 1)$ | |
| | | • RINGS OF DIFFERENTIAL OPERATORS | |

RESULTS II: GENUINE QUANTUM SYMMETRY

QUESTION: GIVEN AN ALGEBRA A, DO WE NEED A HOPF ALG. (≡ QUANTUM GROUP) TO CAPTURE ITS SYMMETRIES? OR WILL GROUPS SUFFICE?

**DEF'N**: WE SAY THAT AN ALGEBRA A

ADMITS GENUINE QUANTUM SYMMETRY

IF $\exists$ $\underset{\text{HOPF ALG}}{H} \curvearrowright A$ DOES NOT FACTOR THROUGH $L \curvearrowright A$

$\Delta_L = \text{flip} \circ \Delta_L \longrightarrow$ COCOMMUTATIVE HOPF ALG.

**THEOREM**: GENUINE QUANTUM SYMMETRY OCCURS FOR

| ACTIONS OF H | ON | ALGEBRAS A | REFERENCES |
|---|---|---|---|
| POINTED (NON-SEMISIMPLE) | | COMMUTATIVE DOMAINS | ETINGOF-W, 2015-2016 |

EX. $U_q(\mathfrak{sl}_2) \curvearrowright \mathbb{C}[x]$ GENUINELY

| ACTIONS OF H | ON | ALGEBRAS A | REFERENCES |
|---|---|---|---|
| FINITE DIMENSIONAL | | PATH ALGEBRAS $\mathbb{C}Q$ OF $\underbrace{\text{QUIVERS}}_{\text{DIRECTED GRAPH}}$ Q | KINSER-W, 2016<br>KINSER-OSWALD 2010.15197 (NEW) |

ALG. GEN. BY VERTICES & ARROWS OF Q, & RELATIONS DETERMINED BY EXISTENCE OF PATHS

EX. HAVE FORMULAS FOR $U_q(\mathfrak{g}) \curvearrowright \mathbb{C}Q$

| ACTIONS OF H | ON | ALGEBRAS A | REFERENCES |
|---|---|---|---|
| FINITE DIMENSIONAL | | ARTIN-SCHELTER REGULAR ALGEBRAS | KIRKMAN-KUZMANOVICH-ZHANG, 2008<br>CHAN-KIRKMAN-W-ZHANG, 2016<br>CHEN-WANG-ZHANG 2008.01289 (NEW) |

NOT NEC. COMMUTATIVE ALGS. THAT ARE GRADED $A = \bigoplus_{i \geq 0} A_i$, CONNECTED $A_0 = \mathbb{C}$, & BEHAVES HOMOLOGICALLY LIKE A COM. POLYNOMIAL ALG.

# RESULTS III: UNIVERSAL QUANTUM SYMMETRY

(CURRENT INTERESTS)

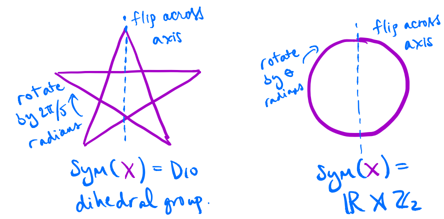
QUESTIONS: GIVEN AN ALGEBRA A.

(a) WHAT IS THE BIGGEST HOPF ALGEBRA THAT CO/ACTS ON A?

I.E. WHAT ARE THE 'QUANTUM AUTOMORPHISM GROUPS'?

(b) DOES SUCH A HOPF ALGEBRA SHARE PROPERTIES WITH A?

Ex. COLLECTION OF SYMMETRIES SHOULD BE DISCRETE $\Leftrightarrow$ OBJECT/ALGEBRA IS DISCRETE




(c) ADDRESS ABOVE FOR OTHER FRAMEWORKS OF SYMMETRY (BROADER THAN HOPF CO/ACTIONS)

∃ GENERALIZATIONS OF HOPF ALGEBRAS: WHAT ARE THEY GOOD FOR SYMMETRY-WISE?

(a) WHAT IS THE BIGGEST HOPF ALGEBRA THAT COACTS ON A?

USE COACTIONS (∗)

(∗) MIGHT DECORATE THIS WITH CONDITIONS IF NEEDED

THIS IS A HOPF ALGEBRA $\mathcal{O}(A)$ THAT (LEFT) COACTS ON A (∗) SO THAT ∀ HOPF ALGEBRAS H (LEFT) COACTING ON A ∃! HOPF ALGEBRA MAP

UNIVERSAL HOPF ALGEBRA COACTING ON A (∗)

$$
\begin{tikzcd}
 & \mathcal{O}(A) \otimes A \arrow[d, dashed, "\pi \otimes id_A"] \\
A \arrow[ur, "\text{coaction of } \mathcal{O}(A) \text{ on } A"] \arrow[r] & H \otimes A
\end{tikzcd}
$$

(left)

$\pi : \mathcal{O}(A) \longrightarrow H$

WHERE DIAGRAM COMMUTES:

COACTION OF H ON A

Have $\mathcal{O}^{right}(A)$
$\mathcal{O}^{both}(A)$

Ex. $\mathcal{O}(GL_n(\mathbb{C}))$, $\mathcal{O}(SL_n(\mathbb{C}))$, $\mathcal{O}(Mat_n(\mathbb{C}))$ IS THE UNIV. BI/HOPF ALG. THAT COACT FROM BOTH SIDES ON $\mathbb{C}[x_1, \dots, x_n]$ SUBJECT TO CONDITIONS: INVERTIBLE DET. / DET = 1 / (NONE)

& THIS DEFORMS NICELY FOR MANY DEFORMATIONS OF A

I.E. $\exists$ "GL-LIKE" HOPF ALG. $\mathcal{O}_A(GL)$

"SL-LIKE" HOPF ALG. $\mathcal{O}_A(SL)$

BIALGEBRA $\mathcal{O}_A(Mat)$

Ex. FOR $A = \mathbb{C}_q[x_1, \dots, x_n]$, $\exists$ CORRESPONDING q-VERSIONS

$\mathcal{O}_q(GL_n(\mathbb{C}))$

$\mathcal{O}_q(SL_n(\mathbb{C}))$

$\mathcal{O}_q(Mat_n(\mathbb{C}))$

NICE REF

MANIN'S 1989 TEXT (2018 REPRINT)

(b) DOES SUCH A HOPF ALGEBRA SHARE PROPERTIES WITH A?

OFTEN TIMES, YES!

Ex. $\mathcal{O}_q(GL_n(\mathbb{C}))$ $\mathcal{O}_q(SL_n(\mathbb{C}))$ $\mathcal{O}_q(Mat_n(\mathbb{C}))$

ARE EACH AS NICE AS $\mathbb{C}_q[x_1, \dots, x_n]$

RING-THEORETICALLY: NOETHERIAN, DOMAIN, NICE GROWTH

HOMOLOGICALLY: FINITE GLOBAL & INJECTIVE DIMENSIONS

A SPECIAL CASE OF THIS QUESTION WAS POSED IN WORK OF ARTIN-SCHELTER-TATE, 1991 BY MANIN

AST, 1991

BROWN-GOODEARL'S TEXT

W-WANG, 2016

PHILOSOPHY

THE MORE SOPHISTICATED
THE OBJECT
THE MORE SOPHISTICATED
THE FRAMEWORK NEEDED
TO ANALYZE ITS SYMMETRIES

(c) ADDRESS ABOVE FOR OTHER
FRAMEWORKS OF SYMMETRY
(BROADER THAN HOPF CO/ACTIONS)

↪ RECENTLY STARTED TO ANALYZE SYMMETRIES OF
<u>NONCONNECTED</u> GRADED ALGEBRAS

$A = \bigoplus_{i \geq 0} A_i$ WITH $A_0$ <u>NOT</u> EQUAL TO $\mathbb{C}$

"EXPANDING THE BASE"

INCLUDE:
- PATH ALGEBRAS $\mathbb{C}Q$ w/ # OF VERTICES OF $Q > 1$

W-WICKS-WON 1911.12847

- $\bigoplus$ OF CONNECTED GRADED ALGEBRAS

HUANG-W-WICKS-WON 2008.00606

- SKEW GROUP ALGEBRAS $B \rtimes G$ FOR $\underset{\text{GROUP}}{G} \curvearrowright \underset{\text{ALG}}{B}$

OTHERS WORKS IN PROGRESS

- SMASH PRODUCT ALGEBRAS $B \# H$ FOR $\underset{\text{HOPF ALG}}{H} \curvearrowright \underset{\text{ALG}}{B}$

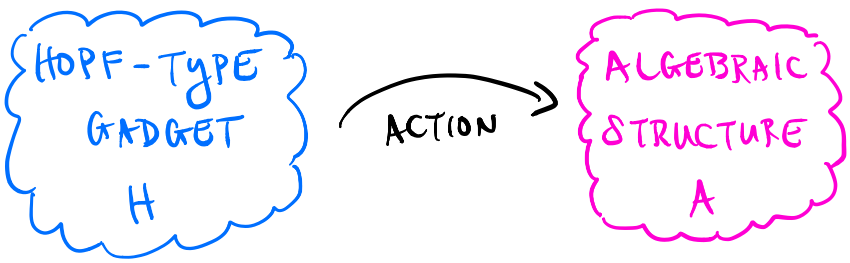
... VIA CO/ACTIONS OF WEAK BI/HOPF ALGEBRAS

$(H, m, u, \Delta, \varepsilon, S)$ WITH $(H, m, u) =$ ALG, $(H, \Delta, \varepsilon) =$ COALG

BUT COMPABILITY CONDITIONS ARE WEAKER THAN THAT FOR BI/HOPF ALGEBRAS

$\equiv$ QUANTUM GROUPOIDS

## ALGEBRAIC QUANTUM SYMMETRY




# Thanks for listening!