DEFORMATIONS OF AN ASSOC. ALGEBRA A_0

(FLAT) FORMAL n-PARAMETER DEFORMATION A of A_0 — $A_{t_1,\dots,t_n}$

= associative alg over $K := k[[t_1,\dots,t_n]]$
topologically free as a K-module ($\simeq A_0[[t_1,\dots,t_n]]$ as k-mods)
equipped with alg isom $A/(t_1,\dots,t_n)A \simeq A_0$

Gerstenhaber

(FLAT) FORMAL (1-parameter) DEFORM'N A of A_0 — A_t $= \varprojlim A_N$

= associative alg over $k[[t]]$, $\simeq A_0[[t]]$ as k-vs.
with multiplication $a * b = \sum_{i \geq 0} t^i \mu_i(a,b)$
for $a, b \in A$, $\mu_i : A_0 \otimes A_0 \to A_0$ k-lin maps ("mult maps") $\mu_0(a,b) = ab$

FORMAL DEFORMATION A of LEVEL/ORDER N — A_N

= associative alg over $k[t]/(t^{N+1})$,
free as a $k[t]/(t^{N+1})$-module ($\simeq A_0[t]/(t^{N+1})$ as $k$vs)
with mult. $a * b = \sum_{i=0}^{N} t^i \mu(a,b)$, $a, b \in A$
$\mu_i : A_0 \otimes A_0 \to A_0$ are also linear / $k[t]/(t^{N+1})$

INFINITESIMAL DEFORMATION A of A_0 — $A_{N=1}$

= formal deformation of level 1, so over $k[t]/(t^2)$

If A_0 is a COMMUTATIVE associative k-alg.

QUANTUM DEFORMATION A of A_0

= formal deformation of A_0

DEFORMATION QUANTIZATION of $(A_0, \{\,\})$

give A_0 a Poisson bracket $\{,\}$:
where $\{a_0, b_0\}$ = image of $[a,b]$ in $tA/t^2A \simeq A_0$
where a, b are any lifts of a_0, b_0 to A
$\rightsquigarrow$ D.Q. of $(A_0, \{,\})$ = formal def of A_0.
(motivated by deformations of Poisson manifolds)
A = quant'n of A_0 $\quad (A_0, \{\}) = $ 'classical limit' of A

ALGEBRAIC DEFORMATION

of an algebraic variety [scheme] X or its coordinate ring or $\mathcal{O}_X$

(as in Hartshorne's Deformation Thy)
$f : X \to T$ flat morphism of schemes.....

Can repeat these definitions for graded A_0, A; set $\deg t_i = 1$

As in Braverman–Gaitsgory —

A_t graded formal def'n of A_0

$/(t-1)$ ↓ ↑ Rees ring $R(B) \simeq A_t$ as graded alg

$B = A_t/(t-1)A_t$
filtered or PBW deform of A_0
↳ $\mathrm{gr}\, B \cong A_0$ as gr alg.

The study of such μ_i that make A associative is done with Hochschild cohom.

μ_1 = Hochschild 2-cocycle

μ_2 = Hochschild cochain

The set of $\simeq$ classes of inf. def. of A_0 is param. by space $Z^2(A_0)$ of Hoch. 2-cocycles of A_0, value in A_0 -or- $H^2(A_0, A_0)$

If $H^2(A_0, A_0) = 0$ then A_0 is RIGID (no def'n)

Given A_N, its obstruction to lift to A_{N+1} lies in $H^3(A_0, A_0)$

If $H^3(A_0, A_0) = 0$, then μ_i can be solved for all $i \geq 0$ and $\exists$ UNIVERSAL DEFORMATION A of A_0 — A_u

---

- M = smooth C^∞ mfld, or smooth affine algebraic variety/$\mathbb{C}$
  $A_0 = \mathcal{O}(M)$
  [Hochschild–Kostant–Rosenberg] $H^i(A_0) = \Gamma(M, \Lambda^i TM)$ as an $A_0 = H^0(A_0)$-module

- [Kontsevich] Any Poisson manifold can be quantized.
  $\exists$, up to equiv, a canonical correspondence between assoc. deform'n of A_0 and formal Poisson structures on A_0

- [Etingof–Kazhdan] Any Lie bialgebra can be quantized.
  That is if $\mathfrak{a}$ is a Lie bialgebra, then $\exists$ Hopf alg. deformation of $U\mathfrak{a}$ whose infin'l is the cocommutator of $\mathfrak{a}$.

C. Walton